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Topological structure of the space of lower semi-continuous functions

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Abstract. Let $\mathrm{L}(X)$ be the space of all lower semi-continuous extended real-valued functions on a Hausdorff space X, where, by identifying each f with the epi-graph $\mathrm{epi}(f)$, $\mathrm{L}(X)$ is regarded the subspace of the space $\mathrm{Cld}^*_F(X \times \mathbb{R})$ of all closed sets in $X \times \mathbb{R}$ with the Fell topology. Let

$$\mathrm{LSC}(X) = \{f \in \mathrm{L}(X) \mid f(X) \cap \mathbb{R} \neq \emptyset,\ f(X) \subset (-\infty, \infty]\} \text{ and}$$
$$\mathrm{LSC_B}(X) = \{f \in \mathrm{L}(X) \mid f(X) \text{ is a bounded subset of } \mathbb{R}\}.$$

We show that $\mathrm{L}(X)$ is homeomorphic to the Hilbert cube $Q = [-1,1]^{\mathbb{N}}$ if and only if X is second countable, locally compact and infinite. In this case, it is proved that $(\mathrm{L}(X), \mathrm{LSC}(X), \mathrm{LSC_B}(X))$ is homeomorphic to $(\mathrm{Cone}\,Q, Q \times (0,1), \Sigma \times (0,1))$ (resp. (Q, s, Σ)) if X is compact (resp. X is non-compact), where $\mathrm{Cone}\,Q = (Q \times \mathbf{I})/(Q \times \{1\})$ is the cone over Q, $s = (-1,1)^{\mathbb{N}}$ is the pseudo-interior, $\Sigma = \{(x_i)_{i\in\mathbb{N}} \in Q \mid \sup_{i\in\mathbb{N}} |x_i| < 1\}$ is the radial-interior.

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Classification: 57N20, 54C35

1. Introduction

The set of all closed sets in a (topological) space X is denoted by $\mathrm{Cld}^*(X)$ and let $\mathrm{Cld}(X) = \mathrm{Cld}^*(X) \setminus \{\emptyset\}$. For each $U \subset X$, we denote

$$U^- = \{A \in \mathrm{Cld}^*(X) \mid A \cap U \neq \emptyset\} \text{ and}$$
$$U^+ = \{A \in \mathrm{Cld}^*(X) \mid A \subset U\}.$$

The *Fell topology* on $\mathrm{Cld}^*(X)$ is the topology generated by

$$\{U^- \mid U \subset X \text{ is open}\} \cup \{(X \setminus K)^+ \mid K \subset X \text{ is compact}\}.$$

By $\mathrm{Cld}^*_F(X)$ (or $\mathrm{Cld}_F(X)$), we denote the space $\mathrm{Cld}^*(X)$ (or $\mathrm{Cld}(X)$) with the Fell topology.[1] In the paper [9], it is proved that $\mathrm{Cld}^*_F(X)$ (resp. $\mathrm{Cld}_F(X)$) is

[1]Note that the hyperspace $\mathrm{Cld}_V(X)$ with the Vietoris topology is metrizable if and only if X is compact metrizable. On the other hand, $\mathrm{Cld}^*_F(X)$ (or $\mathrm{Cld}_F(X)$) is metrizable if and only if X is locally compact and separable metrizable [2, Theorem 5.1.5].

homeomorphic to ($\approx$) the Hilbert cube $Q = [-1,1]^{\mathbb{N}}$ (resp. $Q \setminus \{0\}$) if and only if X is a locally compact, locally connected, separable metrizable space which has no compact components.

By $[-\infty,\infty]$, we denote the extended real line. For an extended real-valued function $f : X \to [-\infty,\infty]$, let

$$\operatorname{epi}(f) = \{(x,t) \in X \times \mathbb{R} \mid t \geq f(x)\},$$

which is called the *epi-graph* of f. Note that

- f is lower semi-continuous if and only if $\operatorname{epi}(f)$ is closed in $X \times \mathbb{R}$,

whence f can be regarded as a lower semi-continuous real-valued function defined on the set $f^{-1}(\mathbb{R}) \subset X$.

