Extending the structural homomorphism of LCC loops

Piroska Csörgő

Abstract. A loop Q is said to be left conjugacy closed if the set $A = \{L_x/x \in Q\}$ is closed under conjugation. Let Q be an LCC loop, let $\mathcal{L}$ and $\mathcal{R}$ be the left and right multiplication groups of Q respectively, and let $I(Q)$ be its inner mapping group, $M(Q)$ its multiplication group. By Drápal's theorem [3, Theorem 2.8] there exists a homomorphism $\Lambda : \mathcal{L} \to I(Q)$ determined by $L_x \to R_x^{-1}L_x$. In this short note we examine different possible extensions of this Λ and the uniqueness of these extensions.

Keywords: LCC loop, multiplication group, inner mapping group, homomorphism

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1. Introduction

Q is a loop if it is a quasigroup with neutral element. The functions $L_a(x) = ax$ (left translation) and $R_a(x) = xa$ (right translation) are permutations on the elements of Q for every $a \in Q$. The permutation group generated by left and right translations $M(Q) = \langle L_a, R_a \ / \ a \in Q\rangle$ is called the multiplication group of the loop Q. Denote $I(Q)$ the stabilizer of the neutral element in $M(Q)$. $I(Q)$ is a subgroup of $M(Q)$ and it is called the inner mapping group of Q. It is clear that $M(Q)$ is a transitive permutation group on the loop Q. Denote $A = \{L_a \ / \ a \in Q\}$ and $B = \{R_a \ / \ a \in Q\}$. It is well known that A and B are left (and right) transversals to $I(Q)$ in $M(Q)$ which satisfy $\langle A, B\rangle = M(Q)$, and the commutator subgroup $[A, B] \leq I(Q)$. Furthermore $\mathrm{core}_{M(Q)}\, I(Q) = 1$ ($\mathrm{core}_{M(Q)}\, I(Q)$ means the largest normal subgroup of $M(Q)$ in $I(Q)$).

The subgroups $\mathcal{L} = \langle L_a \ / \ a \in Q\rangle$ and $\mathcal{R} = \langle R_a \ / \ a \in Q\rangle$ are called left and right multiplication groups, respectively. Denote $\mathcal{L}_1 = \mathcal{L} \cap I(Q)$, $\mathcal{R}_1 = \mathcal{R} \cap I(Q)$ and $T_x = L_x^{-1}R_x$. A standard fact is that $I(Q)$ is generated by $\mathcal{L}_1 \cup \mathcal{R}_1 \cup \{T_x \ / \ x \in Q\}$.

The right nucleus of a loop Q is

$$N_\rho = \{a \in Q \ / \ (xy)a = x(ya) \ \text{ for every } \ x, y \in Q\},$$

the left nucleus of a loop Q is

$$N_\lambda = \{a \in Q \ / \ a(xy) = (ax)y \ \text{ for every } \ x, y \in Q\}.$$

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We have (see [3, Lemma 1.9])

$$C_{M(Q)}(\mathcal{R}) = \{L_a \;/\; a \in N_\lambda\} \text{ and}$$
$$C_{M(Q)}(\mathcal{L}) = \{R_a \;/\; a \in N_\rho\}.$$

A subset A of a group G is said to be closed under conjugation if $a_1^{a_2} \in A$ for all $a_1, a_2 \in A$. This is clearly true if and only if A is a normal subset in $\langle A \rangle$.

A loop Q is said to be conjugacy closed (CC) if the sets $A = \{L_x \;/\; x \in Q\}$ and $B = \{R_x \;/\; x \in Q\}$ are closed under conjugation. The concept of conjugacy closedness was introduced first by Soikis [7] and later independently by Goodaire and Robinson [4].

A loop Q is called left conjugacy closed (LCC) if the set $A = \{L_x \;/\; x \in Q\}$ is closed under conjugation. Thus for all $a, b \in Q$ there exists $c \in Q$ such that $L_a L_b L_a^{-1} = L_c$. Hence in every LCC loop Q we have $L_a L_b L_a^{-1} = L_{T_a(b)}$ for all $a, b \in Q$.

