Two improvements on Tkačenko's addition theorem

J. Gerlits, I. Juhász, Z. Szentmiklóssy

Abstract. We prove that (A) if a countably compact space is the union of countably many D subspaces then it is compact; (B) if a compact T_2 space is the union of fewer than $N(\mathbb{R}) = \mathrm{cov}(\mathcal{M})$ left-separated subspaces then it is scattered. Both (A) and (B) improve results of Tkačenko from 1979; (A) also answers a question that was raised by Arhangel'skiĭ and improves a result of Gruenhage.

Keywords: D-space, left separated, compact, countably compact, scattered space, Novák number

Classification: 54D55, 54G12, 54A25

1. Introduction

We start by recalling a few well-known definitions and introducing some related notation. A space X is said to be a D-space if for any neighbourhood assignment ϕ defined on X there is a closed discrete set $D \subset X$ such that $\phi[D] = \bigcup\{\phi(x) : x \in D\} = X$. For any space X we set

$$D(X) = \min\{|\mathcal{A}| : X = \bigcup \mathcal{A} \text{ and } A \text{ is a } D\text{-space for each } A \in \mathcal{A}\}.$$

The space X is called left-separated if there is a well-ordering $\prec$ on X such that all initial segments w.r.t. $\prec$ are closed in X. Again, we set, for any space X,

$$\mathrm{ls}\,(X) = \min\{|\mathcal{A}| : X = \bigcup \mathcal{A} \text{ and } A \text{ is left-separated for each } A \in \mathcal{A}\}.$$

(Note that both $D(X)$ and $\mathrm{ls}\,(X)$ can be finite.)

It was shown in [7] that left-separated spaces are D-spaces, hence we have $D(X) \le \mathrm{ls}\,(X)$ for any X.

In [6], M. Tkačenko proved the following remarkable result: *If X is a countably compact T_3-space with* $\mathrm{ls}\,(X) \le \omega$ *then*

(i) *X is compact,*
(ii) *X is scattered,*
(iii) *X is sequential.*

Research supported by OTKA grant no. 37758.

It is easy to see that if in a scattered compact T_2-space any countably compact subspace is compact then it is sequential, hence (iii) immediately follows from (i) and (ii), although this is not how (iii) was proved in [6].

The aim of this note is to improve (i) and (ii) as follows.

(A) *Any countably compact space X with $D(X) \leq \omega$ is compact.*

(B) *If X is compact T_2 with* $\mathrm{ls}\,(X) < N(\mathbb{R})$ *then X is scattered.*

Here $N(\mathbb{R})$ denotes the Novák number of the real line $\mathbb{R}$, i.e. the covering number $\mathrm{cov}(\mathcal{M})$ of the ideal $\mathcal{M}$ of all meager subsets of $\mathbb{R}$.

If X is any crowded (i.e. dense-in-itself) space and $Y \subset X$ then we denote by $N(Y, X)$ the relative Novák number of Y in X, that is the smallest number of nowhere dense subsets of X needed to cover Y. In particular, $N(X) = N(X, X)$ is the Novák number of X.

We should also mention that a weaker version of statement (A), in which $D(X) < \omega$ is assumed instead of $D(X) \leq \omega$, has been established in [3].

2. The results

Similarly as in [6], we can actually prove the following higher-cardinal generalization of statement (A) from the introduction.

Theorem 2.1. *Let κ be any infinite cardinal and X be initially κ-compact with $D(X) \leq \kappa$. Then X is actually compact.*

The proof of Theorem 2.1 is based on the following lemma that may have some independent interest in itself.

Lemma 2.2. *Let X be any space and $Y \subset X$ its D subspace. If ρ is a regular cardinal such that X has no closed discrete subset of size ρ (i.e. $\hat{e}(X) \leq \rho$), moreover $\mathcal{U} = \{U_\alpha : \alpha \in \rho\}$ is a strictly increasing open cover of X then there is a closed set $Z \subset X$ such that $Z \cap Y = \emptyset$ and $Z \not\subset U_\alpha$ for all $\alpha \in \rho$.*

Proof: If there is an $\alpha \in \rho$ with $Y \subset U_\alpha$ then $Z = X - U_\alpha$ is clearly as required. So assume from here on that $Y \not\subset U_\alpha$ for all $\alpha \in \rho$.