Let $\mathrm{L}(X)$ be the space of all lower semi-continuous extended real-valued functions on X, where, by identifying each f with $\operatorname{epi}(f)$, $\mathrm{L}(X)$ is considered the subspace of the space $\mathrm{Cld}^*_F(X \times \mathbb{R})$. In this paper, we show the following:

Theorem 1.1. *For a Hausdorff space X, $\mathrm{L}(X) \approx Q$ if and only if X is locally compact, second countable and infinite.*

In this paper, we also study the following subspaces:

$$\begin{aligned}\mathrm{LSC}(X) &= \{f \in \mathrm{L}(X) \mid f(X) \cap \mathbb{R} \neq \emptyset,\ f(X) \subset (-\infty,\infty]\};\\ \mathrm{LSC_B}(X) &= \{f \in \mathrm{L}(X) \mid f(X) \text{ is a bounded subset of } \mathbb{R}\}.\end{aligned}$$

Observe that $\mathrm{L}(X) \supset \mathrm{LSC}(X) \supset \mathrm{LSC_B}(X)$. Each $f \in \mathrm{LSC}(X)$ is called a *proper* lower semi-continuous extended real-valued function. Each $f \in \mathrm{LSC_B}(X)$ is a bounded lower semi-continuous real-valued function defined on the whole space X.

Let $\mathbf{I} = [0,1]$ be the closed unit interval. By $\operatorname{Cone} X$, we denote the *cone* over X which is the quotient space obtained from $X \times \mathbf{I}$ by shrinking $X \times \{1\}$ to a point $*$ (called the *vertex*), that is,

$$\operatorname{Cone} X = (X \times \mathbf{I})/(X \times \{1\}).$$

Throughout this paper, we use the homeomorphism $\theta : [-\infty,\infty] \to \mathbf{I}$ defined as follows:

$$\theta(-\infty) = 0,\ \theta(\infty) = 1 \ \text{ and } \ \theta(t) = \frac{1}{2}\left(\frac{t}{1+|t|} + 1\right).$$

Let Δ^n be the standard n-simplex and $\operatorname{rint}\Delta^n$ the radial interior of Δ^n, i.e.,

$$\begin{aligned}\Delta^n &= \big\{(t_1,\ldots,t_{n+1}) \in \mathbf{I}^{n+1} \mid \textstyle\sum_{i=1}^{n+1} t_i = 1\big\};\\ \operatorname{rint}\Delta^n &= \big\{(t_1,\ldots,t_{n+1}) \in \Delta^n \mid t_i > 0 \text{ for } i = 1,\ldots,n+1\big\}.\end{aligned}$$

In case X is finite, we can easily see that $\mathrm{L}(X) \approx \Delta^n \approx \operatorname{Cone}\Delta^{n-1}$, where $n = \operatorname{card} X$. Indeed, write $X = \{x_1, \dots, x_n\}$ and define $p : \mathrm{L}(X) \to \operatorname{Cone}\Delta^{n-1}$ as follows:

$$p(f) = \begin{cases} * \ \text{(the vertex of } \operatorname{Cone}\Delta^{n-1}) & \text{if } f = \emptyset,^{2} \\ \left(\dfrac{1-\theta(f(x_1))}{\sigma(f)}, \dots, \dfrac{1-\theta(f(x_n))}{\sigma(f)},\ \theta(\min f(X))\right) & \text{otherwise,} \end{cases}$$

where $\sigma(f) = \sum_{i=1}^{n}(1 - \theta(f(x_i)))$. Then, p is a homeomorphism such that

$$p(\mathrm{LSC}(X)) = \Delta^{n-1} \times (0,1) \quad \text{and} \quad p(\mathrm{LSC_B}(X)) = \operatorname{rint}\Delta^{n-1} \times (0,1).$$

Thus, we have the following:

Fact. *For a finite T_1-space X with* $\operatorname{card} X = n$,

$$\begin{aligned}&(\mathrm{L}(X),\, \mathrm{LSC}(X),\, \mathrm{LSC_B}(X)) \\ &\qquad\qquad \approx (\operatorname{Cone}\Delta^{n-1},\, \Delta^{n-1} \times (0,1),\, \operatorname{rint}\Delta^{n-1} \times (0,1)).\end{aligned}$$

In this paper, we generalize this fact into the case X is infinite. Let

$$s = (-1,1)^{\mathbb{N}} \quad \text{and} \quad \Sigma = \big\{(x_i)_{i\in\mathbb{N}} \in Q \mid \sup_{i\in\mathbb{N}} |x_i| < 1\big\},$$

which are called the *pseudo-interior* and the *radial interior* of Q, respectively. We prove the following two generalizations:

Theorem 1.2. *For a Hausdorff space X, the following are equivalent:*
(a) *X is second countable, compact and infinite;*
(b) $(\mathrm{L}(X), \mathrm{LSC}(X)) \approx (\operatorname{Cone} Q, Q \times (0,1))$*;*
(c) $(\mathrm{L}(X), \mathrm{LSC}(X), \mathrm{LSC_B}(X)) \approx (\operatorname{Cone} Q, Q \times (0,1), \Sigma \times (0,1))$.
In the above, the vertex $* \in \operatorname{Cone} Q$ corresponds to the function $\emptyset \in \mathrm{L}(X)$.