LCC loops were also introduced by Soikis [7]. We have to mention Basarab's paper [1], A. Drápal's paper [3] and P. Nagy and K. Strambach's paper [6]. This latter paper is in connection with geometry of LCC loops. As a Bol loop Q is LCC if and only if $x^2 \in N_\lambda$ for all $x \in Q$ we have to emphasize the relevance of the paper of G.P. Nagy with H. Kiechle [5].

A. Drápal studied in [2] the relationship within multiplication groups of conjugacy closed (CC) loops, and in his other paper [3] concerning LCC loops he could transfer some basic facts from CC loops to LCC loops. The following basic result — which has been used in proofs of many statements — can also be found in this latter paper [3]. This result was first obtained for CC loops in Drápal's earlier paper [2].

Theorem 1.1. *Let Q be a left conjugacy closed loop. Denote by $\mathcal{L}$ its left multiplication group. Then there exists a unique homomorphism: $\Lambda : \mathcal{L} \to I(Q)$ that maps L_x to T_x for each $x \in Q$. This homomorphism is the identity on $\mathcal{L}_1$ and its kernel is equal to $Z(\mathcal{L}) = \{R_x \;/\; x \in Q\} \cap \mathcal{L}$; furthermore if $R_x \in Z(\mathcal{L})$, then $x \in Z(N_\rho)$.*

As the kernel of this homomorphism Λ does not contain the whole set $\{R_a \;/\; a \in N_\rho\}$ we cannot conclude — using this kernel — if the loop has non-trivial right nucleus. The purpose of this paper is to extend this homomorphism in such a way that the kernel consists of the set $\{R_a \;/\; a \in N_\rho\}$. Since $C_{M(Q)}(L) = \{R_a \;/\; a \in N_\rho\}$ it seems natural to examine the relationship between $\{g \in M(Q) \;/\; L_a^g \in A$ for every $L_a \in A\}$ and Λ. It turned out that we can really extend this Λ in the required way.

2. Extension

In this section, for the proofs of our theorems we need the following

Lemma 2.1. *Let Q be a loop, $M(Q)$ its multiplication group, $A = \{L_x \,/\, x \in Q\}$, $B = \{R_x \,/\, x \in Q\}$. Denote $H_0 = \{h \in I(Q) \ / \ A^h = A\}$. Then the following statements are true:*

(i) *$H_0 = I(Q) \cap \mathrm{Aut}(Q)$;*
(ii) *$B^h = B$ for some $h \in I(Q)$ if and only if $h \in H_0$.*

Proof: (i) First we show that if $h \in I(Q) \cap \mathrm{Aut}\, Q$, then $h \in H_0$ i.e. $hL_a h^{-1} \in A$ for every $L_a \in A$.

Let $a_0 \in Q$ be arbitrary, and denote $h(a) = a^*$. Then $(hL_a h^{-1})(a_0) = h(ah^{-1}(a_0)) = h(a)a_0 = a^* a_0 = L_{a^*}(a_0)$. Consequently $hL_a h^{-1} = L_{a^*}$.

Conversely, let $h \in H_0$. It suffices to prove $h(xy) = h(x)h(y)$ for arbitrary $x, y \in Q$. Suppose $hL_x h^{-1} = L_{x_1}$, $hL_y h^{-1} = L_{y_1}$. Then

$h(x) = h(x \cdot 1) = (hL_x)(1) = (hL_x h^{-1})(1) = L_{x_1}(1) = x_1$ and
$h(y) = h(y \cdot 1) = (hL_y)(1) = (hL_y h^{-1})(1) = L_{y_1}(1) = y_1$,

further $h(xy) = h(xy \cdot 1) = (hL_x L_y)(1) = (hL_x L_y h^{-1})(1) = L_{x_1} L_{y_1}(1) = x_1 y_1$.

