Birkhoff's Covariety Theorem without limitations

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To my teacher and friend Věra Trnková, from whom I have learned so much, on the occasion of her seventieth birthday.

Abstract. J. Rutten proved, for accessible endofunctors F of **Set**, the dual Birkhoff's Variety Theorem: a collection of F-coalgebras is presentable by coequations (= subobjects of cofree coalgebras) iff it is closed under quotients, subcoalgebras, and coproducts. This result is now proved to hold for all endofunctors F of **Set** provided that coequations are generalized to mean subchains of the cofree-coalgebra chain. For the concept of coequation introduced by H. Porst and the author, which is a subobject of a member of the cofree-coalgebra chain, the analogous result is false, in general. This answers negatively the open problem of A. Kurz and J. Rosický whether every covariety can be presented by equations w.r.t. co-operations.

In contrast, in the category of classes Birkhoff's Covariety Theorem is proved to hold for all endofunctors (using Rutten's original concept of coequations).

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1. Introduction

Coalgebras over a set functor F represent, as demonstrated in the article of J. Rutten [Ru], dynamical systems the type of which is presented by F. Given a coalgebra $A \xrightarrow{\alpha} FA$, we think of A as the state set and, for a state a, the observations we make about a are represented by $\alpha(a)$. For example, a deterministic systems with a binary input and with deadlock states is given by the functor $FX = X \times X + 1$: the function $\alpha\colon A \to A \times A + 1$ assigns to a state a either the pair of next states, or, if a is a deadlock state, the unique element of 1. Another example: labeled transition systems with a set S of actions are coalgebras of the functor $\mathscr{P}(S \times -)$ which is the composite of $FX = S \times X$ and the power-set functor $\mathscr{P}$.

Terminal coalgebras C of the given functor F are coalgebras of "principal behaviors" of states of F-coalgebras. For example, a terminal coalgebra of $FX = X \times X + 1$ is the coalgebra C of all binary trees (finite and infinite). For every coalgebra A the unique homomorphism $f^{\#}\colon A \to C$ assigns to every state the

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binary tree $f^{\#}(a)$ of unfolding a in the deterministic system (e.g., $f^{\#}(a)$ is the singleton tree if a is a deadlock state, $f^{\#}(a)$ is the complete binary tree if no deadlock can be reached from a, etc.). More generally, for a set K of colors, a cofree coalgebra $C(K)$ on K is a coalgebra of "principal behaviors" of states of F-coalgebras colored in K. For any coloring $f\colon A \to K$ the corresponding homomorphism $f^{\#}\colon A \to C(K)$ assigns to every state a the behavior $f^{\#}(a)$, assuming the colors of states are observable. Example: for $F = X \times X + 1$ we have the coalgebra $C(K)$ of all K-colored binary trees. An endofunctor is called a *covarietor* if for every object K a cofree coalgebra exists — this dualizes varietors, see [AT].

J. Rutten proposed in [Ru] to study presentations of collections of coalgebras by subobjects $m\colon M \hookrightarrow C(K)$ of cofree coalgebras (dual to the usual equational presentation of algebras: a system of equations can be substituted by a quotient object of a free algebra). A coalgebra A "satisfies" m provided that all homomorphisms $f^{\#}\colon A \to C(K)$ factorize through m. Now recall the famous Birkhoff's Variety Theorem: a collection of algebras is equationally presentable iff it is an HSP collection, i.e., closed under "homomorphic images" (quotients), subalgebras and products. We will, dually, speak about *HSC collections* of coalgebras: these are collections closed under quotient coalgebras, subcoalgebras, and coproducts. (Here quotient coalgebras are meant to be represented by surjective homomorphisms, and subcoalgebras by one-to-one homomorphisms.) The following was proved by J. Rutten for all accessible (= bounded) functors $F\colon \mathbf{Set} \to \mathbf{Set}$:

> Birkhoff's Covariety Theorem: A collection of coalgebras can be presented by subobjects of cofree coalgebras iff it is an HSC collection.

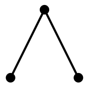
J. Rutten [Ru] assumed that F preserves weak pullbacks, but P. Gumm showed that this assumption is not needed, see [G]. He also observed that instead of general subobjects $M \hookrightarrow C(K)$ we can restrict ourselves to the coatomic ones: $M = C(K) - \{t\}$ for elements t of $C(K)$. (This is analogous to the situation in algebra: we consider an individual equation $u = v$ rather than systems of equations, and this means that we form the atomic quotient of a free algebra given by the equivalence relation with $\{u, v\}$ as the only nontrivial equivalence class.) We call these subobjects $C(K) - \{t\}$ *coequations* and denote them by

$$\boxtimes t \qquad \text{(read: avoid } t\text{)} \qquad \text{for } \ t \in C(K).$$

Thus, a coalgebra A *satisfies* a coequation $\boxtimes t$ provided that for every coloring $f\colon A \to K$ it avoids t in the obvious sense:

$$f^{\#}(a) \neq t \qquad \text{for all } \ a \in A.$$

Example: for $FX = X \times A + 1$ the coequation

$\boxtimes$ 

expresses the property that if a non-deadlock state has both next states deadlock states, then they are equal.

The aim of our paper is to extend Birkhoff's Covariety Theorem from accessible endofunctors to all endofunctors. This needs some explanation since, in general, cofree coalgebras do not exist. One way of avoiding this difficulty is to work with classes rather than small sets. The category

$$\mathbf{Class}$$

of classes and functions has a number of convenient properties, see [AMV], and one of them is that every endofunctor possesses cofree coalgebras. We prove below that the above Birkhoff's Covariety Theorem holds for all $F\colon \mathbf{Class} \to \mathbf{Class}$.