Theorem 1.3. *For a Hausdorff space X, the following are equivalent:*
(a) *X is second countable, locally compact and non-compact;*
(b) $(\mathrm{L}(X), \mathrm{LSC}(X)) \approx (Q, s)$*;*
(c) $(\mathrm{L}(X), \mathrm{LSC}(X), \mathrm{LSC_B}(X)) \approx (Q, s, \Sigma)$.

Remark. It should be remarked that

$$(Q, s, \Sigma) \approx (\operatorname{Cone} Q,\, s \times (0,1),\, \Sigma \times (0,1)).$$

One should also keep in mind that the complement $\mathrm{L}(X)\backslash\mathrm{LSC}(X)$ in Theorem 1.3 is connected, but the one in Theorem 1.2 has two components $\{\emptyset\}$ and $\{f \in \mathrm{L}(X) \mid -\infty \in f(X)\}$.

[2] Here, $f = \emptyset$ means that f is the constant function $x \mapsto \infty$.

2. Metrizability and closedness

The following follows from the result of Fell [5] (cf. [2, Theorem 5.1.3]):

Proposition 2.1. *For every Hausdorff space X, $\mathrm{Cld}_F^*(X \times \mathbb{R})$ is compact. If X is locally compact then $\mathrm{Cld}_F^*(X \times \mathbb{R})$ is a compact Hausdorff space.* □

Let $\mathrm{Cld}_F(X) = \mathrm{Cld}_F^*(X) \setminus \{\emptyset\}$. Then, the hyperspace $\mathrm{Cld}_F(X)$ can be regarded as a subspace of $\mathrm{LSC_B}(X)$ by the embedding $i : \mathrm{Cld}_F(X) \to \mathrm{LSC_B}(X)$ defined by

$$i(A)(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \notin A. \end{cases}$$

Moreover, by identifying $x \in X$ with $\{x\} \in \mathrm{Cld}_F(X)$, we can also regard X as a subspace of $\mathrm{Cld}_F(X)$. Since $\mathrm{Cld}_F^*(X \times \mathbb{R})$ (resp. $\mathrm{Cld}_F(X)$) is metrizable if and only if $X \times \mathbb{R}$ (resp. X) is locally compact and second countable by [2, Theorem 5.1.5], we have the following:

Proposition 2.2. *For a Hausdorff space X, the following are equivalent:*

(a) *X is locally compact and second countable;*
(b) *$\mathrm{Cld}_F^*(X \times \mathbb{R})$ is metrizable;*
(c) *$\mathrm{L}(X)$ is metrizable;*
(d) *$\mathrm{LSC}(X)$ is metrizable;*
(e) *$\mathrm{LSC_B}(X)$ is metrizable;*
(f) *$\mathrm{Cld}_F(X)$ is metrizable.* □

Proposition 2.3. *A Hausdorff space X is locally compact if and only if the space $\mathrm{L}(X)$ is closed in $\mathrm{Cld}_F^*(X \times \mathbb{R})$.*

Proof: To see the "only if" part, assume that X is locally compact. For each $A \in \mathrm{Cld}_F^*(X \times \mathbb{R}) \setminus \mathrm{L}(X)$, we have $x \in X$ and $r_1 < r_2 \in \mathbb{R}$ such that $(x, r_1) \in A$ and $(x, r_2) \notin A$. Choose an open neighborhood V of x in X and $\delta > 0$ so that $\mathrm{cl}\, V$ is compact and

$$\mathrm{cl}\, V \times (r_2 - \delta_1, r_2 + \delta) \subset (X \times \mathbb{R}) \setminus A.$$

Let $K = \mathrm{cl}\, V \times [r_2 - \delta, r_2 + \delta]$ and $U = V \times (-\infty, r_2 - \delta)$. Then,

$$A \in U^- \cap ((X \times \mathbb{R}) \setminus K)^+ \subset \mathrm{Cld}_F^*(X \times \mathbb{R}) \setminus \mathrm{L}(X).$$

Hence, $\mathrm{Cld}_F^*(X \times \mathbb{R}) \setminus \mathrm{L}(X)$ is open in $\mathrm{Cld}_F^*(X \times \mathbb{R})$, that is, $\mathrm{L}(X)$ is closed.