(ii) By (i) it is obvious. □

Lemma 2.2. *Let Q be an LCC loop, $M(Q)$ its multiplication group, $I(Q)$ its inner mapping group, $A = \{L_a \ / \ a \in Q\}$. Let Λ be the homomorphism from Theorem 1.1. Suppose $h \in I(Q) \cap \mathrm{Aut}\, Q$. Then $(\Lambda(L_a))^h = \Lambda(L_a^h)$ for every $L_a \in A$.*

Proof: We have $L_a^h = L_{h^{-1}(a)}$, $T_a^h = T_{h^{-1}(a)}$. Using $\Lambda(L_a) = T_a$ we get our statement. □

We observe that for every element g of $M(Q)$ obviously both $(R_{g(1)})^{-1} g$ and $(L_{g(1)})^{-1} g$ belong to $I(Q)$.

In Drápal's theorem (Theorem 1.1) this homomorphism Λ maps L_x to $T_x = R_x^{-1} L_x$ and it is the identity on $\mathcal{L}_1$ $(= \mathcal{L} \cap I(Q))$. Consequently, $\mathrm{Im}\, \Lambda = \langle T_x \ / \ x \in Q \rangle$. As we have $\mathcal{L} = A\mathcal{L}_1$ and $A \cap \mathcal{L}_1 = \{e\}$, this Λ maps $\mathcal{L}$ into $I(Q)$ in such a way that $\Lambda(\ell) = R_{\ell(1)}^{-1} \ell$ for every $\ell \in \mathcal{L}$.

Extend this function Λ to the whole $M(Q)$. Thus we consider the function Λ_0 which maps $g \in M(Q)$ to $R_{g(1)}^{-1} g \in I(Q)$.

Since Λ_0 is a homomorphism on $\mathcal{L}$, the question arises: which is the largest subgroup of $M(Q)$ such that Λ_0 is a homomorphism on this subgroup. The following theorem gives the answer.

Theorem 2.3. *Let Q be an LCC loop. Let $\Lambda_0 : M(Q) \to I(Q)$ be such that $\Lambda_0(g) = R_{g(1)}^{-1} g$. Then the largest subgroup $\mathcal{L}^*$ of $M(Q)$ such that the restriction of Λ_0 on $\mathcal{L}^*$ is a homomorphism, is the following:*

$$\mathcal{L}^* = \{g \in M(Q) \ / \ L_x^g \in A \ \textit{ for every } \ L_x \in A\},$$
$$\mathcal{L}^* \cap I(Q) = I(Q) \cap \mathrm{Aut}\, Q \quad \textit{and} \quad \mathcal{L}^* = \mathcal{L}(I(Q) \cap \mathrm{Aut}\, Q).$$

Denote Λ^* *the restriction of* Λ_0 *on* $\mathcal{L}^*$. *Then* Λ^* *is the identity on* $\mathcal{L}^* \cap I(Q)$ *and* $\operatorname{Ker}\Lambda^* = \{R_x \ / \ x \in Q\} \cap \mathcal{L}^* = \{R_a \ / \ a \in N_\rho\}$. *Furthermore* $\operatorname{Im}\Lambda^*$ *is generated by* $(\mathcal{L}^* \cap I(Q)) \cup \{T_x \ / \ x \in Q\}$.

Proof: The left conjugacy closedness implies $\mathcal{L}^* \geq \mathcal{L}$. Denote $\mathcal{U} = \{g \in M(Q) \ / \ L_x^g \in A$ for every $L_x \in A\}$. Clearly $\mathcal{U}$ is a subgroup of $M(Q)$.

First we show $\mathcal{L}^* \leq \mathcal{U}$. Since $B \cap I(Q) = \{e\}$ obviously Λ_0 is the identity on $I(Q)$. Let $h \in \mathcal{L}^* \cap I(Q)$, $L_a \in A$. Then $hL_a \in \mathcal{L}^*$ and $\Lambda_0(hL_a) = \Lambda_0(h)\Lambda_0(L_a) = hR_a^{-1}L_a$. On the other hand, $\Lambda_0(hL_a) = R_c^{-1}hL_a$ where $R_c^{-1}hL_a \in I(Q)$. Hence $R_a^{h^{-1}} = R_c$ for every $a \in Q$, $h \in \mathcal{L}^* \cap I(Q)$, consequently $B^h = B$. Using Lemma 2.1 we obtain $\mathcal{L}^* \cap I(Q) \leq \mathcal{U}$. The left conjugacy closedness implies $\mathcal{L} \leq \mathcal{U}$.