Example: the power-set functor $\mathscr{P}$. We can extend $\mathscr{P}$ to an endofunctor $\mathscr{P}^\infty$ of **Class** by

$$\mathscr{P}^\infty X = \text{ the class of all small subsets of } X.$$

A cofree coalgebra $C(K)$ of $\mathscr{P}^\infty$ can be described as follows, see [RT]: if $C'(K)$ is the coalgebra of all small, rooted, non-ordered trees with nodes colored in K (whose coalgebra structure $C'(K) \to \mathscr{P}^\infty C'(K)$ is given by forming the set of all maximum subtrees), then

$$C(K) = C'(K)/\sim$$

is the quotient coalgebra modulo the bisimilarity equivalence $\sim$. As usual, we consider $\mathscr{P}$-coalgebras to be graphs.

For the trivial tree t_0 (the root only) the coequation

$$\boxtimes t_0$$

presents all graphs without leaves, i.e., all graphs in which every node has a neighbor. Let Ω denote the bisimilarity class of the tree consisting of a single path, then the coequation

$$\boxtimes \Omega$$

presents all graphs such that from every node a leaf is reachable by a path.

As the above example demonstrates, one way of presenting collections of coalgebras of a functor $F\colon \mathbf{Set} \to \mathbf{Set}$ is to extend F to $F^\infty\colon \mathbf{Class} \to \mathbf{Class}$. There exists an essentially unique such extension, see [AMV]. We then use coequations $\boxtimes t$ where t are elements of cofree F^∞-coalgebras.

Another approach has been proposed by H. Porst and the present author in [AP]: instead of working with $C(K)$, we can stay entirely within the category **Set** by forming a chain

$$W(K): \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Set}$$

of "approximations" of $C(K)$ as follows:

$$\begin{aligned} W(K)_0 &= 1, \\ W(K)_{i+1} &= FW_i(K) \times K, \\ W(K)_j &= \lim_{i<j} W(K)_i \qquad \text{for limit ordinals } j. \end{aligned}$$

This is the dual of the free algebra chain introduced by the author [A]. Let $A \xrightarrow{\alpha} FA$ be a coalgebra, then for every coloring $f: A \to K$ we have a cone

$$f_i^{\#}: A \to W(K)_i \qquad (i \in \mathbf{Ord})$$

"approximating" the homomorphism $f^{\#}: A \to C(K)$: this is the unique cone such that for every ordinal i we have

$$f_{i+1}^{\#} \equiv A \xrightarrow{\langle F f_i^{\#} \cdot \alpha, f \rangle} FW(K)_i \times K.$$

This makes possible to define a *coequation* as an expression

$$\boxtimes t \qquad \text{for } \ t \in W(K)_i$$

where K is a small set, and i is an ordinal. A coalgebra A *satisfies* $\boxtimes t$ provided that for every coloring $f: A \to K$ we have

$$f_i^{\#}(a) \neq t \qquad \text{for all } \ a \in A.$$

For example, in case of $\mathscr{P}$ we can choose

$$(\emptyset, x) \in \mathscr{P}1 \times \{x\} = W\big(\{x\}\big)_1$$

and we see that

$$\boxtimes(\emptyset, x)$$

presents, again, all graphs without leaves.

Another approach is more global: instead of choosing one element in one member of the chain $W(K)$, we choose a compatible collection $t = (t_i)$ where

$$t_i \in W(K)_i \qquad \text{for every ordinal } i$$

(compatibility means that for all ordinals $i \leq j$ the connecting morphism from $W_j(K)$ to $W_i(K)$ takes t_j to t_i). We call the expression

$$\boxtimes t$$

a *generalized coequation*. A coalgebra A satisfies it iff for every coloring $f\colon A \to K$ we have

$$\big(f_i^{\#}(a)\big)_{i \in \mathbf{Ord}} \neq t \qquad \text{for all} \ \ a \in A.$$

That is: for every $a \in A$ there exists an ordinal i with $f_i^{\#}(a) \neq t_i$. We are going to prove that

(a) Birkhoff's Covariety Theorem holds for every endofunctor of **Set** provided that we consider presentation by generalized coequations,

(b) coequations are in general insufficient, i.e., there exists a set functor F and an HSC-collection of coalgebras which cannot be presented by coequations.

We will also show that presentation by coequations is precisely as "powerful" as presentation by equations between co-operations, as introduced by A. Kurz and J. Rosický [KR]. Therefore, (b) solves negatively the open problem posed in [KR] whether every HSC collection can be presented by equations.

Related Work. As mentioned already, [Ru] and [G] are the original sources for Birkhoff's Covariety Theorem, see also [AH]. Co-operations introduced by A. Kurz and J. Rosický [KR] are a formally different approach to coalgebra — we prove below that the expressive power is equivalent to (non-generalized) coequations.

2. Birkhoff's Covariety Theorem in Class

2.1. We make the usual assumption about a choice of a universe **Set** of small sets. Assuming the Axiom of Choice for all (not necessarily small) sets, we then have a cardinality for every (small or large) set, and we denote by

$$\aleph_\infty$$

the cardinality of the universe **Set** — in the other words $\aleph_\infty$ is the first large cardinal. We can (and will) identify the category **Set** of small sets with the category of all sets with cardinality less than $\aleph_\infty$; the two categories are clearly equivalent. And

$$\mathbf{Class}$$

denotes the category of all sets of cardinality less or equal to $\aleph_\infty$. (This is justified by the intuition that a class is a "property of small sets", i.e., a subset of **Set**. Each such subset either has the full cardinality $\aleph_\infty$, or a smaller one.)

Shortly, our foundations are ZFC with a choice of an inaccessible cardinal $\aleph_\infty$.

2.2 Notation. Let F: **Class** $\to$ **Class** be a functor. The category of coalgebras is denoted by **Coalg** F. Its forgetful functor into **Class** has a right adjoint, see [AMV], which we denote by

$$K \mapsto \big(C(K) \xrightarrow{\gamma_K} FC(K)\big).$$

Thus, $C(K)$ is a cofree coalgebra on K, the couniversal coloring is denoted by $\varepsilon_K: C(K) \to K$.

2.3 Remark. J. Rutten introduced in [Ru] presentation of coalgebras by a subobject

$$m: M \hookrightarrow C(K)$$

of a cofree coalgebra. A coalgebra $A \to FA$ *satisfies* m iff for every coloring $f: A \to K$ the corresponding homomorphism $f^{\#}: A \to C(K)$, defined by $f = \varepsilon_K f^{\#}$, factorizes through m.