Now, to see the "if" part, assume that X is not locally compact, whence we have $x_0 \in X$ which has no compact neighborhoods in X. Let

$$A = (X \times [1, \infty)) \cup \{(x_0, 0)\} \in \mathrm{Cld}_F^*(X \times \mathbb{R}) \setminus L(X).$$

For each neighborhood W of A in $\mathrm{Cld}_F^*(X\times\mathbb{R})$, we can choose open sets $U_1,\dots,U_n \subset X\times\mathbb{R}$ and a compact set $K\subset X\times\mathbb{R}$ so that $(x_0,0)\in U_1$ and

$$A\in U_1^-\cap\dots\cap U_n^-\cap((X\times\mathbb{R})\setminus K)^+\subset W.$$

Since $\mathrm{pr}_X(K)$ is compact, $\mathrm{pr}_X(K)$ is not a neighborhood of x_0 in X, hence $\mathrm{pr}_X(U_1)\not\subset\mathrm{pr}_X(K)$. Thus, we have $x_1\in\mathrm{pr}_X(U_1)\setminus\mathrm{pr}_X(K)$. We define $g\in\mathrm{L}(X)$ by

$$g(x)=\begin{cases}0 & \text{if } x=x_1,\\ 1 & \text{if } x\neq x_1.\end{cases}$$

By identifying g with the epi-graph, we can write as follows:

$$g=(X\times[1,\infty))\cup(\{x_1\}\times[0,\infty)).$$

Then, it is easy to see that

$$g\in U_1^-\cap\dots\cap U_n^-\cap((X\times\mathbb{R})\setminus K)^+\subset W.$$

Hence, $W\cap\mathrm{L}(X)\neq\emptyset$. This means that $A\in\mathrm{cl}\,\mathrm{L}(X)$, that is, $\mathrm{L}(X)$ is not closed in $\mathrm{Cld}_F^*(X\times\mathbb{R})$. □

As corollaries of propositions above, we have the following:

Corollary 2.4. *A Hausdorff space X is locally compact if and only if the space $\mathrm{L}(X)$ is a compact Hausdorff space.* □

Corollary 2.5. *A Hausdorff space X is locally compact and second countable if and only if the space $\mathrm{L}(X)$ is a compact metrizable space.* □

We now consider the subspace:

$$\begin{aligned}\mathrm{L}_{-\infty}(X)&=\{f\in\mathrm{L}(X)\mid -\infty\in f(X)\}\\ &=\mathrm{L}(X)\setminus(\mathrm{LSC}(X)\cup\{\emptyset\})\subset\mathrm{L}(X).\end{aligned}$$

Lemma 2.6. *For a locally compact Hausdorff space X, $\mathrm{L}_{-\infty}(X)$ is compact if and only if X is compact.*

Proof: Assume that X is compact. For each $f\in\mathrm{L}(X)\setminus\mathrm{L}_{-\infty}(X)$, we have $b\in\mathbb{R}$ such that $f(X)\subset(b,\infty]$. Then, f has the following open neighborhood in $\mathrm{L}(X)$:

$$((X\times\mathbb{R})\setminus(X\times\{b\}))^+\cap\mathrm{L}(X)\subset\mathrm{L}(X)\setminus\mathrm{L}_{-\infty}(X).$$

Thus, $\mathrm{L}_{-\infty}(X)$ is closed in $\mathrm{L}(X)$, hence it is compact by Corollary 2.4.