We show Λ_0 is a homomorphism on $\mathcal{U}$. Let $\ell_1, \ell_2 \in \mathcal{U}$, clearly $\ell_1 = L_{a_1}h_1$, $\ell_2 = L_{a_2}h_2$ with $h_1, h_2 \in \mathcal{U} \cap I(Q)$. We prove $\Lambda_0(L_{a_1}h_1L_{a_2}h_2) = \Lambda_0(L_{a_1}h_1)\Lambda_0(L_{a_2}h_2)$. Clearly $\Lambda_0(L_{a_1}h_1) = R_{a_1}^{-1}L_{a_1}h_1 = \Lambda(L_{a_1})h_1$, $\Lambda_0(L_{a_2}h_2) = R_{a_2}^{-1}L_{a_2}h_2 = \Lambda(L_{a_2})h_2$. On the other hand, $\Lambda_0(L_{a_1}h_1L_{a_2}h_2) = \Lambda_0(L_{a_1}L_{a_2}^{h_1^{-1}}h_1h_2) = R_d^{-1}L_{a_1}L_{a_2}^{h_1^{-1}}h_1h_2$ for some $d \in Q$. By the definition of $\mathcal{U}$, $L_{a_2}^{h_1^{-1}} \in A$, whence $L_{a_1}L_{a_2}^{h_1^{-1}} \in \mathcal{L}$, consequently $\Lambda(L_{a_1}L_{a_2}^{h_1^{-1}}) = R_d^{-1}L_{a_1}L_{a_2}^{h_1^{-1}}$, and using that Λ is a homomorphism on $\mathcal{L}$ we get $\Lambda_0(L_{a_1}h_1L_{a_2}h_2) = \Lambda(L_{a_1})\Lambda(L_{a_2}^{h_1^{-1}})h_1h_2$. Thus it suffices to show $\Lambda(L_{a_2}^{h_1^{-1}}) = (\Lambda(L_{a_2}))^{h_1^{-1}}$, but this follows immediately from Lemma 2.2.

Lemma 2.1 implies $\mathcal{L}^* \cap I(Q) = I(Q) \cap \operatorname{Aut} Q$. Since $\mathcal{L} \leq \mathcal{L}^*$, we have $\mathcal{L}^* = \mathcal{L}(I(Q) \cap \operatorname{Aut} Q)$.

As $\operatorname{Im}\mathcal{L} = \langle T_x \ / \ x \in Q\rangle$ and Λ^* is the identity on $\mathcal{L}^* \cap I(Q)$ it follows $\operatorname{Im}\Lambda^*$ is generated by $\mathcal{L}^* \cap I(Q)$ and $\{T_x \ / \ x \in Q\}$. We have $\operatorname{Ker}\Lambda^* = \{f \in \mathcal{L}^* \ / \ \Lambda^*(f) = R_{f(1)}^{-1}f = e\}$. Hence $f = R_{f(1)} \in B \cap \mathcal{L}^*$. From $[A, B] \leq I(Q)$ we get $L_x^{-1}L_x^{R_{f(1)}} \in I(Q)$ for every $L_x \in A$, but $R_{f(1)} \in \mathcal{L}^*$ implies $L_x^{R_{f(1)}} \in A$, consequently $R_{f(1)} \in C_{M(Q)}(A)$, whence $R_{f(1)} \in C_{M(Q)}(\mathcal{L})$. Since $C_{M(Q)}(\mathcal{L}) = \{R_a \ / \ a \in N_\rho\}$, we concluce $\operatorname{Ker}\Lambda^* = B \cap \mathcal{L}^* = \{R_a \ / \ a \in N_\rho\}$. □

Our next question is, whether every homomorphism of $\mathcal{L}^*$ into $I(Q)$ which coincides with Λ on the elements of $\mathcal{L}$ is equal to Λ^*.