Following an idea of H.P. Gumm [G], we can work, instead of with all subsets of $C(K)$, with the maximum subsets $\boxtimes t = C(k) - \{t\}$. In fact, a coalgebra satisfies $M \subseteq C(K)$ iff it satisfies $\boxtimes t$ for all elements t of the complement of M. This leads to the following:

2.4 Definition. Let F be an endofunctor of **Class**. By a *coequation* is meant an expression

$$\boxtimes t$$

where t is an element of a cofree coalgebra $C(K)$. A coalgebra A *satisfies* the coequation provided that for every coloring $f: A \to K$ the homomorphism $f^{\#}: A \to C(K)$ fulfils

$$f^{\#}(a) \neq t \qquad \text{for all} \;\; a \in A.$$

A collection of coalgebras is called a *covariety* if it can be presented by coequations.

2.5 Remark. For an endofunctor F of **Class** it is clear (from the existence of cofree coalgebras) that epimorphisms in **Coalg** F are precisely the surjective homomorphisms. These homomorphisms represent *quotient coalgebras*, i.e., quotient objects in **Coalg** F.

Homomorphisms carried by monomorphisms in **Class** are called *subcoalgebras*.

2.6 Birkhoff's Covariety Theorem in Class. *For every endofunctor F of the category of classes, covarieties are precisely the full subcategories of* **Coalg** F *closed under quotient coalgebras, subcoalgebras, and coproducts.*

Remark. (i) "Coproducts" means class-indexed coproducts, that is: all coproducts existing in **Coalg** F.

(ii) Every covariety can be presented by a single subobject of a cofree algebra.

Proof: It is easy to see that every covariety is closed under the three constructions above, see e.g. [Ru].

Conversely, let $\mathscr{A}$ be a full subcategory closed under coproducts, subcoalgebras, and quotient coalgebras. We observe that every homomorphism $h: B \to A$ in **Coalg** F with $B \neq \emptyset$ factorizes in **Class** as $h = me$ for an epimorphism $e: B \to \bar{B}$ and a split monomorphism $m: \bar{B} \to A$. Then Fm is a monomorphism, thus, there exists a unique coalgebra structure on $\bar{B}$ making both e and m homomorphisms:

$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[d, "e"'] & FB \arrow[d, "Fe"] \\ \bar{B} \arrow[r, dashed, "\bar{\beta}"] \arrow[d, "m"'] & F\bar{B} \arrow[d, "Fm"] \\ A \arrow[r, "\alpha"'] & FA \end{tikzcd}$$

Now choose a proper class K and form the comma-category $\mathscr{A}/C(K)$ of all homomorphisms from coalgebras in $\mathscr{A}$ into a cofree coalgebra $C(K)$. The coproduct

$$B = \coprod A \qquad \text{in } \mathbf{Coalg}\ F$$

indexed by all objects $f: A \to C(K)$ of $\mathscr{A}/C(K)$ exists (since **Class** has class-indexed coproducts and the forgetful functor of **Coalg** F creates them) and we obtain a canonical homomorphism $h: B \to C(K)$. Let us factorize it as above: here we need $B \neq \emptyset$, but the case $B = \emptyset$ is trivial (this would imply that $\mathscr{A}$ contains only the empty coalgebra — and this is a covariety). Since $\mathscr{A}$ is closed under coproducts and quotients, we conclude that $\bar{B}$ lies in $\mathscr{A}$. We claim that $\mathscr{A}$ is presented by the subobject

$$m: \bar{B} \hookrightarrow C(K).$$

Or, equivalently, by all coequations

$$\boxtimes t \qquad \text{for } \ t \in C(K) - m[\bar{B}].$$

In fact, every coalgebra $A \in \mathscr{A}$ clearly satisfies m. Conversely, let D be a coalgebra satisfying $m: \bar{B} \hookrightarrow C(K)$, and let $f: D \to K$ be a monic function (which exists because K is a proper class). The unique homomorphism

$$f^{\#}: D \to C(K) \qquad \text{with} \qquad f = \varepsilon_K f^{\#}$$

factorizes through m:

$$f^{\#} = mg \qquad \text{for some} \qquad g: D \to \bar{B}.$$

Since mg is a homomorphism and Fm is monic (recall our assumption $B \neq \emptyset$), it follows that g is a homomorphism:

$$\begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "g"'] & FD \arrow[d, "Fg"] \\ \bar{B} \arrow[r, "\bar{\beta}"] \arrow[d, "m"'] & F\bar{B} \arrow[d, "Fm"] \\ C(K) \arrow[r, "\gamma_K"'] & FC(K) \end{tikzcd}$$

Consequently, D is a subcoalgebra of $\bar{B}$, thus, $D \in \mathscr{A}$.

This proves that $\mathscr{A}$ is a covariety presented by $m: M \hookrightarrow C(K)$. □

2.7 Remark. (a) In the above proof we constructed a cofree $\mathscr{A}$-coalgebra $\bar{B}$ on colors from K. We can perform this construction with every small set L of colors, and obtain a cofree $\mathscr{A}$-coalgebra $m_L: \bar{B}_L \hookrightarrow C(L)$ on L. Observe that for small F-coalgebras D the following holds: D lies in $\mathscr{A}$ iff it satisfies $m_L: \bar{B}_L \hookrightarrow C(L)$ for every small set L. The proof is as above, except that where we used a monomorphism $f: D \to K$ we now put $L = D$ and $f = \mathrm{id}_D$.

(b) Let $F: \mathbf{Set} \to \mathbf{Set}$ be a covarietor. Then, again, collections of coalgebras presented by coequations are the HSC collections. This has the same proof as above, except the coproduct B is made small by considering the individual elements of $C(K)$. Another proof of this result can be found in [AP, Theorem 6.2].

(c) Theorem 2.5 can also be derived from Theorem 4.1 of A. Kurz and J. Rosický [KR], as we explain in the last section.

