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Weak alg-universality and Q-universality of semigroup quasivarieties

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To Professor Věra Trnková on her 70th birthday.

Abstract. In an earlier paper, the authors showed that standard semigroups $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ play an important role in the classification of weaker versions of alg-universality of semigroup varieties. This paper shows that quasivarieties generated by $\mathbf{M}_2$ and $\mathbf{M}_3$ are neither relatively alg-universal nor Q-universal, while there do exist finite semigroups $\mathbf{S}_2$ and $\mathbf{S}_3$ generating the same semigroup variety as $\mathbf{M}_2$ and $\mathbf{M}_3$ respectively and the quasivarieties generated by $\mathbf{S}_2$ and/or $\mathbf{S}_3$ are quasivar-relatively ff-alg-universal and Q-universal (meaning that their respective lattices of subquasivarieties are quite rich). An analogous result on Q-universality of the variety generated by $\mathbf{M}_2$ was obtained by M.V. Sapir; the size of our semigroup is substantially smaller than that of Sapir's semigroup.

Keywords: semigroup quasivariety, full embedding, ff-alg-universality, Q-universality

Classification: 20M99, 20M07, 08C15, 18B15

1. Introduction

The aim of this paper is to investigate connections between Q-universality and relative ff-alg-universality in semigroup quasivarieties. First we recall some basic notions and facts.

An *algebraic system* $\mathbf{A}$ of a similarity type Δ is a set $X_{\mathbf{A}}$ with a relation $\rho_{\mathbf{A}}$ on X_A of the arity $\mathrm{ar}(\rho)$ for each relation symbol $\rho \in \Delta$ and with an operation $o_{\mathbf{A}}$ on X_A of the arity $\mathrm{ar}(o)$ for each operation symbol $o \in \Delta$. We say that a similarity type Δ is *finite* if Δ contains only finitely many relation and operation symbols and the arity of any relation and operation symbol in Δ is finite. For algebraic systems $\mathbf{A}$ and $\mathbf{B}$ of the same type Δ, a mapping $f : X_{\mathbf{A}} \to X_{\mathbf{B}}$ is called a *homomorphism* from $\mathbf{A}$ into $\mathbf{B}$ (we shall write $f : \mathbf{A} \to \mathbf{B}$) if f maps the relation $\rho_{\mathbf{A}}$ into the relation $\rho_{\mathbf{B}}$ for every relation symbol $\rho \in \Delta$ and f commutes with $o_{\mathbf{A}}$ and $o_{\mathbf{B}}$ for every operation symbol $o \in \Delta$. If Δ contains no relation symbol, then we say that $\mathbf{A}$ is an *algebra*. Let $\mathbb{A}(\Delta)$ denote the category of all algebraic systems of a similarity type Δ and their homomorphisms. A full subcategory $\mathbb{Q}$ of

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$\mathbb{A}(\Delta)$ is called a *quasivariety* if it is closed under products, algebraic subsystems and ultraproducts. If Δ contains no relation symbol then the full subcategory $\mathbb{V}$ of $\mathbb{A}(\Delta)$ closed under products, subalgebras and homomorphic images is called a *variety*. Clearly, any variety is also a quasivariety. For any family $\mathcal{F}$ of algebraic systems of the type Δ, there exists the least quasivariety $\mathrm{QVar}\,\mathcal{F}$ containing $\mathcal{F}$, and if Δ contains no relation symbol then there exists the least variety $\mathrm{Var}\,\mathcal{F}$ containing $\mathcal{F}$, in this case $\mathrm{QVar}\,\mathcal{F}$ is a full subcategory of $\mathrm{Var}\,\mathcal{F}$.

For a quasivariety $\mathbb{Q}$, let $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$ denote the inclusion-ordered lattice of all its subquasivarieties. Properties of $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$ for a given quasivariety $\mathbb{Q}$ were studied in many papers. In particular, great attention was paid to the lattice identities satisfied by $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$. This led M.V. Sapir [18] to call a quasivariety $\mathbb{Q}$ of algebraic systems of a finite type *Q-universal* if the lattice $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{K})$ is a quotient of a sublattice of $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$ for any quasivariety $\mathbb{K}$ of a finite type. Quite a few quasivarieties are Q-universal, see the monograph by V.A. Gorbunov [10], or the paper [3], or the excellent survey paper [2]. Amongst the interesting properties of any Q-universal quasivariety $\mathbb{Q}$ is the fact that $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$ contains an isomorphic copy of the free lattice over the countably infinite set, and that $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$ has $2^{\aleph_0}$ elements. Thus $\mathrm{LAT}_{\mathbf{Q}}(\mathbb{Q})$ satisfies no non-trivial lattice identity. In [3], M.E. Adams and W. Dziobiak derived a sufficient condition for Q-universality. To formulate this condition let us denote $P_f(\omega)$ the set of all finite subsets of the set ω of all natural numbers, and say that an algebraic system $\mathbf{A}$ is *finite* if its underlying set $X_{\mathbf{A}}$ is finite. We also note that the trivial algebraic system is the empty product (the terminal object of $\mathbb{Q}$).

Theorem 1.1 ([3])**.** *A quasivariety $\mathbb{Q}$ of a finite similarity type Δ is Q-universal whenever it contains a family $\{\mathbf{A}_X \mid X \in P_f(\omega)\}$ of finite algebraic systems satisfying these four conditions:*

(p1) *$\mathbf{A}_\emptyset$ is the trivial algebraic system;*
(p2) *if $X = Y \cup Z$ for $X, Y, Z \in P_f(\omega)$ then $\mathbf{A}_X \in \mathrm{QVar}\{\mathbf{A}_Y, \mathbf{A}_Z\}$;*
(p3) *if $X, Y \in P_f(\omega)$ with $X \neq \emptyset$ then $\mathbf{A}_X \in \mathrm{QVar}\{\mathbf{A}_Y\}$ implies $X = Y$;*
(p4) *if $X \in P_f(\omega)$ is such that $\mathbf{A}_X$ is an algebraic subsystem of $\mathbf{B} \times \mathbf{C}$ where $\mathbf{B}$ and $\mathbf{C}$ are finite algebraic systems with $\mathbf{B}, \mathbf{C} \in \mathrm{QVar}\{\mathbf{A}_Y \mid Y \in P_f(\omega)\}$, then there exist subsets $Y, Z \in P_f(\omega)$ with $X = Y \cup Z$, $\mathbf{A}_Y \in \mathrm{QVar}\{\mathbf{B}\}$ and $\mathbf{A}_Z \in \mathrm{QVar}\{\mathbf{C}\}$.* □

To obtain a necessary condition for Q-universality, M.E. Adams and W. Dziobiak [4] called a finite algebra $\mathbf{A}$ *critical* if $\mathbf{A} \notin \mathrm{QVar}\{\mathbf{B} \mid \mathbf{B}$ is a proper subalgebra of $\mathbf{A}\}$, and proved that any Q-universal, locally finite quasivariety $\mathbb{Q}$ of algebras (a quasivariety $\mathbb{Q}$ of algebras is *locally finite* if any finitely generated algebra $\mathbf{A} \in \mathbb{Q}$ is finite) contains infinitely many non-isomorphic critical algebras. Thus

Theorem 1.2 ([4])**.** *If a locally finite quasivariety $\mathbb{Q}$ of algebras contains only finitely many non-isomorphic critical algebras, then $\mathbb{Q}$ is not Q-universal.* □

Let $\mathbb{DG}$ denote the quasivariety of all directed graphs (or digraphs), and let $\mathbb{GRA}$ denote the quasivariety of all undirected graphs (a graph (X, R) is *undirected* if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$). We recall that a concrete category $\mathbb{K}$ is *alg-universal* if $\mathbb{A}(\Delta)$ can be fully embedded into $\mathbb{K}$ for any type Δ. The quasivarieties $\mathbb{GRA}$ and $\mathbb{DG}$ are alg-universal, see [17], and thus a concrete category is alg-universal whenever there exists a full embedding from $\mathbb{GRA}$ or from $\mathbb{DG}$ into $\mathbb{K}$. We say that a concrete category $\mathbb{K}$ is *ff-alg-universal* if there exists a full embedding from $\mathbb{GRA}$ (or from $\mathbb{DG}$) into $\mathbb{K}$ that sends every finite graph (or digraph, respectively) to a finite object of $\mathbb{K}$ (an object $\mathbf{A}$ of $\mathbb{K}$ is called *finite* if its underlying set is finite). In [5], M.E. Adams and W. Dziobiak connected Q-universality to ff-alg-universality as follows.

Theorem 1.3 ([5]). *Any ff-alg-universal quasivariety of a finite type is Q-universal.* □

Numerous varieties of algebras fail to be alg-universal for trivial reasons. For instance, the variety of lattices is not alg-universal because any constant mapping between any two lattices is a homomorphism. Analogously, the variety of monoids or the variety of distributive $(0, 1)$-lattices are not alg-universal because between any two non-trivial monoids or distributive $(0, 1)$-lattices there exists a non-identity trivial homomorphism. On the other hand, by Z. Hedrlín and J. Lambek [11], the variety of semigroups is alg-universal, but it is not ff-alg-universal because for any finite semigroup $\mathbf{S}$ there exists a constant homomorphism from any semigroup into $\mathbf{S}$. This motivates the following notions of weakened alg-universality.

By V. Koubek and J. Sichler [13], for a category $\mathbb{K}$ a class $\mathcal{C}$ of $\mathbb{K}$-morphisms is called an *ideal* if for $\mathbb{K}$-morphisms f and g such that $f \circ g$ is defined we have that $f \circ g \in \mathcal{C}$ whenever $f \in \mathcal{C}$ or $g \in \mathcal{C}$. Observe that if $\mathcal{K}$ is a family of $\mathbb{K}$-objects then the class $\mathcal{C}_{\mathcal{K}}$ of all $\mathbb{K}$-morphisms factorizing through $\mathcal{K}$ is an ideal; precisely,

$$\begin{aligned}\mathcal{C}_{\mathcal{K}} = \{f : A \to B \mid &f \text{ is a } \mathbb{K}\text{-morphism}, \exists\, C \in \mathcal{K} \text{ and } \mathbb{K}\text{-morphisms} \\ &g : A \to C,\ h : C \to B \text{ with } f = h \circ g\}\end{aligned}$$

is an ideal in $\mathbb{K}$.

If $\mathcal{C}$ is an ideal in the category $\mathbb{K}$ then a functor $F : \mathbb{L} \to \mathbb{K}$ is called a *$\mathcal{C}$-relatively full embedding* if the following conditions are satisfied

- F is faithful and $Ff \notin \mathcal{C}$ for every $\mathbb{L}$-morphism f;
- if A and B are $\mathbb{L}$-objects and if $f : FA \to FB$ is a $\mathbb{K}$-morphism then either $f \in \mathcal{C}$ or $f = Fg$ for some $\mathbb{L}$-morphism $g : A \to B$.

We say that a concrete category $\mathbb{K}$ is *$\mathcal{C}$-relatively alg-universal* for an ideal $\mathcal{C}$ in the category $\mathbb{K}$ if there exists a $\mathcal{C}$-relatively full embedding F from an alg-universal category $\mathbb{L}$ into $\mathbb{K}$. If, moreover, $\mathbb{L}$ is ff-alg-universal and F maps finite objects of $\mathbb{L}$ to finite objects of $\mathbb{K}$, then we say that $\mathbb{K}$ is *$\mathcal{C}$-relatively ff-alg-universal*. For quasivarieties and varieties there is a natural modification of these notions. Let $\mathbb{Q}$ be a quasivariety and let $\mathbb{V}$ be a proper subquasivariety of $\mathbb{Q}$. We say that $\mathbb{Q}$ is $\mathbb{V}$-*relatively alg-universal* or $\mathbb{V}$-*relatively ff-alg-universal* if $\mathbb{Q}$ is $\mathcal{C}_{\mathbb{V}}$-relatively alg-universal or $\mathcal{C}_{\mathbb{V}}$-relatively ff-alg-universal. A quasivariety $\mathbb{Q}$ is called *quasivar-relatively alg-universal* or *quasivar-relatively ff-alg-universal* if $\mathbb{Q}$ is $\mathbb{V}$-relatively alg-universal or $\mathbb{V}$-relatively ff-alg-universal for some proper subquasivariety $\mathbb{V}$ of $\mathbb{Q}$. Let $\mathcal{S}_{\mathbb{Q}}$ be the class of all homomorphisms f between algebraic systems in $\mathbb{Q}$ such that f factorizes through an algebraic system belonging to a proper subquasivariety of $\mathbb{Q}$. Then $\mathcal{S}_{\mathbb{Q}}$ is an ideal. If $\mathbb{Q}$ is $\mathcal{S}_{\mathbb{Q}}$-relatively alg-universal or $\mathcal{S}_{\mathbb{Q}}$-relatively ff-alg-universal then we say that $\mathbb{Q}$ is *weakly quasivar-relatively alg-universal* or *weakly quasivar-relatively ff-alg-universal*. Analogous notions are defined for varieties.