We give the answer:

Theorem 2.4. *Let* Q *be an LCC loop,* $M(Q)$ *its multiplication group,* $I(Q)$ *its inner mapping group,* $A = \{L_a \ / \ a \in Q\}$, $\mathcal{L}^* = \{g \in M(Q) \ / \ L_x^g \in A$ *for every* $L_x \in A\}$.

Let $T = \langle T_x \ / \ x \in Q\rangle$, $c \in C_{I(Q)}(T)$. *Define a function* Λ_1 *on* $\mathcal{L}^*$: *if* $\ell = L_ah \in \mathcal{L}^*$ *with* $h \in \mathcal{L}^* \cap I(Q)$, *then* $\Lambda_1(\ell) = \Lambda(L_a)h^c$. *Then* Λ_1 *is a homomorphism on* $\mathcal{L}^*$ *and* Λ_1 *coincides with* Λ *on the elements of* $\mathcal{L}$.

Proof: We have $\operatorname{Im}\Lambda = \langle T_x \ / \ x \in Q\rangle$ and $\mathcal{L}_1 \leq \operatorname{Im}\Lambda$, whence $c \in C_G(\mathcal{L}_1)$, consequently Λ_1 coincides with Λ on the elements of $\mathcal{L}$.

We show Λ_1 is a homomorphism on $\mathcal{L}^*$. Let $\ell_1, \ell_2 \in \mathcal{L}^*$, $\ell_1 = L_{a_1}h_1$, $\ell_2 = L_{a_2}h_2$ with $h_1, h_2 \in \mathcal{L}^* \cap I(Q)$. By the definition of Λ_1, $\Lambda_1(\ell_1) = \Lambda(L_{a_1})h_1^c$, $\Lambda_1(\ell_2) = \Lambda(L_{a_2})h_2^c$. Clearly $\ell_1\ell_2 = L_{a_1}L_{a_2}^{h_1^{-1}}h_1h_2$, since $(L_{a_2})^{h_1^{-1}} \in A$ we have $L_{a_1}(L_{a_2})^{h_1^{-1}} \in \mathcal{L}$, whence $L_{a_1}(L_{a_2})^{h_1^{-1}} = L_{a_3}h_3$ with $h_3 \in \mathcal{L}_1$. Consequently $\Lambda_1(\ell_1\ell_2) = \Lambda(L_{a_3})(h_3h_1h_2)^c$. As $h_3 \in T$ it follows $\Lambda_1(\ell_1\ell_2) = \Lambda(L_{a_3})h_3(h_1h_2)^c = \Lambda(L_{a_3}h_3)(h_1h_2)^c = \Lambda(L_{a_1})\Lambda(L_{a_2})^{h_1^{-1}}(h_1h_2)^c$.

Thus it suffices to prove $(\Lambda(L_{a_2}))^{(h_1^{-1})^c} = \Lambda(L_{a_2}^{h_1^{-1}})$. Using $\operatorname{Im}\Lambda = T$ and $c \in C_{I(Q)}(T)$ it is equivalent to $(\Lambda(L_{a_2}))^{h_1^{-1}} = \Lambda(L_{a_2}^{h_1^{-1}})$. The latter equality follows immediately from Lemma 2.2. □

Corollary 2.5. *Let Q be an LCC loop. Denote $T = \langle T_x \ / \ x \in Q\rangle$ and suppose that $C_{I(Q)}(T) \not\leq C_{I(Q)}(\mathcal{L}^* \cap I(Q))$. Then there exists a homomorphism Λ_1 on $\mathcal{L}^*$ which coincides with Λ on $\mathcal{L}$, but $\Lambda_1 \neq \Lambda^*$, where Λ^* is the homomorphism described in Theorem* 2.3.

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Department of Algebra and Number Theory, Eötvös University,
Pázmány P. sétány 1/C, H–1117 Budapest, Hungary

E-mail: ska@cs.elte.hu

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