3. Birkhoff's Covariety Theorem in Set

3.1 Remark. In algebraic specification initial algebras play a central role. Given an endofunctor F of $\mathbf{Set}$, an initial algebra need not exist, in general, but we always have a transfinite chain $V: \mathbf{Ord} \to \mathbf{Set}$ "of approximations": $V_0 = 0$ is the initial object (empty set), $V_{i+1} = FV_i$, and for limit ordinals j we put $V_j = \mathrm{colim}_{i<j} V_i$. This chain was introduced in [A], and the dual form

$$W: \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Set} \qquad \text{with} \qquad W_0 = 1, \quad W_{i+1} = FW_i, \quad W_j = \lim_{i<j} W_i$$

was later used by M. Barr [B].

When applying this idea to the functor $F(-) \times K$ we obtain the following

3.2 Definition. For a set functor F and a small set K (of colors), we define a *cofree-coalgebra chain*

$$W(K)\colon \mathbf{Ord}^{\mathrm{op}} \to \mathscr{K}$$

to be the essentially unique chain with

$$\begin{aligned} W_0(K) &= 1, \qquad \text{a terminal object,} \\ W_{i+1}(K) &= FW_i(K) \times K \qquad \text{and} \qquad W_{i+1,j+1} = FW_{i,j} \times \mathrm{id}_K, \\ W_j(K) &= \lim_{i<j} W_i(K) \qquad \text{for limit ordinals } j. \end{aligned}$$

For every coalgebra $A \xrightarrow{\alpha} FA$ and every morphism $f\colon A \to K$ there exists a unique cone

$$f_i^{\#}\colon A \to W_i(K) \qquad (i \in \mathbf{Ord})$$

of the above chain with

$$f_{i+1}^{\#} = A \xrightarrow{\langle f_i^{\#}\alpha, f\rangle} W_i(K) \times K \qquad (i \in \mathbf{Ord}).$$

3.3 Remark. (a) Whenever a cofree coalgebra $C(K)$ exists, then the above chain *stops* at some ordinal $i \in \mathbf{Ord}$, i.e., $W_{i,i+1}\colon FW_i(K) \to W_i(K)$ is an isomorphism. Then $C(K) = W_i(K)$ and the inverse of $W_{i,i+1}\colon C(K) \to FC(K) \times K$ has components γ_K and ε_K, respectively. This was proved in [AK].

(b) Conversely, whenever the chain stops, it yields a cofree coalgebra.

(c) Homomorphisms of coalgebras $h\colon A \to B$ preserve the approximation cones: given a coloring $f\colon B \to K$, then the coloring $g = fh\colon A \to K$ fulfils $g_i^{\#} = f_i^{\#}\cdot h$, see [AP, Lemma 3.13].

(d) For an arbitrary endofunctor of **Class** we have a cofree coalgebra $C(K)$, which is in general not a transfinite limit of $W_i(K)$, i.e., the cone

$$\big(\varepsilon_K\big)_i^{\#}\colon C(K) \to W_i(K) \qquad (i \in \mathbf{Ord})$$

is not necessarily a limit cone. It follows however from results of J. Worell that that cone is collectively monic, in other words, a subcone of the limit cone, see [W].

3.4 Remark. In [AP] we defined, for an arbitrary set functor F, a *coequation* to be an expression $\boxtimes t$ where t is an element of $W_i(K)$ (for some set K of colors and some ordinal i). A coalgebra A *satisfies* the coequation $\boxtimes t$ provided that for every coloring $f\colon A \to K$ and every element $a \in A$ we have $f_i^{\#}(a) \neq t$. We will prove in the next section that this concept of coequation is too weak to lead to Birkhoff's Covariety Theorem without limitations. We need a somewhat stronger concept:

3.5 Definition. Let F be an endofunctor of **Set**. By a *generalized coequation* is meant an expression

$$\boxtimes t \qquad \text{(read: avoid } t\text{)}$$

where, for some small set K of colors, $t = (t_i)_{i\in\mathbf{Ord}}$ is a compatible collection of elements of the cofree-coalgebra chain $W(K)$. (That is, $t_i \in W_i(K)$ and if $j \geq i$ then $t_i = W_{j,i}(t_j)$.)

An F-coalgebra A is said to **satisfy** the coequation $\boxtimes t$ provided that for every coloring $f: A \to K$ and every element $a \in A$ the compatible collection $f_i^{\#}(a)$, $i \in \mathbf{Ord}$, is distinct from t.

3.6 Birkhoff's Covariety Theorem in Set. *Let F be an endofunctor of* **Set**. *A full subcategory of* **Coalg** *F can be presented by generalized coequations iff it is closed under quotient coalgebras, subcoalgebras, and coproducts.*

Proof: Denote by F^{∞}: **Class** $\to$ **Class** the essentially unique extension of F. It preserves transfinite colimits (i.e., colimits indexed by **Ord**), see [AMV]. Then **Coalg** F is contained in **Coalg** F^{∞}. Given a full subcategory $\mathscr{A}$ of **Coalg** F closed under (small) coproducts, sucobalgebras and quotient coalgebras, we denote by $\mathscr{A}^{\infty}$ the closure of $\mathscr{A}$ under transfinite colimits in **Coalg** F^{∞}. Then $\mathscr{A}^{\infty}$ will be proved to be closed under class-indexed coproducts, subcoalgebras and quotient coalgebras.

(a) Subcoalgebras: Without loss of generality we can assume that F preserves finite intersections. In fact, as proved in [AT], by changing the value of a set functor at $\emptyset$ we can obtain a functor preserving finite intersections — and the change of $F\emptyset$ does not change the category of F-coalgebras.