From this definition it immediately follows that for any $\mathcal{C}$-relatively alg-universal category $\mathbb{K}$, the class of $\mathbb{K}$-morphisms outside of $\mathcal{C}$ contains an alg-universal category. Viewed informally, this 'weaker' alg-universality is effected by disregarding all members of $\mathcal{C}$; and it is clear that $\emptyset$-relatively alg-universal categories are exactly the alg-universal categories. Interesting ideals of a quasivariety $\mathbb{K}$ consist of all $\mathbb{K}$-morphisms factorizing through an algebraic system from a given proper subquasivariety $\mathbb{Q}$ of $\mathbb{K}$, or even through a member of the union of all proper subquasivarieties of $\mathbb{K}$. When $\mathbb{V}$ is the subquasivariety of all singleton algebraic systems, the quasivariety $\mathbb{K}$ is *almost alg-universal* or *almost ff-alg-universal*. We also note that M.E. Adams and W. Dziobiak specifically asked whether all almost ff-alg-universal quasivarieties are Q-universal. While unable to answer this question, V. Koubek and J. Sichler proved

Theorem 1.4 ([13]). *There is a finitely generated, weakly var-relatively ff-alg-universal variety of distributive double p-algebras that is not Q-universal.* □

We turn our attention to semigroup varieties and quasivarieties. V. Koubek and J. Sichler [12] characterized alg-universal semigroup varieties and a consequence of this result is the fact that no alg-universal semigroup variety is finitely generated. This helped motivate an investigation of weak alg-universality of semigroup varieties. Let us denote $\mathbb{LSN}$ (or $\mathbb{RSN}$, or $\mathbb{LQN}$, or $\mathbb{RQN}$) the variety of bands satisfying the identity $xyz = xyzxz$ (or $xyz = xzxyz$, or $xyxz = xyz$, or $yxzx = yzx$, respectively). Then it was proved

Theorem 1.5. *The variety $\mathbb{LSN}$ is Q-universal and $\mathbb{LQN}$-relatively ff-alg-universal. The variety $\mathbb{RSN}$ is Q-universal and $\mathbb{RQN}$-relatively ff-alg-universal.*

The proof of the fact that $\mathbb{LSN}$ (or $\mathbb{RSN}$) is var-relatively ff-alg-universal is contained in [8]. In [7], M.E. Adams and W. Dziobiak modified this proof to show that $\mathbb{LSN}$ and $\mathbb{RSN}$ are Q-universal. M.V. Sapir then strengthened the result of M.E. Adams and W. Dziobiak [7] by showing that the varieties $\mathbb{LQN}$ and $\mathbb{RQN}$ are Q-universal, see [7].

The proof by M.E. Adams and W. Dziobiak points out a connection between var-relative ff-alg-universality and Q-universality, and suggests that Theorem 1.3 can be further generalized. One of the aims of this paper is to support this contention. The paper [9] studied distinct weak versions of universality of semigroup varieties. The varieties $\mathrm{Var}\{\mathbf{M}_1\}$, $\mathrm{Var}\{(\mathbf{M}_1)^o\}$, $\mathrm{Var}\{\mathbf{M}_2\}$, $\mathrm{Var}\{\mathbf{M}_3\}$, and $\mathrm{Var}\{(\mathbf{M}_3)^o\}$ played an important role there; the semigroups $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ are given by the multiplication tables below, and $(\mathbf{M}_1)^o$ and $(\mathbf{M}_3)^o$ are the semigroups opposite to $\mathbf{M}_1$ and $\mathbf{M}_3$, respectively (we recall that if $\mathbf{S} = (S, \cdot)$ is a semigroup then the *opposite semigroup* $\mathbf{S}^o = (S, \odot)$ is defined by $s \odot t = ts$ for all $s, t \in S$).

M_1	1	a	0
1	1	a	0
a	0	0	0
0	0	0	0

M_2	a	b	c	0
a	0	c	0	0
b	c	0	0	0
c	0	0	0	0
0	0	0	0	0

M_3	d	a	b	c
d	a	a	a	b
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

Since $\mathrm{Var}\{\mathbf{M}_1\}$ and $\mathrm{Var}\{(\mathbf{M}_1)^o\}$ have only finitely many nonisomorphic critical algebras, they are not Q-universal, see [9]. To describe results of the present paper, let us denote $\mathbb{ZS}$ the variety of all zero-semigroups, $\mathbb{LZ}$ the join of the variety $\mathbb{ZS}$ and the variety of all left zero-semigroups, and $\mathbb{RZ}$ the join of $\mathbb{ZS}$ and the variety of all right zero-semigroups. We recall that critical algebras in $\mathbb{LZ}$ and $\mathbb{RZ}$ are exactly two-element semigroups; precisely, the critical algebras in $\mathbb{LZ}$ and $\mathbb{RZ}$ are the two-element zero-semigroup, the two-element left zero-semigroup, and the two-element right zero-semigroup. We prove

Theorem 1.6. *There exists a finite semigroup* $\mathbf{S} \in \mathrm{Var}\{\mathbf{M}_2\}$ *such that* $\mathrm{QVar}\{\mathbf{S}\}$ *is Q-universal and $\mathbb{ZS}$-relatively ff-alg-universal. The quasivariety* $\mathrm{QVar}\{\mathbf{M}_2\}$ *is neither Q-universal nor quasivar-relatively alg-universal.*

There exists a finite semigroup $\mathbf{S} \in \mathrm{Var}\{\mathbf{M}_3\}$ *such that* $\mathrm{QVar}\{\mathbf{S}\}$ *is Q-universal and $\mathbb{LZ}$-relatively ff-alg-universal. The quasivariety* $\mathrm{QVar}\{\mathbf{M}_3\}$ *is neither Q-universal nor quasivar-relatively alg-universal.*

There exists a finite semigroup $\mathbf{S} \in \mathrm{Var}\{(\mathbf{M}_3)^o\}$ *such that* $\mathrm{QVar}\{\mathbf{S}\}$ *is Q-universal and $\mathbb{RZ}$-relatively ff-alg-universal. The quasivariety* $\mathrm{QVar}\{(\mathbf{M}_3)^o\}$ *is neither Q-universal nor quasivar-relatively alg-universal.*

In [18], M.V. Sapir proved that there exists a finite semigroup $\mathbf{S}' \in \mathrm{Var}(\mathbf{M}_2)$ for which $\mathrm{QVar}\{\mathbf{S}'\}$ is Q-universal (he formulated his results for the variety of commutative three-nilpotent semigroups but, in fact his proof works in the variety $\mathrm{Var}\{\mathbf{M}_2\}$). The semigroup $\mathbf{S}$ from the first statement of Theorem 1.6 has 23 elements, substantially less than the estimated size of Sapir's semigroup $\mathbf{S}'$.

The proof of Theorem 1.6 is based on

Theorem 1.7 ([9]). *The variety* $\mathrm{Var}\{\mathbf{M}_2\}$ *is* $\mathbb{ZS}$*-relatively* ff*-alg-universal,* $\mathrm{Var}\{\mathbf{M}_3\}$ *is* $\mathbb{ILZ}$*-relatively* ff*-alg-universal, and* $\mathrm{Var}\{(\mathbf{M}_3)^o\}$ *is* $\mathbb{IRZ}$*-relatively* ff*-alg-universal.* □

We use the method from the paper by V. Koubek and J. Sichler [14], where they proved that the variety of 0-lattices generated by the five-element modular non-distributive lattice is almost ff-alg-universal and Q-universal. This method transforms the proof of a relative ff-alg universality into the proof of Q-universality, and may possibly lead to generalizations of Theorem 1.3.

Next we make several observations. The auxiliary Section 2 is devoted to undirected and directed graphs. S.V. Sizyi [19] proved that the quasivariety of directed antireflexive graphs is Q-universal (a directed graph (X, R) is *antireflexive* if $|X| = 1$ or $(x, x) \notin R$ for all $x \in X$). Sizyi's result extends that of M.E. Adams and W. Dziobiak [6], who proved that the quasivariety of all undirected 3-colourable antireflexive graphs is Q-universal. Both results can be proved using a combination of full embeddings constructed in the monograph by A. Pultr and V. Trnková [17] with Theorem 1.3. A stronger version of Adams-Dziobiak result of [6] was obtained by A. Kravchenko [15]. He proved that a quasivariety of antireflexive undirected graphs is Q-universal exactly when it contains a non-bipartite graph.

In the second section we construct a particular family of undirected antireflexive 3-colourable graphs satisfying conditions (p1)–(p4). This gives a new proof of the result of [6] quoted in the previous paragraph. The quasivariety of undirected antireflexive 3-colourable graphs is contained in the quasivariety $\mathrm{QVar}\{\mathbf{K}_4\}$ (which is generated by the complete undirected antireflexive graph $\mathbf{K}_4$ on the four-element set). We give an analogous result for directed antireflexive digraphs, and show that the quasivariety of antisymmetric directed graphs is Q-universal (recall that a directed graph (X, R) is *antisymmetric* if $(x, y) \in R$ implies $(y, x) \notin R$ for all $x, y \in X$, and observe that any antisymmetric graph is antireflexive). In Section 3, we apply these results to the quasivarieties $\mathrm{Var}\{\mathbf{M}_2\}$, $\mathrm{Var}\{\mathbf{M}_3\}$ and $\mathrm{Var}\{(\mathbf{M}_3)^o\}$ to obtain the proof of Theorem 1.6.

We recall several facts about factorization systems, see [1]. Let $\mathbb{K}$ be a category, then a $\mathbb{K}$-morphism $f : A \to B$ is called an *extremal epimorphism* if any $\mathbb{K}$-monomorphism h such that $f = h \circ g$ for some $\mathbb{K}$-morphism g is an isomorphism. We recall that a graph homomorphism f is an extremal epimorphism if and only if f is surjective on vertices and edges. Precisely, a homomorphism $f : (V, E) \to (V', E')$ of undirected graphs is an extremal epimorphism if and only if $f(V) = V'$ and $E' = \{\{f(u), f(v)\} \mid \{u, v\} \in E\}$ and a digraph homomorphism $f : (X, R) \to (X', R')$ is an extremal epimorphism if and only if $f(X) = X'$ and $R' = \{(f(x), f(y)) \mid (x, y) \in R\}$. If $\mathbf{A}$ and $\mathbf{B}$ are algebras of the same similarity type, then a homomorphism $f : \mathbf{A} \to \mathbf{B}$ is an extremal epimorphism if and only if f is surjective. If $\mathbb{Q}$ is a quasivariety of algebraic systems, then the pair (extremal epimorphisms, monomorphisms) is a factorization system of $\mathbb{Q}$ (we recall that monomorphisms in any quasivariety of algebraic systems are exactly injective homomorphisms). A family $\{f : X \to X_i \mid i \in I\}$ of mappings is called a *separating family* if for every pair $\{x, y\}$ of distinct elements of X there exists $i \in I$ with $f_i(x) \neq f_i(y)$. We recall an important property of factorization systems. Let $\mathbb{Q}$ be a quasivariety of algebraic systems and let $f : A \to B$, $g : A \to C$, $f_i : B \to D_i$, $g_i : C \to D_i$ for $i \in I$ be $\mathbb{Q}$-homomorphisms with $f_i \circ f = g_i \circ g$ for all $i \in I$. If f is an extremal epimorphism and $\{g_i \mid i \in I\}$ is a separating family, then there exists a $\mathbb{Q}$-homomorphism $h : B \to C$ with $h \circ f = g$ and $g_i \circ h = f_i$ for all $i \in I$. If, moreover, g is an extremal epimorphism, then h is also an extremal epimorphism, if $\{f_i \mid i \in I\}$ is a separating family, then h is a monomorphism (and thus it is injective). This property is called the *diagonalization property* of a factorization system.

We recall that if $\{\mathbf{A}_i \mid i \in I\}$ is a finite family of finite algebraic systems of the same type Δ, then an algebraic system $\mathbf{B}$ of the type Δ belongs to $\mathrm{QVar}\{\mathbf{A}_i \mid i \in I\}$ if and only if the family $\mathcal{F}$ consisting of all homomorphisms $f : \mathbf{B} \to \mathbf{A}_i$ for some $i \in I$ is separating.

For the sake of simplicity we identify any natural number n with the set $\{0, 1, \dots, n-1\}$ of natural numbers of size n. The set of all natural numbers is denoted by ω. Let $P_f(\omega)$ denote the set of all finite subsets of ω, and let $P_{nf}(\omega) = P_f(\omega) \setminus \{\emptyset\}$.