For every point $y \in Y$ let $\alpha(y)$ be the *minimal* ordinal α such that $y \in U_\alpha$ and then consider the neighbourhood assignment ϕ on Y defined by

$$\phi(y) = U_{\alpha(y)}.$$

Since Y is a D-space there is a set $E \subset Y$, closed and discrete in Y, such that $Y \subset \phi[E]$. We claim that $Z = E'$, the derived set of E, is now as required.

Indeed, Z is closed in X and $Z \cap Y = \emptyset$ as E has no limit point within Y. It remains to show that $Z \not\subset U_\alpha$ for all $\alpha \in \rho$. Assume, indirectly, that $Z \subset U_\alpha$ for some $\alpha \in \rho$. Note first that for any point $y \in Y \cap U_\alpha$ we have $\alpha(y) \leq \alpha$, consequently $\phi[E \cap U_\alpha] \subset U_\alpha$. On the other hand, $Z = E' \subset U_\alpha$ implies that $E - U_\alpha$ is closed discrete in X, hence $|E - U_\alpha| < \rho$ by our assumption. But then

$$\beta = \sup\{\alpha(y) : y \in E - U_\alpha\} < \rho$$

because ρ is regular, consequently we have

$$Y \subset \phi[E] = \phi[E \cap U_\alpha] \cup \phi[E - U_\alpha] \subset U_\alpha \cup U_\beta = U_{\max\{\alpha,\beta\}},$$

contradicting that no member of $\mathcal{U}$ covers Y. □

Now, we can turn to the proof of our theorem.

Proof of Theorem 2.1: It suffices to prove that for no regular cardinal ρ is there a strictly increasing open cover of X of the form $\mathcal{U} = \{U_\alpha : \alpha \in \rho\}$. For $\rho \leq \kappa$ this is clear, for X is initially κ-compact. So assume now that $\rho > \kappa$, and assume indirectly that $\mathcal{U} = \{U_\alpha : \alpha \in \rho\}$ is a strictly increasing open cover of X. Note also that X has no closed discrete subset of size $\rho > \kappa$ because X is initially κ-compact.

By $D(X) \leq \kappa$ we have $X = \bigcup\{Y_\nu : \nu \in \kappa\}$, where Y_ν is a D subspace of X for each $\nu \in \kappa$. Using Lemma 2.2 then we may define by a straightforward transfinite recursion on $\nu \in \kappa$ closed sets $Z_\nu \subset X$ such that for each $\nu \in \kappa$ we have $Z_\nu \cap Y_\nu = \emptyset$, $Z_\nu \not\subset U_\alpha$ for all $\alpha \in \rho$, moreover $\nu_1 < \nu_2$ implies $Z_{\nu_1} \supset Z_{\nu_2}$. In this we make use of the fact that if $\nu < \kappa$ and $\{Z_\eta : \eta \in \nu\}$ is a decreasing sequence of closed sets in X such that $\bigcap\{Z_\eta : \eta \in \nu\} \subset U$ for some open $U \subset X$ then there is an $\eta \in \nu$ with $Z_\eta \subset U$ as well, using again the initial κ-compactness of X.

But then, applying once more that X is initially κ-compact, we conclude that

$$\bigcap\{Z_\nu : \nu \in \kappa\} \neq \emptyset,$$

contradicting that $X = \bigcup\{Y_\nu : \nu \in \kappa\}$. □

It should be noted that in the above result no separation axiom is needed. This is in contrast with Tkačenko's result from [6].

Let us now turn to our second statement (B). Again, we need to first give a preparatory result. For this we recall the cardinal function $\delta(X)$ that was introduced in [8]:

$$\delta(X) = \sup\{d(S) : \ S \ \text{ is dense in } X\}.$$

Let us note here that if X is a compact T_2-space then $\delta(X) = \pi(X)$, as was shown in [4].

Lemma 2.3. *Assume that X is an arbitrary crowded topological space and $Y \subset X$ is its left-separated subspace. Then we have*

$$N(Y, X) \leq \delta(X),$$

consequently

$$N(X) \leq \mathrm{ls}\,(X) \cdot \delta(X).$$

Proof: We shall prove $N(Y, X) \leq \delta(X)$ by transfinite induction on the order type of the well-ordering that left-separates Y. So assume that $\prec$ is a left-separating well-ordering of Y such that if Z is any proper initial segment of Y, w.r.t. $\prec$, then $N(Z, X) \leq \delta(X)$.