Let $A = \operatorname{colim}_{i\in\mathbf{Ord}} A_i$ be an arbitrary object of $\mathscr{A}^{\infty}$, where (A_i) is a chain in $\mathscr{A}$ with colimit maps $c_i: A_i \to A$, $i \in \mathbf{Ord}$. Let B be a subcoalgebra of the coalgebra A. We can assume that the colimit cocone is formed by inclusions of subalgebras $c_i: A_i \hookrightarrow A$ ($i \in \mathbf{Ord}$) — if not, use epi-mono factorizations of the colimit morphisms c_i and the fact that $\mathscr{A}$ is closed under quotient coalgebras. Since F preserves finite intersections, an intersection of two subcoalgebras is a subcoalgebra. Thus, we obtain a transfinite chain of subcoalgebras $B \cap A_i$ of B, and we know that $B \cap A_i$ lies in $\mathscr{A}$ since A_i does. Obviously, B is a colimit of this chain, thus, $B \in \mathscr{A}^{\infty}$.

(b) Quotient coalgebras: For $A = \operatorname{colim} A_i$ as above, let $e: A \to E$ be a quotient coalgebra and let

$$e \cdot c_i \equiv A_i \overset{e_i}{\twoheadrightarrow} E_i \overset{m_i}{\rightarrowtail} E \qquad (i \in \mathbf{Ord})$$

be an epi-mono factorization. Then $E_i \in \mathscr{A}$ since $A_i \in \mathscr{A}$, and we obtain a chain $(E_i)_{i\in\mathbf{Ord}}$ which is easily seen to have E as a colimit.

(c) Class-indexed coproducts: For every ordinal $j \in \mathbf{Ord}$ let

$$A^j = \operatorname{colim}_{i\in\mathbf{Ord}} A_i^j$$

be an object of $\mathscr{A}^{\infty}$ expressed as a transfinite colimit of objects A_i^j in $\mathscr{A}$. Form a new transfinite chain in $\mathscr{A}$

$$B_i = \coprod_{j \le i} A_i^j \qquad (i \in \mathbf{Ord})$$

with the obvious connecting morphisms. It is clear that this chain has a colimit $\coprod_{j \in \mathbf{Ord}} A^j$ (in **Class**, thus also in **Coalg** F^{∞}) — consequently, the last coproduct lies in $\mathscr{A}^{\infty}$.

By 2.6, 2.7 and 2.3 there exists a collection of coequations $\boxtimes t$ for $t \in C(L)$, L a small set, presenting precisely the family of all small coalgebras of $\mathscr{A}^{\infty}$. Now, every small coalgebra of $\mathscr{A}^{\infty}$ obviously lies in $\mathscr{A}$, thus, the coequations $\boxtimes t$ above form a presentation of $\mathscr{A}$.

The final step is the observation that each coequation $\boxtimes t$ where $t \in C(L)$ can be substituted by the generalized coequation $\boxtimes (t_i)_{i \in \mathbf{Ord}}$ where for the universal coloring (2.2) we put

$$t_i = \big(\varepsilon_L\big)_i^{\#}(t) \qquad \text{for } \ i \in \mathbf{Ord}.$$

In fact, this follows from Remark 3.3(d): let $\widetilde{C}(L)$ be a limit of $W(L)$. For every F-coalgebra A and every coloring $f\colon A \to L$ the cone $f_i^{\#}\colon A \to W_i(L)$ yields the unique factorization $\tilde{f}\colon A \to \widetilde{C}(L)$ which avoids t (i.e. $\tilde{f}(a) \neq t$ for all $a \in A$) iff $A \in \mathscr{A}^{\infty}$; since $\mathscr{A}$ is A small, the latter means $A \in \mathscr{A}$. Due to

$$f_i^{\#} = \big(\varepsilon_L\big)_i^{\#} \cdot \tilde{f} \qquad (i \in \mathbf{Ord})$$

we see that, since the right-hand cone is a subcone of the limit cone, $\tilde{f}$ avoids t iff

$$\big(f_i^{\#}(a)\big)_{i \in \mathbf{Ord}} \neq \big(t_i\big)_{i \in \mathbf{Ord}} \qquad \text{for all } \ a \in A.$$

$\square$

4. A counterexample

4.1. We present an example of a category of coalgebras which is not presentable by coequations, but is presentable by generalized coequations.

We work with the *reduced power-set functor* $\hat{\mathscr{P}}\colon \mathbf{Set} \to \mathbf{Set}$ defined on objects by

$$\hat{\mathscr{P}} X = \exp X$$

and on morphisms $f\colon X \to Y$ by

$$\hat{\mathscr{P}} f\colon M \mapsto \begin{cases} f[M] & \text{if } f/M \text{ is monic} \\ \emptyset & \text{else} \end{cases}$$

for all $M \subseteq X$.

There is a unique compatible family $s_i \in W_i$ ($i \in \mathbf{Ord}$) such that

$$s_{i+1} = \emptyset \in \hat{\mathscr{P}} W_i(K) \qquad \text{(for all } i \in \mathbf{Ord}).$$

Denote by $\mathscr{A}$ the collection of $\hat{\mathscr{P}}$-coalgebras presented by the generalized coequation $\boxtimes(s_i)_{i\in\mathbf{Ord}}$. We will prove that $\mathscr{A}$ cannot be presented by coequations.

The objects of **Coalg** $\hat{\mathscr{P}}$ are, as usual, considered as graphs: given $\alpha: A \to \hat{\mathscr{P}} A$, then A is the set of all vertices, and for a vertex a the set of all neighbor vertices is $\alpha(a)$.

For every small set K of colors and every color $x \in K$ denote by $s(x)_i \in W_i(K)$ the unique compatible family with

$$s(x)_{i+1} = (\emptyset, x) \in \hat{\mathscr{P}} W_i(K) \times K$$

for all $i \in \mathbf{Ord}$. Observe that if B is a nonempty discrete graph (i.e., $\alpha: B \to \hat{\mathscr{P}} B$ is the constant function with value $\emptyset$), then for every coloring $f: B \to K$ and every node b we have

$$f_i^{\#}(b) = s(x)_i \qquad \text{where } x = f(b) \qquad (i \in \mathbf{Ord}).$$

In particular, B does not lie in $\mathscr{A}$.