We say that a functor $F : \mathbb{K} \to \mathbb{L}$ *preserves extremal epimorphisms* if Ff is an extremal epimorphism of $\mathbb{L}$ whenever f is an extremal epimorphism of $\mathbb{K}$. If $\mathbb{K}$ and $\mathbb{L}$ are concrete categories then F *preserves separating families* whenever $\{Ff_i : FX \to FX_i \mid i \in I\}$ is a separating family of $\mathbb{L}$-morphisms for any separating family $\{f_i : X \to X_i \mid i \in I\}$ of $\mathbb{K}$-morphisms.

2. Graph constructions

The aim of this section is to construct a family $\{\mathbf{G}_A \mid A \in P_{nf}(\omega)\}$ of undirected graphs such that

(q1) if $B \subseteq A$ then there exists an extremal epimorphism $g_{A,B} : \mathbf{G}_A \to \mathbf{G}_B$;
(q2) if $f : \mathbf{G}_A \to \mathbf{G}_B$ is a graph homomorphism for $A, B \in P_{nf}(\omega)$ then $B \subseteq A$ and $f = g_{A,B}$;
(q3) if $B \in P_{nf}(\omega)$ and if $\{A_i \mid i \in I\} \subseteq P_{nf}(\omega)$ is finite with $A_i \subseteq B$ for all $i \in I$, then $\{g_{B,A_i} \mid i \in I\}$ is a separating family if and only if $B = \bigcup_{i \in I} A_i$.

Observe that $g_{A,A}$ is the identity mapping, and that $g_{A,C} = g_{B,C} \circ g_{A,B}$ for $A, B, C \in P_{nf}(\omega)$ with $C \subseteq B \subseteq A$.

First we consider undirected graphs $\mathbf{H}_0 = (W_0, F_0)$ from Figure 1, $\mathbf{H}_1 = (W_1, F_1)$ from Figure 2, and $\mathbf{H}_2 = (W_2, F_2)$ from Figure 3.

To apply techniques from the monograph by A. Pultr and V. Trnková [17], let us recall that a *cycle* of an undirected graph $\mathbf{G}$ is a sequence $\{v_i\}_{i=0}^{n-1}$ of

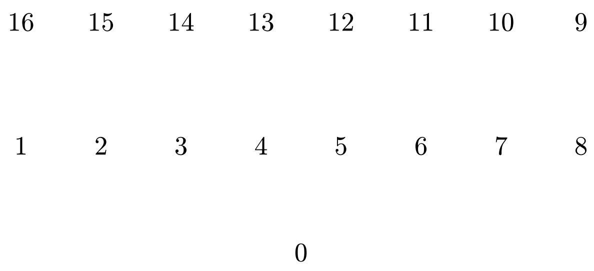

Figure 1. The graph $\mathbf{H}_0$

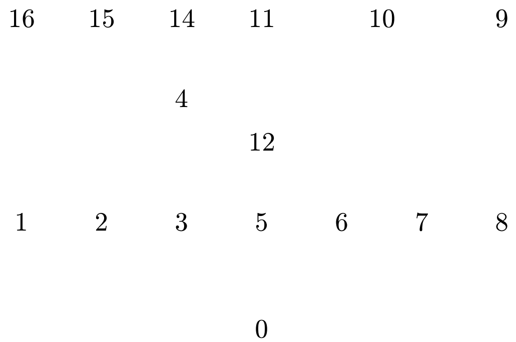

Figure 2. The graph $\mathbf{H}_1$

vertices of $\mathbf{G}$ such that $n \geq 3$, $\{v_0, v_{n-1}\}$ is an edge of $\mathbf{G}$, and $\{v_i, v_{i+1}\}$ is an edge of $\mathbf{G}$ for all $i = 0, 1, \ldots, n-2$. Then n is the *length* of this cycle. For an undirected graph $\mathbf{G}$, let $c(\mathbf{G})$ denote the length of its shortest cycle, and let $\chi(\mathbf{G})$ denote the chromatic number of $\mathbf{G}$. The following technical lemma describes graph homomorphisms between $\mathbf{H}_0$, $\mathbf{H}_1$ and $\mathbf{H}_2$. For $i, j \in 3$ with $i \leq j$ define a mapping $g_{i,j} : W_i \longrightarrow W_j$ so that $g_{i,j}$ is the inclusion mapping if $i > 0$ or $j = 0$; if $i = 0$ and $j > 0$ then $g_{i,j}(k) = k$ for $k \neq 13$ and $g_{i,j}(13) = 11$.

Lemma 2.1. *If $f : \mathbf{H}_i \longrightarrow \mathbf{H}_j$ is a graph homomorphism for $i, j \in 3$ then $i \leq j$ and $f = g_{i,j}$. Conversely, if $i, j \in 3$ with $i \leq j$ then $g_{i,j} : \mathbf{H}_i \longrightarrow \mathbf{H}_j$ is a graph homomorphism. Further $\chi(\mathbf{H}_i) = 3$ for all $i \in 3$.*

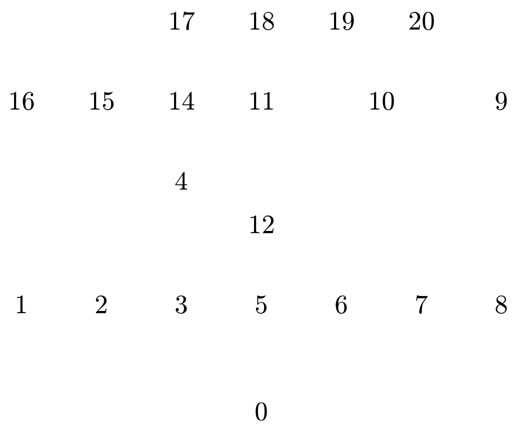

FIGURE 3. The graph $\mathbf{H}_2$

PROOF: Let $i, j \in 3$ and let $f : \mathbf{H}_i \longrightarrow \mathbf{H}_j$ be a graph homomorphism. We recall that if $g : \mathbf{G} \longrightarrow \mathbf{G}'$ is a graph homomorphism between undirected antireflexive graphs $\mathbf{G}$ and $\mathbf{G}'$ such that $c(\mathbf{G}) = c(\mathbf{G}')$ then g is injective on every shortest length cycle of $\mathbf{G}$. Clearly, $\mathbf{H}_0$, $\mathbf{H}_1$ and $\mathbf{H}_2$ are antireflexive,

$$\begin{aligned}\{C_0 =&\{0,1,2,3,4,5,6,7,8\}, C_1 = \{4,5,6,7,8,9,10,11,12\},\\ &C_2 = \{1,2,3,4,12,13,14,15,16\}\}\end{aligned}$$

is the list of all shortest cycles in $\mathbf{H}_0$,

$$\{C_0, C_1, C_3 = \{1,2,3,4,12,11,14,15,16\}\}$$

is the list of all shortest cycles in $\mathbf{H}_1$, and

$$\{C_0, C_1, C_3, C_4 = \{9,10,11,14,15,17,18,19,20\}\}$$

is the list of all shortest cycles in $\mathbf{H}_2$. Thus $c(\mathbf{H}_i) = 9$ for all $i \in 3$ and 4 is the unique element of the intersection of C_l for $l = 0,1,2$ or $l = 0,1,3$ and 11 is the unique element of the intersection of C_l for $l = 1,3,4$ and the cycles C_0 and C_4 are vertex-disjoint. Hence if $i = 0$ then f is injective on C_l for $l \in 3$ and on the set $\{3,4,5,12\}$, thus $f(C_l) \neq f(C_{l'})$ for distinct $l, l' \in 3$; if $i = 1$ then f is injective on C_l for $l = 0,1,3$ and on the set $\{3,4,5,12\}$, thus $f(C_l) \neq f(C_{l'})$ for distinct $l, l' \in \{0,1,3\}$; if $i = 2$ then f is injective on C_l for $l = 0,1,3,4$ and on the sets $\{3,4,5,12\}$ and $\{10,11,12,14\}$.

Assume that $i, j \in 2$. Then $f(4) = 4$ and $f\{3,5,12\} = \{3,5,12\}$. If $f(5) = 3$ then necessarily, $f(6) = 2$ and $f(7) = 1$. Hence $f(8) = 0$ and together $f(8) = 16$ — this is a contradiction. If $f(5) = 12$ and $j = 0$ then $f(6) = 11$ and together $f(6) = 13$ — a contradiction, if $f(5) = 12$ and $j = 1$ then $f(6) = 11$, hence $f(7) = 10$ and together $f(7) = 14$ — a contradiction. Thus $f(5) = 5$ and $\{f(C_0), f(C_1)\} = \{C_0, C_1\}$. If $f(3) = 12$ then $f(C_0) = C_1$. Then $j = 0$ implies $f(2) = 11$ and together $f(2) = 13$ — a contradiction, and $j = 1$ implies $f(2) = 11$, hence $f(1) = 10$ and together $f(1) = 14$ — a contradiction. Therefore $f(3) = 3$, $f(12) = 12$, $f(C_0) = C_0$ and $f(C_1) = C_1$. If $i = 1$ and $j = 0$ then $f(11) = 11$ and together $f(11) = 13$ — a contradiction. Thus we can summarize: $i \leq j$ and $f = g_{i,j}$ for $i, j \in 2$.

Assume that $i = 0, 1$ and $j = 2$. Then we have either $f\{3,4,5,12\} = \{3,4,5,12\}$ or $f\{3,4,5,12\} = \{10,11,12,14\}$. In the first case $f(4) = 4$, in the second case $f(4) = 11$. The intersection of C_0 and C_1 is the set $\{4,5,6,7,8\}$ but the intersection of C_l and $C_{l'}$ for distinct $l, l' \in \{1,3,4\}$ has at most three elements, therefore $f(4) \neq 11$ and whence $f(4) = 4$ and $f = \iota \circ f'$ for a graph homomorphism $f' : \mathbf{H}_i \to \mathbf{H}_1$ and the inclusion $\iota : \mathbf{H}_1 \to \mathbf{H}_2$. Thus, by the foregoing part of the proof, $f = g_{i,j}$.

Assume that $i = 3$. Then $f \circ \iota$ is a graph homomorphism from $\mathbf{H}_1$ into $\mathbf{H}_j$ and, by the foregoing part, $j \neq 0$ and $f(k) = k$ for all $k = \{0, 1, \ldots, 16\} \setminus \{13\}$. Then necessarily $f(C_4) = C_4$, $j = 3$, and $f(k) = k$ for $k = 17, 18, 19, 20$ and the first statement is proved.

The second statement is obtained, by a direct inspection of Figure 1, Figure 2, and Figure 3. It is well-known that if an undirected graph $\mathbf{G}$ contains an independent set A such that, by a deletion of all vertices from A, we obtain a forest, then $\chi(\mathbf{G}) \leq 3$. For graphs $\mathbf{H}_0$ and $\mathbf{H}_1$ the set $\{0, 4\}$ has this property and for $\mathbf{H}_2$ the set $\{0, 4, 20\}$ has this property. From $c(\mathbf{H}_i) = 9$ it follows $\chi(\mathbf{H}_i) = 3$ for all $i \in 3$. $\square$

Let $\{p_i\}_{i=0}^{\infty}$ be an increasing sequence of primes greater than 9. For every natural number i define an undirected graph $\mathbf{G}_i = (V_i, E_i)$ such that $V_i = W_0 \times p_i = \{(v, j) \mid v \in W_0, j \in p_i\}$ and

$$E_i = \{\{(v,j),(w,j)\} \mid \{v,w\} \in F_0, j \in p_i\} \cup \{\{(13,j),(10,j+1 \bmod p_i)\} \mid j \in p_i\}.$$

Next we prove an auxiliary lemma.

Lemma 2.2. *If $f : \mathbf{G}_i \to \mathbf{G}_j$ is a graph homomorphism for some $i, j \in \omega$, then $i = j$, f is an automorphism of $\mathbf{G}_i$ and there exists $l \in \omega$ with $f(v,k) = (v, k + l \bmod p_i)$ for all $v \in W_0$ and all $k \in p_i$.*

A mapping $f : V_i \to W_1$ is a graph homomorphism from $\mathbf{G}_i$ to $\mathbf{H}_1$ for some $i \in \omega$ if and only if $f(v,j) = g_{0,1}(v)$ for all $v \in W_0$ and all $j \in p_i$.

For every $i \in \omega$, $\chi(\mathbf{G}_i) = 3$.