On the other hand, if X is not compact then it contains an infinite and discrete set $\{x_i \mid i \in \mathbb{N}\}$, where $x_i \neq x_j$ if $i \neq j$. For each $i \in \mathbb{N}$, we define $f_i \in \mathrm{L}_{-\infty}(X)$ by

$$f_i(x) = \begin{cases} -\infty & \text{if } x = x_i, \\ \infty & \text{if } x \neq x_i, \end{cases}$$

that is, $f_i = \mathrm{epi}(f_i) = \{x_i\} \times \mathbb{R}$. For each neighborhood W of $\emptyset$ in $\mathrm{L}(X)$, we have a compact set $K \subset X$ such that $((X \times \mathbb{R}) \setminus K)^+ \subset W$. Since $\{x_i \mid i \in \mathbb{N}\}$ is discrete in X and $\mathrm{pr}_X(K)$ is compact, we have $n \in \mathbb{N}$ such that if $i \geq n$ then $x_i \notin \mathrm{pr}_X(K)$, hence $f_i \in ((X \times \mathbb{R}) \setminus K)^+ \subset W$. Thus, the sequence $(f_i)_{i \in \mathbb{N}}$ converges to the function $\emptyset$. Therefore, $\mathrm{L}_{-\infty}(X)$ is not compact. $\square$

Proposition 2.7. *Let X be a locally compact Hausdorff space.*

(1) *If X is σ-compact then $\mathrm{LSC}(X)$ is absolutely G_δ.*
(2) *If X is compact then $\mathrm{LSC}(X)$ is open in $\mathrm{L}(X)$, hence it is locally compact.*
(3) *If X is non-compact then $\mathrm{LSC}(X)$ is nowhere locally compact.*

Proof: (1) Since $\mathrm{L}(X)$ is a compact Hausdorff space, it suffices to see that $\mathrm{LSC}(X)$ is G_δ in $\mathrm{L}(X)$. Let $X = \bigcup_{n \in \mathbb{N}} X_n$, where each X_n is compact. For each $n \in \mathbb{N}$, let

$$W_n = \{f \in \mathrm{L}(X) \mid -\infty \notin f(X_n)\}.$$

Then, $\mathrm{LSC}(X) = \bigcap_{n \in \mathbb{N}} W_n \setminus \{\emptyset\}$. For each $f \in W_n$, since X_n is compact, we have $r \in \mathbb{R}$ such that $f(X_n) \subset (r, \infty]$, which implies

$$f \in ((X \times \mathbb{R}) \setminus (X_n \times \{r\}))^+ \cap \mathrm{L}(X) \subset W_n.$$

This means that W_n is open in $\mathrm{L}(X)$.

(2) For each $f \in \mathrm{LSC}(X)$, since X is compact, we have $r \in \mathbb{R}$ such that $f(X) \subset (r, \infty]$. Then,

$$f \in ((X \times \mathbb{R}) \setminus (X \times \{r\}))^+ \cap \mathrm{L}(X) \setminus \{\emptyset\} \subset \mathrm{LSC}(X).$$

Hence, $\mathrm{LSC}(X)$ is open in $\mathrm{L}(X)$.

(3) For each $f \in \mathrm{LSC}(X)$ and each neighborhood of W in $\mathrm{LSC}(X)$, we have open sets $U_1, \dots, U_n \subset X \times \mathbb{R}$ and a compact set $K \subset X \times \mathbb{R}$ such that

$$f \in U_1^- \cap \dots \cap U_n^- \cap ((X \times \mathbb{R}) \setminus K)^+ \cap \mathrm{LSC}(X) \subset W.$$

Since X is non-compact, we have $x_0 \in X \setminus \mathrm{pr}_X(K)$. For each $i \in \mathbb{N}$, we define $f_i \in W$ as follows:

$$f_i(x) = \begin{cases} f(x_0) - i & \text{if } x = x_0, \\ f(x) & \text{if } x \neq x_0. \end{cases}$$

Then, $(f_i)_{i\in\mathbb{N}}$ converges to $f_\infty \in \mathrm{L}_{-\infty}(X)$ defined as follows:

$$f_\infty(x) = \begin{cases} -\infty & \text{if } x = x_0, \\ f(x) & \text{if } x \neq x_0. \end{cases}$$

Since $\mathrm{L}(X)$ is Hausdorff, $\{f_i \mid i \in \mathbb{N}\}$ is discrete in $\mathrm{LSC}(X) \cap \mathrm{cl}\, W$. Therefore, $\mathrm{LSC}(X) \cap \mathrm{cl}\, W$ is not compact. □