We are going to construct coalgebras A^i in $\mathscr{A}$ with distinguished vertices $c^i \in A^i$ such that the coloring

$$f^i: A^i \to K \qquad \text{constant with value } x$$

fulfils

$$(f^i)_i^{\#}(c^i) = s(x)_i \qquad (i \in \mathbf{Ord}).$$

This proves that $\mathscr{A}$ is not presentable by coequations. In fact, let $\mathscr{C}$ be a class of coequations presenting $\mathscr{A}$. Then given $\boxtimes t$ in $\mathscr{C}$ for $t \in W_i(K)$, we know that $t \neq s(x)_i$ for every $x \in K$ (because A^i fulfils $\boxtimes t$, but it does not satisfy $\boxtimes s(x)_i$). This implies, for any discrete graph $B \neq \emptyset$, that B fulfills $\boxtimes t$. However, $B \notin \mathscr{A}$ — a contradiction.

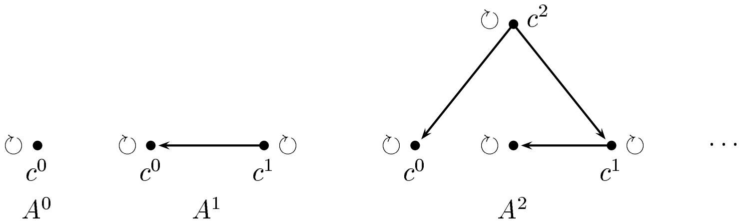
The graphs A^i are constructed by transfinite induction on $i \in \mathbf{Ord}$: form the coproduct $\coprod_{j<i} A^j$ (i.e., the disjoint union of the already constructed graphs) and add a new element $c^i \notin \coprod_{j<i} A^j$. Denote by $C^i \subseteq A^i$ the set of all the distinguished nodes c^j $(j < i)$ in the summands A^j of A^i plus c^i:

$$C^i = \{c^j; j \leq i\}.$$

The edges of each A^j are precisely those of the graph A^j, i.e., the function $\alpha^i\colon A^i \to \hat{\mathscr{P}}A^i$ has on A^j the domain-codomain restriction α^j (for all $j < i$). The neighbors of c^i are all the distinguished nodes:

$$\alpha^i(c^i) = C^i.$$

The first graphs are as follows:



We denote by $f^i\colon A^i \to K$ the constant function with value x (for all $i \in \mathbf{Ord}$). Put

$$(f^i)^{\#}_k(c^i) = r^i_k \qquad \text{for all} \quad i, k \in \mathbf{Ord}.$$

Observe that for each $j < i$ the j-th summand A^j of A^i is a subcoalgebra. Consequently, the cone $(f^j)^{\#}_k$ is a domain-restriction of the cone $(f^i)^{\#}_k$, see 3.3(c). This means that

$$(f^i)^{\#}_k(c^j) = r^j_k \qquad \text{(independent of } i).$$

Since every vertex of A^i is the distinguished vertex in some copy of A^j, $j \leq i$ (where the copy A^j can be all of A^i, or a summand in $A^i = \coprod_{j<i} A^j + \{c^i\}$, or a summand of a summand, etc.), we conclude that the values that $(f^i)^{\#}_k$ takes are just r^j_k for $j \leq i$. We are going to prove that for every ordinal $k \in \mathbf{Ord}$

(a) the values r^i_k are pairwise distinct for $i \leq k$

and

(b)

$$r^i_k = \begin{cases} s(x)_i & \text{if } k \leq i, \\ \left(\{r^j_{k-1}\}_{j \leq i}, x\right) & \text{if } k > i,\ k \text{ isolated}, \\ \left(r^i_l\right)_{l<k} & \text{if } k > i \text{ is a limit ordinal.} \end{cases}$$

This implies that $(f^i)^{\#}_i(c^i) = r^i_i = s(x)_i$, as desired. It also implies that $A^i \in \mathscr{A}$. In fact, choose $K = 1$ and x the unique element of 1. Then the value of $(f^i)^{\#}_{i+1}$ at any node $c \in A^i$ is equal to r^j_{i+1}, where j is the smallest ordinal such that c appears in a copy of A^j (embedded as A^i, or as a summand of A^i, or a summand of a summand, etc.) — in fact, c is the distinguished element c^j of that copy, thus, $(f^i)^{\#}_{i+1}(c) = (f^i)^{\#}_{i+1}(c^j) = r^j_{i+1}$. Therefore,

$$(f^i)^{\#}_{i+1}(c) = r^j_{i+1} = \Big(\{r^t_i\}_{t\le j}, x\Big) \ne (\emptyset, x).$$

Thus, $(f^i)^{\#}_{i+1}$ does not take c to $s(x)_{i+1} = (\emptyset, x)$, which implies

$$\Big((f^i)^{\#}_k(c)\Big)_{k\in\mathbf{Ord}} \ne \big(s(x)_k\big)_{k\in\mathbf{Ord}}.$$

Proof of (a) and (b)**.** We proceed by transfinite induction on k.

First step. For $k = 0$ both (a) and (b) hold by default.

Isolated step. Since

$$(f^i)^{\#}_{k+1} = \Big\langle \hat{\mathscr{P}}(f^i)^{\#}_k\cdot\alpha^i, f^i\Big\rangle$$

and since $\alpha^i(c^i)$ is the set of the distinguished elements c^j of the copies A^j $(j \le i)$ canonically embedded to A^i, to which $(f^i)^{\#}_k$ assigns r^j_k, we have

$$r^i_{k+1} = \begin{cases} \Big(\{r^j_k; j \le i\}, x\Big) & \text{if } r^j_k \text{ are pairwise distinct,} \\ (\emptyset, x) & \text{else.} \end{cases}$$

The latter case happens for every $i > k$ because

$$r^k_k = s(x)_k = r^i_k$$

by (b) in the induction hypothesis. The first case happens for every $i \le k$ by (a) in the induction hypothesis. This proves (b) for $k+1$. And (a) too: if $i < j < k+1$ then r^j_{k+1} contains r^j_k, while r^i_{k+1} does not (due to (a) in the induction hypothesis). Also, for $i < j = k+1$ we have $r^i_{k+1} \ne (\emptyset, x) = r^j_{k+1}$.