Proof: First observe that $c(\mathbf{G}_k) = 9$ and $\{\{(v,l) \mid v \in W_0\} \mid l \in p_k\}$ is the set of all components of the subgraph (V_k, E'_k) of $\mathbf{G}_k$ where E'_k consists of all edges of $\mathbf{G}_k$ contained in a cycle of length 9 in $\mathbf{G}_k$. Hence if $f : \mathbf{G}_i \to \mathbf{G}_j$ is a graph homomorphism for $i, j \in \omega$ then, by the first statement of Lemma 2.1, there exists a mapping h from p_i into p_j with $f(v,l) = (v, h(l))$ for all $v \in W_0$ and all $l \in p_i$. Observe that if $l, l' \in p_i$ are distinct and $\{(13,l),(v,l')\}$ is an edge of $\mathbf{G}_i$, then $v = 10$ and $l' = l+1 \bmod p_i$ and hence $h(l+1) = h(l) + 1 \bmod p_j$ for all $l \in p_i$. Since p_i is a prime, we have $i = j$ and $f(v,k) = (v, k + l \bmod p_i)$ for some $l \in p_i$ and for all $v \in W_0$ and all $k \in p_i$. Whence f is an automorphism of $\mathbf{G}_i$.

Since the induced subgraph of $\mathbf{G}_i$ on the set $\{(v,j) \mid v \in W_0\}$ is isomorphic to $\mathbf{H}_0$ for all $j \in p_i$ we conclude, by the first statement of Lemma 2.1, that if $f : \mathbf{G}_i \to \mathbf{H}_1$ is a graph homomorphism then $f(v,j) = g_{0,1}(v)$ for all $v \in W_0$ and all $j \in p_i$. The converse follows from the second statement of Lemma 2.1.

The third statement follows from the third statement of Lemma 2.1 and the fact that $\{0,4,10\}$ is an independent set of $\mathbf{H}_0$. $\square$

Remark. Observe that for every pair of natural numbers $\{i,l\}$, the mapping $f : V_i \to V_i$ defined by $f(v,k) = (v, k + l \bmod p_i)$ for all $v \in W_0$ and all $k \in p_i$ is an automorphism of $\mathbf{G}_i$.

For $A \in P_{nf}(\omega)$, let $\pi(A)$ denote the increasing sequence of all elements of A. For $A \in P_{nf}(\omega)$ with $\pi(A) = \{a_i\}_{i=0}^{n-1}$ we now define an undirected graph $\mathbf{G}_A = (V_A, E_A)$ as follows:

$$\begin{aligned} V_A = &\{(v,i,a) \mid v \in W_2,\ i = 0,1\} \cup \{(v,i,b) \mid v \in W_1, i \in n+1\} \cup \\ &\{(v,j,i,c) \mid v \in W_0,\ j \in p_{a_i},\ i \in n\} \end{aligned}$$

where a, b and c are distinct fixed new elements and

$$\begin{aligned} E_A = &\{\{(v,i,a),(w,i,a)\} \mid \{v,w\} \in F_2,\ i = 0,1\} \cup \\ &\{\{(v,i,b),(w,i,b)\} \mid \{v,w\} \in F_1,\ i \in n+1\} \cup \\ &\{\{(v,j,i,c),(w,j,i,c)\} \mid \{v,w\} \in F_0,\ j \in p_{a_i},\ i \in n\} \cup \\ &\{\{(13,j,i,c),(10, j+1 \bmod p_{a_i}, i, c)\} \mid j \in p_{a_i},\ i \in n\} \cup \\ &\{\{(0,0,i,c),(1,i,b)\},\{(16,0,i,c),(1,i+1,b)\} \mid i \in n\} \cup \\ &\{\{(4,0,a),(1,0,b)\},\{(20,1,a),(1,n,b)\}\}. \end{aligned}$$

Let $A, B \in P_{nf}(\omega)$ be sets with $A \subseteq B$, $\pi(A) = \{a_i\}_{i=0}^{n-1}$, and $\pi(B) = \{b_i\}_{i=0}^{m-1}$. Define a mapping $\phi : \{0,1,\dots,m\} \to \{0,1,\dots,n\}$ so that

$$\phi(i) = \begin{cases} \min\{j \mid b_i \le a_j\} & \text{if } b_i \le a_{n-1}, \\ n & \text{if } i = m \text{ or } b_i > a_{n-1} \end{cases}$$

for all $i = 0, 1, \dots, m$. Define a mapping $g_{B,A} : V_B \longrightarrow V_A$ by

$$g_{B,A}(x) = \begin{cases} (v, i, a) & \text{if } x = (v, i, a) \in V_B, \\ (v, \phi(i), b) & \text{if } x = (v, i, b) \in V_B, \\ (g_{0,1}(v), \phi(i), b) & \text{if } x = (v, j, i, c) \in V_B \text{ and } b_i \notin A, \\ (v, j, \phi(i), c) & \text{if } x = (v, j, i, c) \in V_B \text{ and } b_i \in A. \end{cases}$$

We prove

Proposition 2.3. *For $A, B, C \in P_{nf}(\omega)$ we have*

(1) *if $B \subseteq A$ then $g_{A,B} : \mathbf{G}_A \longrightarrow \mathbf{G}_B$ is an extremal epimorphism;*
(2) *if $f : \mathbf{G}_A \longrightarrow \mathbf{G}_B$ is a graph homomorphism then $B \subseteq A$ and $f = g_{A,B}$;*
(3) *if $B, C \subseteq A$ then $\{g_{A,B}, g_{A,C}\}$ is a separating family if and only if $A = B \cup C$;*
(4) $\chi(\mathbf{G}_A) = 3$.

Proof: First we list several important properties of $\mathbf{G}_A$ for $A \in P_{nf}(\omega)$. Assume that $\pi(A) = \{a_i\}_{i=0}^{n-1}$. Then

(i) $\mathbf{G}_A$ is antireflexive, $c(\mathbf{G}_A) = 9$, and any vertex of $\mathbf{G}_A$ belongs to a cycle of length 9;
(ii)

$$\{\{(v, i, a) \mid v \in W_2\} \mid i \in 2\} \cup \{\{(v, i, b) \mid v \in W_1\} \mid i \in n+1\}$$
$$\cup \{\{(v, j, i, c) \mid v \in W_0\} \mid i \in n,\ j \in p_{a_i}\}$$

is the list of all components of the subgraph (V_A, E'_A) of $\mathbf{G}_A$ — where E'_A consists of all edges of $\mathbf{G}_A$ contained in a cycle of length 9 in $\mathbf{G}_A$;
(iii) for every $i \in \{0, 1\}$ the induced subgraph of $\mathbf{G}_A$ on the set $\{(v, i, a) \mid v \in W_2\}$ is isomorphic to $\mathbf{H}_2$, for every $i \in n+1$ the induced subgraph of $\mathbf{G}_A$ on the set $\{(v, i, b) \mid v \in W_1\}$ is isomorphic to $\mathbf{H}_1$, for every $i \in n$ and $j \in p_{a_i}$ the induced subgraph of $\mathbf{G}_A$ on the set $\{(v, j, i, c) \mid v \in W_0\}$ is isomorphic to $\mathbf{H}_0$, for every $i \in n$ the induced subgraph of $\mathbf{G}_A$ on the set $\{(v, j, i, c) \mid v \in W_0, j \in p_{a_i}\}$ is isomorphic to $\mathbf{G}_{a_i}$;
(iv) if $\{x, y\}$ is an edge of $\mathbf{G}_A$ such that $\{x, y\} \cap \{(v, i, a) \mid v \in W_2\}$ is a singleton for some $i \in \{0, 1\}$, then either $\{x, y\} = \{(4, 0, a), (1, 0, b)\}$ or $\{x, y\} = \{(20, 1, a), (1, n, b)\}$;
(v) if $\{x, y\}$ is an edge of $\mathbf{G}_A$ such that $\{x, y\} \cap \{(v, i, b) \mid v \in W_1\}$ is a singleton for some $i \in n+1$, then either $\{x, y\} = \{(1, i, b), (0, 0, i, c)\}$ and $i < n$, or $\{x, y\} = \{(1, i, b), (16, 0, i-1, c)\}$ and $i > 0$, or $i = 0$ and $\{x, y\} = \{(4, 0, a), (1, 0, b)\}$, or $i = n$ and $\{x, y\} = \{(20, 1, a), (1, n, b)\}$;
(vi) if $\{x, y\}$ is an edge of $\mathbf{G}_A$ such that $\{x, y\} \cap \{(v, j, i, c) \mid v \in W_0\}$ is a singleton for some $i \in n$ and $j \in p_{a_i}$, then either $\{x, y\} = \{(13, j, i, c), (10, j+$

$1 \bmod p_{a_i}, i, c)\}$, or $\{x,y\} = \{(13, j-1 \bmod p_{a_i}, i, c), (10, j, i, c)\}$, or $j = 0$ and $\{x,y\} = \{(1, i+1, b), (16, 0, i, c)\}$, or $j = 0$ and $\{x,y\} = \{(1, i, b), (0, 0, i, c)\}$;

(vii) the graph $\mathbf{G}_A$ is connected and if we delete the vertex $(0,0,i,c)$ for some $i \in n$, then the sets $\{(v,0,a) \mid v \in W_2\}$ and $\{(v,1,a) \mid v \in W_2\}$ belong to distinct components of the resulting graph.

Assume that $\pi(A) = \{a_i\}_{i=0}^{n-1}$, $\pi(B) = \{b_i\}_{i=0}^{m-1}$, and $\pi(C) = \{c_i\}_{i=0}^{q-1}$ for $A, B, C \in P_{nf}(\omega)$. Then $\mathbf{G}_A$ (or $\mathbf{G}_B$, or $\mathbf{G}_C$) consists of two disjoint copies of $\mathbf{H}_2$, of $n+1$ (or $m+1$, or $q+1$) disjoint copies of $\mathbf{H}_1$ and of one copy of $\mathbf{G}_y$ for each $y \in A$ (or $y \in B$, or $y \in C$, respectively).

By Lemmas 2.1 and 2.2, $g_{B,A}$ is a graph homomorphism on any copy of $\mathbf{H}_2$, any copy of $\mathbf{H}_1$ and any copy of $\mathbf{G}_{a_i}$ with $a_i \in A$. To demonstrate that f is a graph homomorphism it suffices to observe that edges between distinct copies of these subgraphs are preserved. Direct inspection shows this, and that $g_{B,A}$ is an extremal epimorphism; thus (1) is proved.

To prove (2) we assume that $f : \mathbf{G}_A \to \mathbf{G}_B$ is a graph homomorphism. By (i), (ii), (iii) and Lemma 2.1, we obtain that

- for each $i \in \{0,1\}$ there exists $j \in \{0,1\}$ such that $f(v,i,a) = (v,j,a)$ for all $v \in W_2$,
- for every $i \in n+1$ either there exists $j \in \{0,1\}$ such that $f(v,i,b) = (v,j,a)$ for all $v \in W_1$, or there exists $j \in m+1$ such that $f(v,i,b) = (v,j,b)$ for all $v \in W_1$,
- for every pair $i \in n$ and $j \in p_{a_i}$ either there exists $k \in \{0,1\}$ such that $f(v,j,i,c) = (g_{0,2}(v), k, a)$ for all $v \in W_0$, or there exists $k \in m+1$ such that $f(v,j,i,c) = (g_{0,1}(v), k, b)$ for all $v \in W_0$, or there exist $k \in m$ and $l \in p_{b_k}$ such that $f(v,j,i,c) = (v,l,k,c)$ for all $v \in W_0$.

By (iv), $\{(4,0,a),(1,0,b)\}$ and $\{(20,1,a),(1,n,b)\}$ are edges of $\mathbf{G}_A$, if $\{(4,1,a),y\}$ is an edge of $\mathbf{G}_A$, then $y \in \{(3,1,a),(5,1,a),(12,1,a)\}$, and if $\{(20,0,a),y\}$ is an edge of $\mathbf{G}_A$, then $y \in \{(9,0,a),(19,0,a)\}$. Hence we deduce $f(v,i,a) = (v,i,a)$ for all $v \in W_2$ and $i = 0,1$, and $f(v,0,b) = (v,0,b)$, $f(v,n,b) = (v,m,b)$ for all $v \in W_1$.