3. Homotopy dense subsets and AR property

A subset Y of a space X is said to be *homotopy dense* in X if there exists a homotopy $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}_X$ and $h_t(X) \subset Y$ for every $t > 0$, where $h_t : X \to X$ is defined by $h_t(x) = h(x,t)$. Let $\eta, \zeta : \mathrm{L}(X) \times \mathbf{I} \to \mathrm{L}(X)$ be the homotopies defined as follows:

$$\eta_t(f)(x) = \begin{cases} f(x) & \text{if } t = 0, \\ \min\{f(x), 1/t\} & \text{if } t > 0; \end{cases}$$

$$\zeta_t(f)(x) = \begin{cases} f(x) & \text{if } t = 0, \\ \max\{f(x), -1/t\} & \text{if } t > 0. \end{cases}$$

By identifying $\eta_t(f)$ and $\zeta_t(f)$ with the epi-graphs, we can write

$$\eta_t(f) = f \cup X \times [1/t, \infty) \quad \text{and} \quad \zeta_t(f) = f \cap X \times [-1/t, \infty).$$

We shall verify the continuity of η and ζ.

Continuity of η : Let $V \subset X \times \mathbb{R}$ be open. For each $(f,t) \in \eta^{-1}(V^-)$, $f \cap V \neq \emptyset$ or $X \times [1/t,\infty) \cap V \neq \emptyset$ (the latter does not occur if $t = 0$). When $f \cap V \neq \emptyset$, $V^- \cap \mathrm{L}(X)$ is a neighborhood of f in $\mathrm{L}(X)$ and $\eta_s(g) \in V^-$ for every $g \in V^- \cap \mathrm{L}(X)$ and $s \in \mathbf{I}$. When $X \times [1/t,\infty) \cap V \neq \emptyset$ ($t > 0$), it follows that $X \times [a,\infty) \cap V \neq \emptyset$ for some $a > 1/t$. Then, $t \in (1/a, 1]$ and $X \times [1/s,\infty) \cap V \neq \emptyset$ for every $s \in (1/a, 1]$, which implies that $\eta_s(g) \in V^-$ for every $g \in \mathrm{L}(X)$ and $s \in (1/a, 1]$. Hence, $\eta^{-1}(V^-)$ is open in $\mathrm{L}(X) \times \mathbf{I}$.

Now, let $K \subset X \times \mathbb{R}$ be compact. For each $(f,t) \in \eta^{-1}(((X \times \mathbb{R}) \setminus K)^+)$, $f \cap K = \emptyset$ and $X \times [1/t,\infty) \cap K = \emptyset$, whence $((X \times \mathbb{R}) \setminus K)^+ \cap \mathrm{L}(X)$ is a neighborhood of f in $\mathrm{L}(X)$ and $X \times [a,\infty) \cap K = \emptyset$ for some $0 < a < 1/t$. Then, $t \in [0, 1/a)$ and $X \times [1/s,\infty) \cap K = \emptyset$ if $0 < s < 1/a$. It follows that $\eta_s(g) \in ((X \times \mathbb{R}) \setminus K)^+$ for every $g \in ((X \times \mathbb{R}) \setminus K)^+ \cap \mathrm{L}(X)$ and $s \in [0, 1/a)$. Thus, $\eta^{-1}(((X \times \mathbb{R}) \setminus K)^+)$ is open in $\mathrm{L}(X) \times \mathbf{I}$. □

Continuity of ζ : Let $V \subset X \times \mathbb{R}$ be open. For each $(f,t) \in \zeta^{-1}(V^-)$, we have $(x,r) \in V$ such that $r \geq \max\{f(x), -1/t\}$ ($r \geq f(x)$ if $t = 0$). Since V is open in $X \times \mathbb{R}$, $(x, r_0) \in V$ for some $r_0 > r$. Let $r < r_1 < r_0$ and $W = V \cap X \times (r_1, \infty)$. Then, $W^- \cap \mathrm{L}(X)$ is a neighborhood of f in $\mathrm{L}(X)$. Since $-1/t < r_1$, we have

$a > t$ so that $-1/s < r_1$ if $0 < s < a$. Then, $t \in [0, a)$. Let $g \in W^-$ and $s \in [0, a)$. Then, we have $(x', r') \in W$ with $r' \geq g(x')$. Since $r' > r_1 > -1/s$, it follows that $r' \geq \max\{g(x'), -1/s\}$, which means $\zeta_s(g) \in W^- \subset V^-$. Therefore, $\zeta^{-1}(V^-)$ is open in $\mathrm{L}(X) \times \mathbf{I}$.