Let G be the union of all those open sets U in X for which Y (or more precisely: $U \cap Y$) is dense in U. Clearly, then $Y \setminus G$ is nowhere dense in X and $Y \cap G$ is dense in G. The latter then implies

$$d(Y \cap G) \leq \delta(G) \leq \delta(X).$$

On the other hand, since $\prec$ left-separates $Y \cap G$, any dense subset of $Y \cap G$ must be cofinal in $Y \cap G$ w.r.t. $\prec$, hence we clearly have

$$\mathrm{cf}\,(Y \cap G, \prec) \leq d(Y \cap G) \leq \delta(X).$$

But any proper $\prec$-initial segment of $Y \cap G$ may be covered by $\delta(X)$ many nowhere dense sets, by the inductive hypothesis, hence we have

$$N(Y, X) \leq 1 + \delta(X) \cdot \delta(X) = \delta(X),$$

because $d(X)$ and so $\delta(X)$ is always infinite by definition. The second part now follows immediately. □

Note that again absolutely no separation axiom was needed in the above result. However, in the proof of the following theorem the assumption of Hausdorffness is essential.

Theorem 2.4. *Let X be a compact T_2-space satisfying $\mathrm{ls}\,(X) < N(\mathbb{R})$. Then X must be scattered.*

Proof: We actually prove the contrapositive form of this statement. So assume that X is not scattered, then it is well-known that some closed subspace $F \subset X$ admits an irreducible continuous closed map $f : F \to \mathbb{C}$ onto the Cantor set $\mathbb{C}$.

It is also well-known and easy to check that then we have $\delta(F) = \delta(\mathbb{C}) = \omega$, moreover $N(F) = N(\mathbb{C}) = N(\mathbb{R}) > \omega$. But then from Lemma 2.3 we conclude that

$$\mathrm{ls}\,(X) \geq \mathrm{ls}\,(F) = \mathrm{ls}\,(F) \cdot \omega \geq N(F) = N(\mathbb{R}).$$

□

We would like to mention that 2.3 and 2.4 were motivated by the treatment of Tkačenko's results given in [5]. We also point out that Theorems 2.1 and 2.4 yield a slight strengthening of Tkačenko's theorem in that the T_3 separation axiom may be replaced by T_2 in it. This is new even in the case of left-separated spaces (i.e. the assumption $\mathrm{ls}\,(X) = 1$) that preceded Tkačenko's result in [2].

Corollary 2.5. *Let X be a countably compact T_2 space that satisfies $\mathrm{ls}\,(X) \leq \omega$. Then X is compact, scattered, and sequential.*

We finish by formulating a couple of natural problems concerning our results.

Problem 2.6. *Is the upper bound $N(\mathbb{R})$ in Theorem 2.4 sharp? Can it actually be replaced by the cardinality of the continuum (in ZFC, of course)?*

Note that as metric or compact spaces are all D-spaces, in Theorem 2.4 one clearly cannot replace $\mathrm{ls}\,(X)$ with $D(X)$. Also, a compact (D-)space may fail to be sequential. Being left-separated, however, is clearly a hereditary property, hence left-separated spaces are actually hereditary D-spaces. Thus the following problems may be raised.

Problem 2.7. *Is a compact T_2 hereditary D-space sequential? Does it contain a point of countable character?*

Concerning this problem we note that it follows easily from Theorem 2.1 that a compact T_2-space X satisfying $D(Y) \leq \omega$ for all $Y \subset X$ has countable tightness.

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Alfréd Rényi Institute of Mathematics, P.O.Box 127, 1364 Budapest, Hungary

E-mail: gerlits@renyi.hu

Alfréd Rényi Institute of Mathematics, P.O.Box 127, 1364 Budapest, Hungary

E-mail: juhasz@renyi.hu

Eötvös Loránt University, Department of Analysis, 1117 Budapest, Pázmány Péter sétány 1/A, Hungary

E-mail: zoli@renyi.hu

(*Received* April 7, 2005, *revised* April 7, 2005)