To prove that

$$f^{-1}(\{(v,i,a) \mid v \in W_2,\, i = 0,1\}) = \{(v,i,a) \mid v \in W_2,\ i = 0,1\},$$

assume that i is the least natural number with

$$f(\{(v,i,b) \mid v \in W_1\} \cup \{(v,j,i,c) \mid v \in W_0,\ j \in p_{a_i}\}) \cap \{(v,0,a) \mid v \in W_2\} \neq \emptyset.$$

If there exists $v \in W_1$ with $f(v,i,b) \in \{(v,0,a) \mid v \in W_2\}$ then, by the foregoing part of the proof, $f(v,i,b) = (v,0,a)$ for all $v \in W_1$. From the choice of i it follows that

$$f(\{(v,0,i-1,c) \mid v \in W_0\}) \cap \{(v,0,a) \mid v \in W_2\} = \emptyset$$

and hence, by (iv), $\{f(16,0,i-1,c), f(1,i,b)\} = \{(4,0,a),(1,0,b)\}$. This implies $(1,0,a) = f(1,i,b) = (4,0,a)$ — a contradiction. Thus $f(\{(v,i,b) \mid v \in W_1\}) \cap \{(v,0,a) \mid v \in W_2\} = \emptyset$. If there exists $v \in W_0$ with $f(v,0,i,c) \in \{(v,0,a) \mid v \in W_2\}$ then, by the foregoing part of the proof, $f(v,0,i,c) = (g_{0,2}(v),0,a)$ for all $v \in W_0$. By (iv), $\{f(0,0,i,c), f(1,i,b)\} = \{(4,0,a),(1,0,b)\}$ and thus $(0,0,a) = (g_{0,2}(0),0,a) = f(0,0,i,c) = (4,0,a)$ — this is a contradiction. Therefore

$$f(\{(v,0,i,c) \mid v \in W_0\}) \cap \{(v,0,a) \mid v \in W_2\} = \emptyset.$$

Let $j \in p_{a_i}$ be the least number such that $f(v,j,i,c) \in \{(v,0,a) \mid v \in W_2\}$ for some $v \in W_0$. By the foregoing part of the proof, $j > 0$ and $f(v,j,i,c) = (g_{0,2}(v),0,a)$ for all $v \in W_0$. Hence, by (iv), $\{f(13,j-1,i,c), f(10,j,i,c)\} = \{(4,0,a),(1,0,b)\}$ and thus

$$(10,0,a) = (g_{0,2}(10),0,a) = f(10,j,i,c) = (4,0,a),$$

which is a contradiction. Thus $f^{-1}(\{(v,0,a) \mid v \in W_2\}) = \{(v,0,a) \mid v \in W_2\}$ and, in an analogous way, we conclude that $f^{-1}(\{(v,1,a) \mid v \in W_2\}) = \{(v,1,a) \mid v \in W_2\}$. Whence for every $i \in n+1$ there exists $k \in m+1$ with $f(v,i,b) = (v,k,b)$ for all $v \in W_1$. If for $i \in n$ there exists $k \in m+1$ such that $f(v,i,b) = (v,k,b)$ for all $v \in W_1$ then either $k < m$ and $f(v,0,i,c) = (v,0,k,c)$ for all $v \in W_0$ and $f(v,i+1,b) = (v,k+1,b)$ for all $v \in W_1$, or $f(v,0,i,c) = (g_{0,1}(v),k,b)$ for all $v \in W_0$ and $f(v,i+1,b) = (v,k,b)$ for all $v \in W_1$. By (v) and Lemma 2.2, if for $i \in n$ there exists $k \in m+1$ such that $f(v,0,i,c) = (g_{0,1}(v),k,b)$ for all $v \in W_0$, then $f(v,j,i,c) = (g_{0,1}(v),k,b)$ for all $v \in W_0$ and for $j \in p_{a_i}$ because $g_{0,1}(13) = 11$ and $\{(11,k,b),y\} \in E_B$ implies $y \in \{(10,k,b),(12,k,b),(14,k,b)\}$. If for $i \in n$ there exists $k \in m$ such that $f(v,0,i,c) = (v,0,k,c)$ for all $v \in W_0$ then, by Lemma 2.2 and (vi), $a_i = b_k$ and $f(v,j,i,c) = (v,j,k,c)$ for all $v \in V$ and $j \in p_{a_i}$ because $\{10,13\} \notin F_0$. Thus for every $i \in n$ either there exists $k \in m+1$ such that $f(v,j,i,c) = (g_{0,1}(v),k,b)$ for all $v \in W_0$ and $j \in p_{a_i}$, or there exists $k \in m$ with $p_{a_i} = p_{b_k}$ and $f(v,j,i,c) = (v,j,k,c)$ for all $v \in W_0$ and $j \in p_{a_i}$. From (vii) it follows that $\{(0,0,k,c) \mid 0 \le k < m\} \subseteq \mathrm{Im}(f)$ and this implies that $B \subseteq A$. To complete the proof of (2), we prove, by induction, that for every $i = 0,1,\ldots,n$

$$f(x) = \begin{cases} (v,0,a) & \text{if } x = (v,0,a) \in V_A, \\ (v,\phi(k),b) & \text{if } x = (v,k,b) \in V_A, \text{ and } k \in i+1, \\ (g_{0,1}(v),\phi(k),b) & \text{if } x = (v,j,k,c) \in V_A,\ a_k \notin B \text{ and } k \in i, \\ (v,j,\phi(k),c) & \text{if } x = (v,j,k,c) \in V_A,\ a_k \in B \text{ and } k \in i. \end{cases}$$

Clearly, the statement holds for $i = 0$. Assuming that it holds for $i-1$, we prove it for i. If $a_i \notin B$ then $f(v,i,b) = (v,\phi(i),b)$ for all $v \in W_1$ and, by Lemma 2.2, there exists no graph homomorphism between $\mathbf{G}_{a_i}$ and $\mathbf{G}_{b_{\phi(i)}}$. Thus, by the foregoing part of the proof, $f(v,j,i,c) = (g_{0,1}(v), \phi(i), b)$ for all $v \in W_0$ and all $j \in p_{a_i}$ and $f(v, i+1, b) = (v, \phi(i), b)$ for all $v \in W_1$. If $a_i \in B$ then, by the foregoing part of the proof, either $f(v,j,i,c) = (g_{0,1}(v), \phi(i), b)$ for all $v \in W_0$ and all $j \in p_{a_i}$ and $f(v,i+1,b) = (v, \phi(i), b)$ for all $v \in W_1$, or $f(v,j,i,c) = (v, j, \phi(i), c)$ for all $v \in W_0$ and all $j \in p_{a_i}$ and $f(v, i+1, b) = (v, \phi(i)+1, b)$ for all $v \in W_1$. Since $a_l > a_i$ for all $l > i$ we find that in the first case $(0, 0, \phi(i), c) \notin \operatorname{Im}(f)$ — this is a contradiction. Thus the second case occurs and the induction step is proved because if $a_i = b_{\phi(i)}$ then $\phi(i) + 1 = \phi(i+1)$. Whence (2) is proved.

To prove (3), first observe that $g^{-1}_{B,A}(v,i,a)$ is a singleton for all $v \in W_2$ and $i \in \{0,1\}$. Further $g_{B,A}$ is injective on the set $\{(v,i,b) \mid v \in W_1\}$ for all $i \in n+1$. If $a_i \in B$ then $g^{-1}_{B,A}(g_{b,A}(v,j,i,c))$ is a singleton for all $v \in W_0$ and all $j \in p_{a_i}$ and $g^{-1}_{B,A}(g_{B,A}(\{(v,i,b) \mid v \in W_1\})) \subseteq \{(v,k,b) \mid v \in W_1,\ k \in i+1\} \cup \{(v,j,k,c) \mid v \in V,\ k \in i,\ j \in p_{a_k}\}$. Thus if $A = B \cup C$ then $\{g_{B,A}, g_{C,A}\}$ is a separating family. Conversely, if $a_i \notin B \cup C$ then $g_{B,A}(v,i,b) = g_{B,A}(v,i+1,b)$ and $g_{C,A}(v,i,b) = g_{C,A}(v,i+1,b)$ for all $v \in W_1$ and the proof of (3) is complete.

The proof of (4) follows from Lemmas 2.1 and 2.2. □

We extend the family $\{\mathbf{G}_A \mid A \in P_{nf}(\omega)\}$ so that $\mathbf{G}_\emptyset$ is the trivial undirected graph. We prove

Proposition 2.4. *The family of undirected graphs* $\{\mathbf{G}_A \mid A \in P_f(\omega)\}$ *satisfies conditions* (p1)–(p4).

Proof: Condition (p1) follows from the definition of $\mathbf{G}_\emptyset$. Proposition 2.3(3) implies condition (p2). By Proposition 2.3(1) and (2), if $f : \mathbf{G}_A \to \mathbf{G}_B$ for $A, B \in P_{nf}(\omega)$ is a graph homomorphism then $B \subseteq A$ and $f = g_{A,B}$, and $g_{A,B}$ is injective exactly when $A = B$, hence condition (p3) follows. To prove condition (p4), assume that $\mathbf{F}_1$ and $\mathbf{F}_2$ are finite undirected graphs such that $\mathbf{F}_1, \mathbf{F}_2 \in \operatorname{QVar}\{\mathbf{G}_A \mid A \in P_{nf}(\omega)\}$ and $A \in P_{nf}(\omega)$ such that $\mathbf{G}_A$ is a subgraph of $\mathbf{F}_1 \times \mathbf{F}_2$. Let us define

$$\mathcal{A}_i = \{B \in P_{nf}(\omega) \mid \exists \text{ a graph homomorphism } g : \mathbf{F}_i \to \mathbf{G}_B\}$$

for $i = 1, 2$. Then the family of all graph homomorphisms from $\mathbf{F}_i$ into $\mathbf{G}_D$ for $D \in \mathcal{A}_i$ is separating and thus $\mathbf{F}_i \in \operatorname{QVar}\{\mathbf{G}_D \mid D \in \mathcal{A}_i\}$ for $i = 1, 2$. Since there exists a graph homomorphism from $\mathbf{G}_A$ to $\mathbf{G}_B$ for any $B \in \mathcal{A}_1 \cup \mathcal{A}_2$ we conclude that $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite and that $B = \bigcup_{X \in \mathcal{A}_1} X, C = \bigcup_{X \in \mathcal{A}_2} X \subseteq A$. By Proposition 2.3(1), $g_{A,B}$ and $g_{A,C}$ are extremal epimorphisms in $\mathbb{GRA}$. Since $\mathbf{G}_A$ is a subgraph of $\mathbf{F}_1 \times \mathbf{F}_2$ we find, by Proposition 2.3(3), that $\mathbf{G}_A \in \operatorname{QVar}(\{\mathbf{G}_B, \mathbf{G}_C\})$ and thus $A = B \cup C$. Then the families $\{g_{B,D} \mid D \in \mathcal{A}_1\}$ and $\{g_{C,D} \mid D \in \mathcal{A}_2\}$ are separating. Let $f : \mathbf{G}_A \to \mathbf{F}_1 \times \mathbf{F}_2$ be an injective graph homomorphism and let $\pi_i : \mathbf{F}_1 \times \mathbf{F}_2 \to \mathbf{F}_i$ be a projection for $i = 1, 2$. We can assume that $\pi_i \circ f : \mathbf{G}_A \longrightarrow \mathbf{F}_i$ is an extremal epimorphism for $i = 1, 2$ (for else we can replace $\mathbf{F}_i$ by the graph $\pi_i \circ f(\mathbf{G}_A)$). From Proposition 2.3(1) and (2) it follows that $h \circ \pi_1 \circ f = g_{A,D} = g_{B,D} \circ g_{A,B}$ for every graph homomorphism $h : \mathbf{F}_1 \longrightarrow \mathbf{G}_D$ and $D \in \mathcal{A}_1$, and $h \circ \pi_2 \circ f = g_{A,D} = g_{C,D} \circ g_{A,C}$ for every graph homomorphism $h : \mathbf{F}_2 \longrightarrow \mathbf{G}_D$ and $D \in \mathcal{A}_2$. Now the diagonalization property completes the proof of condition (p4). $\square$

Let $\mathbf{K}_4$ be the complete antireflexive graph on the four-element set. Then the family of all graph homomorphisms from $\mathbf{G}$ into $\mathbf{K}_4$ is separating for every undirected graph $\mathbf{G}$ with $\chi(\mathbf{G}) \leq 3$. Thus $\mathbf{G}$ belongs to the quasivariety generated by $\mathbf{K}_4$ and since the full subcategory of $\mathbb{GRA}$ determined by graphs $\mathbf{G}$ with $\chi(\mathbf{G}) = 3$ is ff-alg-universal, see [17], we can summarize as follows.