Limit step. Since for every limit ordinal k

$$(f^i)^{\#}_k = \Big\langle (f^i)^{\#}_l \Big\rangle_{l<k}$$

we have

$$r^i_k = \big(r^i_l\big)_{l<k}.$$

This proves (b): if $k \le i$ then $r^i_l = s(x)_l$, thus

$$r^i_k = \big(s(x)_l\big)_{l<k} = s(x)_k$$

and if $k > i$ the desired formula follows. It also proves (a): if $i < j < k$, then $r^i_{j+1} \ne r^j_{j+1}$ by induction hypothesis — thus, $r^i_k \ne r^j_k$. If $i < j = k$ then $r^i_{i+1} \ne s(x)_{i+1} = r^j_{i+1}$, thus, $r^i_k \ne r^j_k$. □

5. Equations for co-operations

5.1. The categories of coalgebras defined by Kurz and Rosický [KR] are based on co-operations on a set A which are functions of the form $\sigma_A\colon X^A \to Y^A$. The pair (X, Y) of sets is called the arity of σ. More precisely: a *coalgebraic signature* is defined to be a class Σ of "co-operations symbols" together with an "arity map" which to every symbol $\sigma \in \Sigma$ assigns a pair of small sets. We now recall the basic concept of equation from [KR] — except that those authors called their concept "coequation".

5.2 Definition (see [KR]). (1) A Σ-*coalgebra* is a small set A together with a function $\sigma_A\colon [A, X] \to [A, Y]$ for every (X, Y)-ary operation symbol $\sigma \in \Sigma$. A *homomorphism* from A to a Σ-coalgebra B is a function $f\colon A \to B$ such that $\sigma_A{\cdot}[f, X] = [f, Y]{\cdot}\sigma_B$, i.e., given $u\colon B \to X$ then $\sigma_A(uf) = f{\cdot}\sigma_B(u)$ for every $\sigma \in \Sigma$.
(2) The class of all Σ-*terms* is defined as the smallest class such that

(i) every (X, Y)-ary operation symbol is an (X, Y)-ary term,
(ii) every function $u\colon X \to Y$ defines an (X, Y)-ary term $[u]$ (denoted by x_u in [KR]),

and

(iii) given an (X, Y)-ary term t and a (Y, Z)-ary term s we have an (X, Z)-ary term $s{\cdot}t$.

The interpretation $t_A\colon [A, X] \to [A, Y]$ of (X, Y)-ary terms in a coalgebra A is the expected one: (i) σ_A is given, (ii) $[u]_A(-) = u{\cdot}(-)$, and (iii) $(s{\cdot}t)_A = s_A{\cdot}t_A$.

(3) A pair of Σ-terms is called an *equation*, notation: $t = s$. A coalgebra *satisfies* the equation $t = s$ provided that $t_A = s_A$.

(4) Given a class $\mathscr{E}$ of equations between Σ-terms,

$$\mathbf{Coalg}\ (\Sigma, \mathscr{E})$$

denotes the category of all Σ-algebras satisfying all equations in $\mathscr{E}$ (and all homomorphisms). We say that this category is *presented* by the equations of $\mathscr{E}$.

5.3 Remark (see [KR]). (a) For small signatures Σ the category of Σ-coalgebras has the form **Coalg** F for the set functor

$$FA = \prod_{\substack{\sigma\in\Sigma \\ \operatorname{ar}\sigma=(X,Y)}} [[A, X], Y].$$

This follows from the description of a function $\sigma_A\colon [A, X] \to [A, Y]$ via the function $A \to [[A, X], Y]$ obtained by the obvious curry-uncurry map.

(b) For every set functor F denote by

$$\Sigma_F$$

the signature of one operation symbol σ^X of arity (X, FX) for every small set X. Then every F-coalgebra $A \xrightarrow{\alpha} FA$ defines a Σ_F-coalgebra by

$$\sigma_A^X(f) \equiv A \xrightarrow{\alpha} FA \xrightarrow{Ff} FX \qquad \text{for all } \; f \in [A, X]. \tag{5.1}$$

This coalgebra satisfies, for every function $u\colon X \to Y$, the equation

$$\sigma^Y \cdot [u] = [Fu] \cdot \sigma^X. \tag{5.2}$$

Conversely, Σ_F-coalgebras satisfying the equations (5.2) form a category equivalent to **Coalg** F.

5.4 Theorem. *For every set functor F coequations are as strong as equations: every collection of F-coalgebras presentable by equations in the signature Σ_F is also presentable by coequations.*

Proof: (1) With every Σ_F-term t of arity (X, Y) we associate a function

$$\tilde{t}\colon W_i(X) \to Y \qquad \text{where } \; i \in \mathbf{Ord},$$

see 3.2, by the following structural induction: the term σ^X (where $Y = FX$) is translated as

$$\widetilde{\sigma^X} \equiv W_2(X) = F(F1 \times X) \times A \xrightarrow{\text{outl}} F(F1 \times X) \xrightarrow{F\text{outr}} FX$$

and the term $[u]$ (for $u\colon X \to Y$) as

$$\widetilde{[u]} \equiv W_1(X) = F1 \times X \xrightarrow{\text{outr}} X \xrightarrow{u} Y.$$

The translation of $t \cdot s$, where s is an (X, Y)-term and t a (Y, Z)-term, is again defined by induction on t; thus, we only need to consider

(a) the case $t = \sigma^Y$ (and $Z = FY$) for which we put, given $\tilde{s}\colon W_i(X) \to Y$,

$$\widetilde{\sigma^Y \cdot s} \equiv W_{i+1}(X) = FW_i(X) \times X \xrightarrow{\text{outl}} FW_i(X) \xrightarrow{F\tilde{s}} FY$$

and

(b) the case $t = [u]$ (where $u\colon Y \to Z$) with

$$\widetilde{[u] \cdot s} \equiv W_i(X) \xrightarrow{\tilde{s}} Y \xrightarrow{u} Z.$$