Corollary 2.5. *The quasivariety generated by $\mathbf{K}_4$ is finitely generated, ff-alg-universal and Q-universal.* $\square$

Next we apply this result to digraphs. We recall a construction from the second section of [9]. In the paper [9], an ff-alg-universal full subcategory $\mathbb{DG}_s$ of $\mathbb{DG}$ is constructed such that

- (c1) any digraph $\mathbf{G}$ from $\mathbb{DG}_s$ is a strongly connected antisymmetric digraph, and any arc of $\mathbf{G}$ belongs to an oriented cycle of length 3 in $\mathbf{G}$;
- (c2) any digraph $\mathbf{G}$ from $\mathbb{DG}_s$ contains two distinct nodes $a_{\mathbf{G}}$ and $b_{\mathbf{G}}$ such that there exists no arc between $a_{\mathbf{G}}$ and $b_{\mathbf{G}}$ in $\mathbf{G}$ and any graph homomorphism $f : \mathbf{G} \longrightarrow \mathbf{G}'$ from $\mathbb{DG}_s$ satisfies $f(a_{\mathbf{G}}) = a_{\mathbf{G}}$ and $f(b_{\mathbf{G}}) = b_{\mathbf{G}'}$;
- (c3) for any pair of digraphs (X, R) and (Y, S) from $\mathbb{DG}_s$ there exists no graph homomorphism $f : (X, R) \longrightarrow (Y, S^d)$ where $S^d = \{(x, y) \mid (y, x) \in S\}$.

In [9], two functors Λ and Ω are constructed such that Ω is a full embedding from the quasivariety $\mathbb{GRA}$ into the quasivariety $\mathbb{DG}(2)$ of two binary relations and homomorphisms preserving both relations. Precisely, for an undirected graph (V, E), define $\Omega(V, E) = (X, R_1, R_2)$ where $X = V \cup \{v, w\}$ for two new nodes v and w, $R_1 = \{(x, y) \mid \{x, y\} \in E\}$ and R_2 is the least ordering such that $(x, v), (v, w) \in R_2$ for all $x \in V$. For a graph homomorphism f, Ωf is an extension of f by $\Omega f(v) = v$ and $\Omega f(w) = w$. It is easy to see that Ω is a full embedding preserving finiteness, extremal epimorphisms and separating families. The functor Λ is a šíp-construction, see [16] or [17], from $\mathbb{DG}(2)$ into $\mathbb{DG}_s$. We use two šíps that are finite, rigid, pairwise rigid, antisymmetric relations (both have 11 nodes and 21 arcs). Therefore Λ is a full embedding from $\mathbb{DG}(2)$ into $\mathbb{DG}_s$ preserving finiteness, extremal epimorphisms, and separating families. Analogously as in the proof of Proposition 2.4 we obtain that the family $\{\mathbf{H}_\emptyset\} \cup \{\mathbf{H}_A = \Lambda \circ \Omega \mathbf{G}_A \mid A \in P_{nf}(\omega)\}$ (where $\mathbf{H}_\emptyset$ is the trivial digraph) satisfies conditions (p1)–(p4) and for any $A \in P_f(\omega)$, $\mathbf{H}_A$ belongs to the quasivariety generated by $\Lambda \circ \Omega \mathbf{K}_4$. We recall that $\Lambda \circ \Omega \mathbf{K}_4$ has 195 nodes and 441 arcs. Thus we can summarize

Corollary 2.6. *The quasivariety* $\mathrm{QVar}(\Lambda \circ \Omega(\mathbf{K}_4))$ *is a finitely generated, ff-alg-universal and Q-universal.* $\square$

Clearly, $\mathrm{QVar}(\Lambda \circ \Omega(\mathbf{K}_4))$ is a subquasivariety of the quasivariety of all antisymmetric graphs.

3. Varieties of semigroups

We will apply the results of the second section to the semigroup varieties $\mathrm{Var}\{\mathbf{M}_2\}$, $\mathrm{Var}\{\mathbf{M}_3\}$ and $\mathrm{Var}\{(\mathbf{M}_3)^o\}$, by exploiting the functors from [9]. First we shall investigate the variety $\mathrm{Var}\{\mathbf{M}_2\}$. By [9], this variety is given by the identities $xyz = uu$ and $xy = yx$. M.V. Sapir [18] proved that there exists a finite semigroup $\mathbf{S} \in \mathrm{Var}\{\mathbf{M}_2\}$ such that $\mathrm{QVar}\{\mathbf{S}\}$ is Q-universal. We strengthen his result: we construct a finite semigroup $\mathbf{S}$ of size 23 such that $\mathrm{QVar}(\mathbf{S})$ is $\mathbb{ZS}$-relatively ff-alg-universal and Q-universal (Sapir's semigroup is a subdirect product of a finite collection of semigroups of size at most 2^{40}). This also strengthens the results of [9].

We recall the definition of the functor $\Phi : \mathbb{GRA} \longrightarrow \mathrm{Var}\{\mathbf{M}_2\}$ from [9]. For an undirected graph (V, E), let us denote $\mathtt{C}(E) = \{\{u, v\} \mid u, v \in V,\ u \neq v,\ \{u, v\} \notin E\}$ and let $\Phi(V, E)$ be a groupoid $(\Phi_0(V, E), \cdot)$ where $\Phi_0(V, E) = (V \times \{0, 1, 2\}) \cup \{t_i \mid i \in 9\} \cup \{0, u\} \cup \mathtt{C}(E)$ (we assume that $\{t_i \mid i \in 9\} \cup \{0, u\}$ is a set of pairwise distinct elements disjoint with $V \times \{0, 1, 2\} \cup \mathtt{C}(E)$) and for $x, y \in \Phi_0(V, E)$ define

- $xy = 0$ if either $x = y$, or $\{x, y\} = \{t_i, t_{i+1}\}$ for $i \in 9$, or $\{x, y\} = \{t_0, t_4\}$, or $\{x, y\} = \{t_8, (v, 2)\}$ for some $v \in V$, or $\{x, y\} = \{t_0, (v, 0)\}$ for some $v \in V$, or $\{x, y\} = \{(v, i), (v, i + 1)\}$ for some $v \in V$ and some $i \in 2$, or $\{x, y\} \cap (\{0, u\} \cup \mathtt{C}(E)) \neq \emptyset$;
- $xy = u$ if either $\{x, y\} = \{t_i, t_j\}$ for $i, j \in 9$ with $j \neq i - 1, i, i + 1$ and $\{i, j\} \neq \{0, 4\}$, or $\{x, y\} = \{t_i, (v, j)\}$ for $i \in 9$, $v \in V$ and $j \in 3$ with $(i, j) \neq (0, 0)$ and $(i, j) \neq (8, 2)$, or $\{x, y\} = \{(v, 0), (w, 2)\}$ for $v, w \in V$ with $v \neq w$, or $\{x, y\} = \{(v, i), (w, j)\}$ for $\{v, w\} \in E$, $i, j \in 3$ with $|i - j| \leq 1$;
- $xy = z$ for $z \in \mathtt{C}(E)$ if either $\{x, y\} = \{(v, i), (w, i+1)\}$, $z = \{v, w\} \in \mathtt{C}(E)$, and $i \in 2$, or $\{x, y\} = \{(v, i), (w, i)\}$, $z = \{v, w\} \in \mathtt{C}(E)$, and $i \in 3$.

For a graph homomorphism $f : (V, E) \longrightarrow (W, F) \in \mathbb{GRA}$ let us define $\Phi f : \Phi(V, E) \longrightarrow \Phi(W, F)$ so that for $x \in \Phi_0(V, E)$

- $\Phi f(x) = x$ if $x \in \{0, u\} \cup \{t_i \mid i \in 8\}$;
- $\Phi f(v, i) = (f(v), i)$ for all $v \in V$ and all $i \in 3$;
- if $x = \{v, w\} \in \mathtt{C}(E)$ then

$$\Phi f(x) = \begin{cases} 0 & \text{if } f(v) = f(w), \\ u & \text{if } f(v) \neq f(w) \text{ and } \{f(v), f(w)\} \in F, \\ \{f(v), f(w)\} & \text{if } f(v) \neq f(w) \text{ and } \{f(v), f(w)\} \in \mathtt{C}(F). \end{cases}$$

The functor Φ is denoted as Φ_2 in [9]. The next theorem gives its properties.

Theorem 3.1 ([9]). *For any undirected graph (V,E) the groupoid $\Phi(V,E)$ is a semigroup from* $\mathrm{Var}\{\mathbf{M}_2\}$*, for any graph homomorphism $f:(V,E)\to(W,F)$, $\Phi f:\Phi(V,E)\to\Phi(W,F)$ is a semigroup homomorphism. The functor $\Phi:\mathbb{GRA}\to$* $\mathrm{Var}\{\mathbf{M}_2\}$ *is a $\mathcal{S}_{\mathbb{ZS}}$-relative full embedding preserving finiteness, separating families, and extremal epimorphisms. If $f:\Phi(V,E)\to\Phi(W,F)$ is a semigroup homomorphism then either $f=\Phi g$ for a graph homomorphism $g:(V,E)\to(W,F)$, or* $\mathrm{Im}(f)$ *is a zero-semigroup and $f(\{0,u\}\cup\complement(E))=\{0\}$.* $\square$

Let $\mathbf{C}_\emptyset$ be the trivial semigroup and $\mathbf{C}_A=\Phi\mathbf{G}_A$ for $A\in P_{nf}(\omega)$. We prove an auxiliary lemma.

Lemma 3.2. *The family $\{\mathbf{C}_A\mid A\in P_f(\omega)\}$ of finite semigroups from the variety* $\mathrm{Var}\{\mathbf{M}_2\}$ *satisfies conditions* (p1)–(p4).

Proof: Condition (p1) follows from the definition of $\mathbf{C}_\emptyset$, conditions (p2) and (p3) are consequences of Proposition 2.3, Theorem 3.1 and the fact that Φ preserves separating families. To prove (p4), assume that $\mathbf{S},\mathbf{T}\in\mathrm{QVar}\{\mathbf{C}_B\mid B\in P_{nf}(\omega)\}$ are finite semigroups and $A\in P_{nf}(\omega)$ is such that $\mathbf{C}_A$ is a subsemigroup of $\mathbf{S}\times\mathbf{T}$. Then there exist finite separating families $\{f_i:\mathbf{S}\to\mathbf{C}_{B_i}\mid i\in I\}$ and $\{g_j:\mathbf{T}\to\mathbf{C}_{C_j}\mid j\in J\}$ of homomorphisms. Let $\psi:\mathbf{C}_A\to\mathbf{S}\times\mathbf{T}$ be an injective homomorphism and let $\pi_1:\mathbf{S}\times\mathbf{T}\to\mathbf{S}$ and $\pi_2:\mathbf{S}\times\mathbf{T}\to\mathbf{T}$ be the projections. With no loss of generality we can assume that $\pi_i\circ\psi$ is surjective for $i=1,2$ — for else we replace $\mathbf{S}$ and/or $\mathbf{T}$ by their subsemigroups $\mathrm{Im}(\pi_1\circ\psi)$ and/or $\mathrm{Im}(\pi_2\circ\psi)$. If condition (p4) is true for these new semigroups then it is also true for $\mathbf{S}$ and $\mathbf{T}$ and condition (p4) will then be proved. Let I_1 denote the subset of I consisting of all $i\in I$ with $f_i\circ\pi_1\circ\psi(u)\neq 0$ and J_1 the subset of J consisting of all $j\in J$ with $g_j\circ\pi_2\circ\psi(u)\neq 0$. Since ψ is injective and $\{\pi_1,\pi_2\}$, $\{f_i\mid i\in I\}$ and $\{g_j\mid j\in J\}$ are separating families we conclude that either $I_1\neq\emptyset$ or $J_1\neq\emptyset$. By Theorem 3.1, for every $i\in I_1$ we have $f_i\circ\pi_1\circ\psi=\Phi g_{A,B_i}$ and for every $j\in J_1$ we have $g_j\circ\pi_2\circ\psi=\Phi g_{A,C_j}$. Thus, by Proposition 2.3, $B=\bigcup_{i\in I_1}B_i\subseteq A$, $C=\bigcup_{j\in J_1}C_j\subseteq A$, and $\{g_{B,B_i}\mid i\in I_1\}$ and $\{g_{C,C_j}\mid j\in J_1\}$ are separating families. By Theorem 3.1, for $i\in I\setminus I_1$ and $j\in J\setminus J_1$, if $\mathbf{G}_A=(V,E)$ then $f_i\circ\pi_1\circ\psi(\{u,0\}\cup\complement(E))=\{0\}=g_j\circ\pi_2\circ\psi(\{u,0\}\cup\complement(E))$. Thus $\{f_i\mid i\in I_1\}\cup\{g_j\mid j\in J_1\}$ is a separating family on the set $\{u,0\}\cup\complement(E)$. Hence $\Phi g_{A,B\cup C}$ is injective on the set $\{u,0\}\cup\complement(E)$ because $\{\Phi g_{B\cup C,B_i}\mid i\in I_1\}\cup\{\Phi g_{B\cup C,C_j}\mid j\in J_1\}$ is a separating family of $\mathbf{C}_{B\cup C}$ and $f_i\circ\pi_1\circ\psi=\Phi g_{B\cup C,B_i}\circ\Phi g_{A,B\cup C}=\Phi g_{B,B_i}\circ\Phi g_{A,B}$ for all $i\in I_1$ and $g_j\circ\pi_2\circ\psi=\Phi g_{B\cup C,C_i}\circ\Phi g_{A,B\cup C}=\Phi g_{C,C_i}\circ\Phi g_{A,C}$ for all $j\in J_1$. From the fact that $\Phi g_{A,B\cup C}$ is an isomorphism if and only if $\Phi g_{A,B\cup C}$ is injective on the set $\{u,0\}\cup\complement(E)$ (this follows from the definition of Φ and Theorem 3.1) we conclude, by Proposition 2.3(3), that $A=B\cup C$. Assume that $B\neq\emptyset$. Then, by the diagonalization property, there exists a semigroup homomorphism $h:\mathbf{S}\to\mathbf{C}_B$ with $h\circ\pi_1\circ\psi=\Phi g_{A,B}$ and $\Phi g_{B,B_i}\circ h=f_i$ for all $i\in I_1$ and h is surjective because $\Phi g_{A,B}$ is surjective (by Proposition 2.3(1), $g_{A,B}$ is an extremal epimorphism and Φf is surjective for any extremal epimorphism f). Since $\{f_i\mid i\in I\}$ is a separating family for distinct elements x and y of $\mathbf{S}$ such that $h(x) = h(y)$ there exists $i \in I \setminus I_1$ with $f_i(x) \neq f_i(y)$. From $f_i \circ \pi_1 \circ \psi(\{u,0\} \cup \complement(E)) = \{0\}$ it follows that x or y is an irreducible element because $\Phi_0(V,E) \setminus (\{u,0\} \cup \complement(E))$ is the set of all irreducible elements of $\mathbf{C}_A$. Since $h(x) = h(y)$ and since $\Phi g_{A,B} = h \circ \pi_1 \circ \psi$ implies that $h(x)$ is irreducible we conclude that both x and y are irreducible in $\mathbf{S}$. For every element z of $\mathbf{S}$ we have $xz, yz \in \pi_1 \circ \psi(\{u,0\} \cup \complement(E))$ and from the fact that $h^{-1}(z)$ is a singleton for all $z \in h \circ \pi_1 \circ \psi(\{u,0\} \cup \complement(E))$ it follows that $xz = yz$ for all elements x, y, and z from $\mathbf{S}$ with $h(x) = h(y)$. Thus a mapping $k : \mathbf{C}_B \to \mathbf{S}$ such that $h \circ k$ is the identity mapping is a homomorphism and whence $\mathbf{C}_B \in \mathrm{QVar}\{\mathbf{S}\}$ because k is injective. Analogously, we obtain that $\mathbf{C}_C \in \mathrm{QVar}\{\mathbf{T}\}$ if $C \neq \emptyset$. Since $\mathbf{C}_\emptyset$ is the trivial semigroup the statement is true also if $C = \emptyset$ or $B = \emptyset$, and condition (p4) is proved. □

Since for every undirected graph (V,E) with $\chi(V,E) = 3$ the family of graph homomorphisms from (V,E) into $\mathbf{K}_4$ is separating and since the full subcategory of $\mathbb{GRA}$ consisting of all graphs $\mathbf{G}$ with $\chi(\mathbf{G}) = 3$ is ff-alg-universal (see, [17]), we deduce

Theorem 3.3. *The quasivariety generated by the semigroup $\Phi\mathbf{K}_4$ is contained in $\mathrm{Var}\{\mathbf{M}_2\}$ and it is $\mathbb{ZS}$-relatively ff-alg-universal and Q-universal. The size of $\Phi\mathbf{K}_4$ is* 23.

Proof: From $\Phi\mathbf{K}_4 \in \mathrm{Var}\{\mathbf{M}_2\}$ it follows that $\mathrm{QVar}\{\Phi\mathbf{K}_4\} \subseteq \mathrm{Var}\{\mathbf{M}_2\}$. Since for every undirected graph (V,E) with $\chi(V,E) = 3$ we have $\Phi(V,E) \in \mathrm{QVar}\{\Phi\mathbf{K}_4\}$, Theorem 3.1 completes the proof that $\mathrm{QVar}\{\Phi\mathbf{K}_4\}$ is $\mathbb{ZS}$-relatively ff-alg-universal. Lemma 3.2 and Theorem 1.1 imply that $\mathrm{QVar}\{\Phi\mathbf{K}_4\}$ is Q-universal because, according to the definition, Φ preserves extremal epimorphisms and separating families. Direct inspection shows that the size of $\Phi\mathbf{K}_4$ is 23. □

We recall the $\mathbb{ILZ}$-relatively full embedding $\Gamma : \mathbb{DG}_s \to \mathrm{Var}\{\mathbf{M}_3\}$ from [9]. For a digraph (X,R) from $\mathbb{DG}_s$, let $\Gamma(X,R)$ be a groupoid $(\Gamma_0(X,R),\cdot)$ where $\Gamma_0(X,R) = X \cup R \cup \{a,b,a_1,b_1,e\}$ (a, b, a_1, b_1, e are pairwise distinct elements with $(X \cup R) \cap \{a,b,a_1,b_1,e\} = \emptyset$) and

- $a(x,y) = x$ and $b(x,y) = y$ for all $(x,y) \in R$;
- $ax = bx = e$ for all $x \in X \cup \{e,a,b\}$;
- $aa_1 = a_{\mathbf{G}}$, $bb_1 = b_{\mathbf{G}}$, $ab_1 = ba_1 = e$ (where $a_{\mathbf{G}}$ and $b_{\mathbf{G}}$ are determined by condition (c2) on $\mathbb{DG}_s$);
- $xy = x$ for all $x \in X \cup R \cup \{a_1,b_1,e\}$ and all $y \in \Gamma_0(X,R)$.

For a graph homomorphism $f : (X,R) \to (Y,S)$ from $\mathbb{DG}_s$ let $\Gamma f : \Gamma_0(X,R) \to \Gamma_0(Y,S)$ be a mapping such that

$$\Gamma f(x) = \begin{cases} x & \text{if } x \in \{a,b,a_1,b_1,e\}, \\ f(x) & \text{if } x \in X, \\ (f(y),f(z)) & \text{if } x = (y,z) \in R. \end{cases}$$

The functor Γ is denoted as Γ_2 in [9]. Then we have

Theorem 3.4 ([9])**.** *The groupoid $\Gamma(X,R)$ is a semigroup from the variety* $\mathrm{Var}\{\mathbf{M}_3\}$*, Γf is a semigroup homomorphism for every graph homomorphism from $\mathbb{DG}_s$. The functor $\Gamma : \mathbb{DG}_s \to \mathrm{Var}\{\mathbf{M}_3\}$ is an $\mathbb{ILZ}$-relatively full embedding preserving finiteness, extremal epimorphisms and separating families of graph homomorphisms. If $\mathbf{G}$ and $\mathbf{G}'$ are digraphs from $\mathbb{DG}_s$ and if $f : \Gamma\mathbf{G} \to \Gamma\mathbf{G}'$ is a semigroup homomorphism then either $f = \Gamma g$ for some graph homomorphism $g : \mathbf{G} \to \mathbf{G}'$ or $f(a_{\mathbf{G}}) = f(b_{\mathbf{G}})$. If $\mathbf{G} = (X,R)$ and if $f(a_{\mathbf{G}}) = f(b_{\mathbf{G}})$ for a semigroup homomorphism $f : \Gamma\mathbf{G} \to \Gamma\mathbf{G}'$ then $f(X \cup \{e\})$ is a singleton.* $\square$

Define $\mathbf{I}_\emptyset$ as the trivial semigroup, and for $A \in P_{nf}(\omega)$ define $\mathbf{I}_A = (\Gamma \circ \Lambda \circ \Omega)\mathbf{G}_A$. We prove

Lemma 3.5. *The family $\{\mathbf{I}_A \mid A \in P_f(\omega)\}$ of finite semigroups from the variety* $\mathrm{Var}\{\mathbf{M}_3\}$ *satisfies conditions* (p1)–(p4).

Proof: Condition (p1) follows from the definition of $\mathbf{I}_\emptyset$. Since Γ, Λ and Ω preserve separating families and extremal epimorphisms we conclude, by Lemma 2.2, Proposition 2.3 and Theorem 3.4, that conditions (p2) and (p3) are satisfied. The proof of condition (p4) is analogous to the proof of Lemma 3.2. Let $A \in P_{nf}(\omega)$ be a set and let $\mathbf{S}$ and $\mathbf{T}$ be finite semigroups such that $\mathbf{S}, \mathbf{T} \in \mathrm{QVar}\{\mathbf{I}_B \mid B \in P_{nf}(\omega)\}$ and that $\mathbf{I}_A$ is isomorphic to a subsemigroup of $\mathbf{S} \times \mathbf{T}$. By the finiteness of $\mathbf{S}$ and $\mathbf{T}$, there exist finite separating families $\{f_i : \mathbf{S} \to \mathbf{I}_{A_i} \mid i \in I\}$ and $\{g_j : \mathbf{T} \to \mathbf{I}_{C_j} \mid j \in J\}$. Let $\psi : \mathbf{I}_A \to \mathbf{S} \times \mathbf{T}$ be an injective semigroup homomorphism and let $\pi_1 : \mathbf{S}\times\mathbf{T} \to \mathbf{S}$ and $\pi_2 : \mathbf{S}\times\mathbf{T} \to \mathbf{T}$ be the projections. Analogously to Lemma 3.2, we can assume that $\pi_1 \circ \psi$ and $\pi_2 \circ \psi$ are surjective. Let I' be the subset of I consisting of all i with $f_i \circ \pi_1 \circ \psi(a_{\mathbf{I}_A}) \neq f_i \circ \pi_1 \circ \psi(b_{\mathbf{I}_A})$ and let J' be the subset of J consisting of all j with $g_j \circ \pi_2 \circ \psi(a_{\mathbf{I}_A}) \neq g_j \circ \pi_2 \circ \psi(b_{\mathbf{I}_A})$. By Theorem 3.4, $(\Gamma\circ\Lambda\circ\Omega)g_{A,B_i} = f_i \circ \pi_1 \circ \psi$ for all $i \in I'$ and $(\Gamma\circ\Lambda\circ\Omega)g_{A,C_j} = g_j \circ \pi_2 \circ \psi$ for all $j \in J'$. Let us define $B = \bigcup_{i\in I'} B_i$ and $C = \bigcup_{j\in J'} C_j$. Clearly, $B, C \subseteq A$, $(\Gamma \circ \Lambda \circ \Omega)g_{B\cup C,B_i} \circ (\Gamma \circ \Lambda \circ \Omega)g_{A,B\cup C} = (\Gamma \circ \Lambda \circ \Omega)g_{A,B_i}$ for all $i \in I'$, and $(\Gamma \circ \Lambda \circ \Omega)g_{B\cup C,C_j} \circ (\Gamma \circ \Lambda \circ \Omega)g_{A,B\cup C} = (\Gamma \circ \Lambda \circ \Omega)g_{A,C_j}$ for all $j \in J'$. Since $(\Lambda \circ \Omega)g_{A,B\cup C}$ is an extremal epimorphism and because Λ and Ω are full embeddings we conclude, by Proposition 2.3(3), that $(\Lambda \circ \Omega)g_{A,B\cup C}$ is injective if and only if $A = B \cup C$. If $B \cup C \neq A$ then there exist distinct nodes x and y in $(\Lambda \circ \Omega)\mathbf{G}_A$ with $(\Lambda \circ \Omega)g_{A,B\cup C}(x) = (\Lambda \circ \Omega)g_{A,B\cup C}(y)$. Thus $(\Gamma\circ\Lambda\circ\Omega)g_{A,B\cup C}(x) = (\Gamma\circ\Lambda\circ\Omega)g_{A,B\cup C}(y)$, and whence $f_i\circ\pi_1\circ\psi(x) = f_i\circ\pi_1\circ\psi(y)$ for all $i \in I'$ and $g_j \circ \pi_2 \circ \psi(x) = g_j \circ \pi_2 \circ \psi(y)$ for all $j \in J'$. By Theorem 3.4, $f_i \circ \pi_1 \circ \psi(x) = f_i \circ \pi_1 \circ \psi(y)$ for all $i \in I \setminus I'$ and $g_j \circ \pi_2 \circ \psi(x) = g_j \circ \pi_2 \circ \psi(y)$ for all $j \in J \setminus J'$. Therefore $\{f_i \circ \pi_1 \circ \psi \mid i \in I\} \cup \{g_j \circ \pi_2 \circ \psi \mid j \in J\}$ is not a separating family and this is a contradiction because ψ is injective and $\{\pi_1, \pi_2\}$, $\{f_i \mid i \in I\}$ and $\{g_j \mid j \in J\}$ are separating families. Thus $A = B \cup C$. From $(\Gamma \circ \Lambda \circ \Omega)g_{B,B_i} \circ (\Gamma \circ \Lambda \circ \Omega)g_{A,B} = (\Gamma \circ \Lambda \circ \Omega)g_{A,B_i}$ for all $i \in I'$ and