(2) The interpretation t_A of an (X,Y)-term in a coalgebra $A \xrightarrow{\alpha} FA$ is given by the following formula (w.r.t. $\tilde{t}\colon W_i(X) \to Y$ above):

$$\begin{tikzcd} A \arrow[r, "t_A(f)"] \arrow[d, "f_i^{\#}"'] & Y \\ W_i(X) \arrow[ur, "\tilde{t}"'] & \end{tikzcd}$$

for all colorings $f \in [A, X]$. The proof is an easy structural induction in t:

(i) If $t = \sigma^X$ then $t_A(f) = Ff{\cdot}\alpha$, see (5.2.2), and

$$\begin{aligned} \tilde{t}{\cdot}f_2^{\#} &= F\mathrm{outr}{\cdot}\mathrm{outl}{\cdot}\langle Ff_1^{\#}{\cdot}\alpha, f\rangle \\ &= F\mathrm{outr}{\cdot}Ff_1^{\#}{\cdot}\alpha \\ &= F\big(\mathrm{outr}{\cdot}\langle Ff_0^{\#}{\cdot}\alpha, f\rangle\big) \\ &= Ff. \end{aligned}$$

(ii) If $t = [u]$ then $t_A(f) = u{\cdot}f$, and

$$\tilde{t}{\cdot}f_1^{\#} = u{\cdot}\mathrm{outr}{\cdot}\langle Ff_0^{\#}{\cdot}\alpha, f\rangle = u{\cdot}f.$$

(iii) If $t = \sigma^Y{\cdot}s$ with $\tilde{s}\colon W_i(X) \to Y$ fulfilling the above formula, then

$$\begin{aligned} \tilde{t}{\cdot}f_{i+1}^{\#} &= F\tilde{s}{\cdot}\mathrm{outl}{\cdot}\langle Ff_i^{\#}{\cdot}\alpha, f\rangle && \text{by definition of } (-)_{i+1}^{\#} \\ &= F\tilde{s}{\cdot}Ff_i^{\#}{\cdot}\alpha \\ &= F\big(s_A(f)\big){\cdot}\alpha && \text{by induction hypothesis} \\ &= \sigma_A^Y\big(s_A(f)\big) && \text{by definition of } \sigma_A^Y \\ &= \big(\sigma_Y{\cdot}s\big)_A(f). \end{aligned}$$

(iv) If $t = [u]{\cdot}s$ with $\tilde{s}\colon W_i(X) \to Y$, then

$$\tilde{t}{\cdot}f_i^{\#} = [u]{\cdot}\tilde{s}{\cdot}f_i^{\#} = \big([u]{\cdot}s\big)_A(f).$$

(3) Every equation $t = s$ between Σ_F-terms can be substituted by coequations as follows: let $\tilde{t}\colon W_i(X) \to Y$ and $\tilde{s}\colon W_j(X) \to Y$ be given with, say, $j \le i$. Form an equalizer (in **Set**) of $\tilde{t}$ and $\tilde{s}{\cdot}W_{ij}$:

$$\begin{tikzcd} E \arrow[r, hook] & W_i(X) \arrow[rr, "\tilde{t}"] \arrow[dr, "W_{ij}"'] & & Y \\ & & W_j(X) \arrow[ur, "\tilde{s}"'] & \end{tikzcd}$$

An F-coalgebra A satisfies $t = s$ iff every coloring $f\colon A \to X$ fulfils $\tilde{t}\cdot f_i^{\#} = t_A(f) = s_A(f) = \tilde{s}\cdot f_j^{\#} = \tilde{s}\cdot W_{ij}\cdot f_i^{\#}$, i.e., iff $f_i^{\#}$ factorizes through E. Thus $t = s$ is logically equivalent to the conjunction of all $\boxtimes r$ for $r \in W_i(X) - E$. $\square$

5.5 Corollary. *For the signature $\Sigma_{\hat{\mathscr{P}}}$ of the reduced power-set functor $\hat{\mathscr{P}}$ there exists no equational presentation (for co-operations) of the covariety $\mathscr{A}$ of Section 4.*

In fact, if such a presentation would exist, then the corresponding coequations for $\hat{\mathscr{P}}$ together with those corresponding to (5.2) would yield a coequational presentation of $\mathscr{A}$.

5.6 Remark. Conversely, coequations are not actually stronger than equations:

For every set functor F define a coalgebraic signature $\bar{\Sigma}_F$ extending the above Σ_F by symbols

$$\varrho_i^X \qquad \text{of arity } \big(X, W_i(X)\big)$$

for all sets X and all $i \in \mathbf{Ord}$. Every F-coalgebra A yields a $\bar{\Sigma}_F$-coalgebra with $(\varrho_i^X)_A\colon [A,X] \to [A, W_i(X)]$ given by $f \mapsto f_i^{\#}$. It satisfies the following equations, where π_1, π_2 are the projections of $W_{i+1}(X) = FW_i(X) \times X$:

$$[\pi_1]\cdot\varrho_{i+1}^X = \sigma^{W_i(X)}\cdot\varrho_i^X \tag{5.3}$$

$$[\pi_2]\cdot\varrho_{i+1}^X = [\mathrm{id}_X] \tag{5.4}$$

for all ordinals i, and

$$[W_{j,i}]\cdot\varrho_j^X = \varrho_i^X \tag{5.5}$$

for all limit ordinals j and all $i < j$. Conversely, whenever a $\bar{\Sigma}_F$-coalgebra satisfies (5.2)–(5.5) then it stems from an F-coalgebra.

Given a coequation $\boxtimes t$ where $t \in W_i(K)$, let $u, v\colon W_i(K) \to \{0,1\}$ be functions whose equalizer is $W_i(K) - \{t\}$. We can substitute the coequation $\boxtimes t$ by the equation

$$[u]\cdot\varrho_i^K = [v]\cdot\varrho_i^K.$$

An F-coalgebra A represented as a $\bar{\Sigma}$-coalgebra satisfies this equation iff for every coloring $f\colon A \to K$ we have $uf_i^{\#} = vf_i^{\#}$. Or, equivalently, iff $f_i^{\#}$ factorizes through $W_i(K) - \{t\}$. This tells us precisely that A satisfies the coequation $\boxtimes t$.

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