$(\Gamma\circ\Lambda\circ\Omega)g_{C,C_j}\circ(\Gamma\circ\Lambda\circ\Omega)g_{A,C}=(\Gamma\circ\Lambda\circ\Omega)g_{A,C_j}$ for all $j\in J'$ it follows, by the diagonalization property, the existence of homomorphisms $h_{\mathbf{S}}:\mathbf{S}\longrightarrow(\Gamma\circ\Lambda\circ\Omega)\mathbf{G}_B$ and $h_{\mathbf{T}}:\mathbf{T}\longrightarrow(\Gamma\circ\Lambda\circ\Omega)\mathbf{G}_C$ with $(\Gamma\circ\Lambda\circ\Omega)g_{A,B}=h_{\mathbf{S}}\circ\pi_1\circ\psi$, $(\Gamma\circ\Lambda\circ\Omega)g_{A,C}=h_{\mathbf{T}}\circ\pi_2\circ\psi$, $(\Gamma\circ\Lambda\circ\Omega)g_{B,B_i}\circ h_{\mathbf{S}}=f_i$ for all $i\in I'$, and $(\Gamma\circ\Lambda\circ\Omega)g_{C,C_j}\circ h_{\mathbf{T}}=g_j$ for all $j\in J'$ because $\pi_1\circ\psi$ and $\pi_2\circ\psi$ are surjective, and $\{(\Gamma\circ\Lambda\circ\Omega)g_{B,B_i}\mid i\in I'\}$ and $\{(\Gamma\circ\Lambda\circ\Omega)g_{C,C_j}\mid j\in J'\}$ are separating families. Since $(\Gamma\circ\Lambda\circ\Omega)g_{A,B}$ and $(\Gamma\circ\Lambda\circ\Omega)g_{A,C}$ are surjective, we deduce that $h_{\mathbf{S}}$ and $h_{\mathbf{T}}$ are also surjective. Observe that if $(\Lambda\circ\Omega)\mathbf{G}_A=(X,R)$ then $((\Gamma\circ\Lambda\circ\Omega)g_{A,B})^{-1}(x)$ and $((\Gamma\circ\Lambda\circ\Omega)g_{A,C})^{-1}(x)$ are singletons for $x\in\{a,b,a_1,b_1,e\}$,

$$((\Gamma\circ\Lambda\circ\Omega)g_{A,B})^{-1}((\Gamma\circ\Lambda\circ\Omega)g_{A,B}(X))=X=$$
$$=((\Gamma\circ\Lambda\circ\Omega)g_{A,C})^{-1}((\Gamma\circ\Lambda\circ\Omega)g_{A,C}(X)).$$

Hence, by Theorem 3.4, $h_{\mathbf{S}}^{-1}((\Gamma\circ\Lambda\circ\Omega)g_{A,B}(z))$ and $h_{\mathbf{T}}^{-1}((\Gamma\circ\Lambda\circ\Omega)g_{A,C}(z))$ are singletons for all $z\in X\cup\{a,b,a_1,b_1,e\}$. Let $k_{\mathbf{S}}:(\Gamma\circ\Lambda\circ\Omega)\mathbf{G}_B\longrightarrow\mathbf{S}$ and $k_{\mathbf{T}}:(\Gamma\circ\Lambda\circ\Omega)\mathbf{G}_C\longrightarrow\mathbf{T}$ be mappings such that $h_{\mathbf{S}}\circ k_{\mathbf{S}}$ and $h_{\mathbf{T}}\circ k_{\mathbf{T}}$ are the identity mappings. We prove that $k_{\mathbf{S}}$ is a semigroup homomorphism from $\mathbf{I}_B$ into $\mathbf{S}$. If $\mathbf{S}=(S,\cdot)$ then $S^2=\{st\mid s,t\in S\}$ is a left-zero semigroup consisting of all left zeros of $\mathbf{S}$, see [9]. Since $h_{\mathbf{S}}$ is surjective we deduce that $h_{\mathbf{S}}(S^2)$ is the greatest left-zero subsemigroup of $\mathbf{I}_B$. Since the elements a and b are uniquely determined in $\mathbf{I}_B$ and $\mathbf{I}_A$ we conclude that $k_{\mathbf{S}}(h(S^2))\subseteq S^2$ and any element of $h_{\mathbf{S}}(S^2)$ is a left zero of $\mathbf{I}_B$. Thus $k_{\mathbf{S}}(x)k_{\mathbf{S}}(y)=k_{\mathbf{S}}(xy)$ for all $x\in h_{\mathbf{S}}(S^2)$. If $x\notin h_{\mathbf{S}}(S^2)$ then $xy\in(\Gamma\circ\Lambda\circ\Omega)g_{A,B}(X\cup\{a_1,b_1,e\})$ and hence

$$h_{\mathbf{S}}\circ k_{\mathbf{S}}(xy)=xy=h_{\mathbf{S}}(k_{\mathbf{S}}(x))h_{\mathbf{S}}(k_{\mathbf{S}}(y))=h_{\mathbf{S}}(k_{\mathbf{S}}(x)k_{\mathbf{S}}(y)).$$

Thus $k_{\mathbf{S}}(xy)=k_{\mathbf{S}}(x)k_{\mathbf{S}}(y)$ and $k_{\mathbf{S}}$ is an injective homomorphism from $\mathbf{I}_B$ into $\mathbf{S}$. Whence $\mathbf{I}_B\in\mathrm{QVar}\{\mathbf{S}\}$. Analogously we prove that $k_{\mathbf{S}}:\mathbf{I}_C\longrightarrow\mathbf{T}$ is an injective homomorphism and condition (p4) is proved. □

Theorem 3.6. *The quasivariety* $\mathrm{QVar}\{(\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4\}$ *generated by the semigroup* $(\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4$ *is contained in* $\mathrm{Var}\{\mathbf{M}_3\}$ *and is* $\mathbb{ILZ}$*-relatively* ff*-alg-universal and* Q*-universal. The size of* $(\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4$ *is* 641.

Proof: Since Λ and Ω are full embeddings preserving finiteness, extremal epimorphisms and separating families and since Γ is an $\mathbb{ILZ}$-relatively full embedding preserving finiteness, extremal epimorphisms and separating families we conclude that $\Gamma\circ\Lambda\circ\Omega$ is an $\mathbb{ILZ}$-relatively full embedding preserving finiteness, extremal epimorphisms and separating families. Since any graph $\mathbf{G}$ with $\chi(\mathbf{G})=3$ is a subdirect power of $\mathbf{K}_4$ we conclude that the quasivariety $\mathrm{QVar}\{(\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4\}$ is $\mathbb{ILZ}$-relatively ff-alg-universal. By Lemma 3.5, Proposition 2.4, and Theorem 1.1, the quasivariety $\mathrm{QVar}\{(\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4\}$ is Q-universal. The digraph $(\Lambda\circ\Omega)\mathbf{K}_4$ has 195 nodes and 441 arcs and thus the semigroup $(\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4$ has 641 elements. □

By symmetry, we immediately obtain

Corollary 3.7. *The quasivariety* $\mathrm{QVar}\{((\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4)^o\}$ *generated by the semigroup* $((\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4)^o$ *is contained in* $\mathrm{Var}\{(\mathbf{M}_3)^o\}$ *and is* $\mathbb{IRZ}$*-relatively* ff*-alg-universal and* Q*-universal. The size of* $((\Gamma\circ\Lambda\circ\Omega)\mathbf{K}_4)^o$ *is* 641. $\square$

Finally, we investigate the quasivarieties generated by $\mathbf{M}_2$, or $\mathbf{M}_3$ or $(\mathbf{M}_3)^o$. We recall that any semigroup from $\mathbb{ILZ}$ or $\mathbb{IRZ}$ is a subdirect product of two-element semigroups. We exploit Theorem 1.2 saying that any locally finite, Q-universal quasivariety has infinitely many non-isomorphic critical algebras.

Theorem 3.8. *The quasivarieties* $\mathrm{QVar}\{\mathbf{M}_2\}$, $\mathrm{QVar}\{\mathbf{M}_3\}$, *and* $\mathrm{QVar}\{(\mathbf{M}_3)^o\}$ *are neither quasivar-relatively alg-universal nor* Q*-universal.*

Proof: Let $\mathbf{S}$ be a semigroup from $\mathrm{Var}\{\mathbf{M}_2\}$. By a direct inspection, either $\mathbf{S}$ is a subdirect product of two-element algebras (i.e., $\mathbf{S}\in\mathbb{ZS}$), or there exists a subsemigroup of $\mathbf{S}$ isomorphic to $\mathbf{M}_2$. Hence any critical algebra of $\mathrm{QVar}\{\mathbf{M}_2\}$ either has only two elements or is isomorphic to $\mathbf{M}_2$. Thus any set of non-isomorphic critical semigroups from $\mathrm{QVar}\{\mathbf{M}_2\}$ has at most two elements and hence, by Theorem 1.2, $\mathrm{QVar}\{\mathbf{M}_2\}$ is not Q-universal. Since for any semigroup $\mathbf{S}\in\mathrm{QVar}\{\mathbf{M}_2\}$ either $\mathbf{S}\in\mathbb{ZS}$ or $\mathbf{M}_2$ is a quotient semigroup of $\mathbf{S}$, we conclude that any semigroup $\mathbf{S}\in\mathrm{QVar}\{\mathbf{M}_2\}\setminus\mathbb{ZS}$ with at least five elements has two distinct endomorphisms f of $\mathbf{S}$ such that the subsemigroup $\mathrm{Im}(f)$ of $\mathbf{S}$ generates the quasivariety $\mathrm{QVar}\{\mathbf{M}_2\}$. If $\mathrm{QVar}\{\mathbf{M}_2\}$ is $\mathbb{V}$-relatively alg-universal for some proper subquasivariety $\mathbb{V}$ of $\mathrm{QVar}\{\mathbf{M}_2\}$, then there exists a proper class $\mathcal{C}$ of nonisomorphic semigroups from $\mathrm{QVar}\{\mathbf{M}_2\}$ such that any endomorphism of any semigroup from $\mathcal{C}$ whose image-subsemigroup does not belong to $\mathbb{V}$ is the identity mapping. Since $\mathbb{ZS}$ is not quasivar-relatively alg-universal and since $\mathbb{ZS}$ is the greatest proper subquasivariety of $\mathrm{QVar}\{\mathbf{M}_2\}$, we conclude that $\mathrm{QVar}\{\mathbf{M}_2\}$ is not quasivar-relatively alg-universal.

The proof for $\mathrm{QVar}\{\mathbf{M}_3\}$ is similar. Let $\mathbf{S}\in\mathrm{Var}\{\mathbf{M}_3\}$ be a semigroup. Then either $\mathbf{S}\in\mathbb{ILZ}$ or $\mathbf{M}_3$ is isomorphic to a subsemigroup of $\mathbf{S}$. Thus any critical semigroup in $\mathrm{QVar}\{\mathbf{M}_3\}$ either has two element or is isomorphic to $\mathbf{M}_3$. Therefore, by Theorem 1.2, $\mathrm{QVar}\{\mathbf{M}_3\}$ is not Q-universal because there exist at most three nonisomorphic critical semigroups in $\mathrm{QVar}\{\mathbf{M}_3\}$. If $\mathbf{S}\in\mathrm{QVar}\{\mathbf{M}_3\}$ then either $\mathbf{S}\in\mathbb{ILZ}$ or $\mathbf{M}_3$ is a quotient semigroup of $\mathbf{S}$. Thus any semigroup $\mathbf{S}\in\mathrm{QVar}\{\mathbf{M}_3\}\setminus\mathbb{IZS}$ with at least five elements has two distinct endomorphisms f of $\mathbf{S}$ such that the subsemigroup $\mathrm{Im}(f)$ of $\mathbf{S}$ generates the quasivariety $\mathrm{QVar}\{\mathbf{M}_3\}$. Hence $\mathrm{QVar}\{\mathbf{M}_3\}$ is not quasivar-relatively alg-universal because $\mathbb{ILZ}$ is the greatest proper subquasivariety of $\mathrm{QVar}\{\mathbf{M}_3\}$ and it is not quasivar-relatively alg-universal.

The proof for $(\mathbf{M}_3)^o$ follows by symmetry. $\square$

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