Let $K \subset X \times \mathbb{R}$ be compact and $(f, t) \in \zeta^{-1}(((X \times \mathbb{R}) \setminus K)^+)$, that is, $f \cap X \times [-1/t, \infty) \cap K = \emptyset$. Observe that

$$f \cap X \times \{c\} = f^{-1}((-\infty, c]) \times \{c\} \quad \text{for each} \quad c \in \mathbb{R}.$$

By this fact, it is easy to see that

$$c < d \Rightarrow f \cap X \times [c, \infty) \subset f^{-1}((-\infty, d]) \times [c, d] \cup (f \cap X \times [d, \infty)).$$

Then, it follows that $f \cap X \times [a, \infty) \cap K = \emptyset$ for some $a < -1/t$ because K is compact. Let

$$\mathcal{W} = ((X \times \mathbb{R}) \setminus (X \times [a, \infty) \cap K))^+ \cap \mathrm{L}(X).$$

Then, $\mathcal{W}$ is a neighborhood of f in $\mathrm{L}(X)$ and $t \in (1/|a|, 1]$. For each $g \in \mathcal{W}$ and $s \in (1/|a|, 1]$, $g \cap X \times [-1/s, \infty) \cap K = \emptyset$, which means that $\zeta(g, s) \in ((X \times \mathbb{R}) \setminus K)^+$. Hence, $\zeta^{-1}(((X \times \mathbb{R}) \setminus K)^+)$ is open in $X \times \mathbb{R}$. □

We define the homotopy $\xi : \mathrm{L}(X) \times \mathbf{I} \to \mathrm{L}(X)$ by $\xi_t = \eta_t \zeta_t = \zeta_t \eta_t$ for every $t \in \mathbf{I}$, that is,

$$\xi_t(f) = (f \cap X \times [-1/t, \infty)) \cup X \times [1/t, \infty) \subset X \times \mathbb{R}.$$

Since $\xi_t(\mathrm{L}(X)) \subset \mathrm{LSC_B}(X)$ for $t > 0$, we have the following:

Proposition 3.1. *The subspace* $\mathrm{LSC_B}(X)$ *is homotopy dense in* $\mathrm{L}(X)$. □

It can be shown that the complement $\mathrm{LSC}(X) \setminus \mathrm{LSC_B}(X)$ is homotopy dense in $\mathrm{LSC}(X)$. At the same time, we shall prove that some other subspaces of $\mathrm{L}(X)$ are homotopy dense in $\mathrm{L}(X)$ and they are AR's.[3] To this end, we use the result on Lawson semilattices.

A *topological semilattice* is a topological space S equipped with a continuous operator $\vee : S \times S \to S$ which is idempotent, commutative and associative (i.e., $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$). A topological semilattice S is called a *Lawson semilattice* if S admits an open basis consisting of subsemilattices ([7]). A subspace Y of X is called *relatively* LC^0 *in* X if every neighborhood U of each $x \in X$ contains a neighborhood V of x in X such that any two points $y, z \in V \cap Y$ can be connected by a path in $V \cap Y$. The following is proved in [6, Theorem 5.1].

[3] AR = absolute retract; ANR = absolute neighborhood retract.

Proposition 3.2. *Let X be a metrizable Lawson semilattice and $Y \subset X$ a subsemilattice. If Y is relatively* LC^0 *in X (and Y is connected), then X is an ANR (an AR) and Y is homotopy dense in X, hence Y is also an ANR (an AR).* □

To apply Proposition 3.2 above, we show the following:

Proposition 3.3. *For a Hausdorff space X, the space* $\mathrm{Cld}^*_F(X)$ *is a Lawson semilattice with the union operator $\cup$. The spaces* $\mathrm{L}(X)$, $\mathrm{LSC}(X)$, $\mathrm{LSC_B}(X)$ *and* $\mathrm{L}_{-\infty}(X)$ *are subsemilattices of* $\mathrm{Cld}^*_F(X)$.

PROOF: For each open set $U \subset X$ and each compact set $K \subset X$, U^- and $(X \setminus K)^+$ are subsemilattices of $\mathrm{Cld}^*_F(X)$. Hence, $\mathrm{Cld}^*_F(X)$ has an open basis consisting of subsemilattices. The continuity of $\cup$ is easily observed. The second statement is evident. □

We consider the following subspace:

$$\begin{aligned} F(X) &= \{f \in \mathrm{LSC}(X) \mid f(x) = \infty \text{ except for finitely many } x \in X\} \\ &= \{f \in \mathrm{LSC}(X) \mid f^{-1}(\mathbb{R}) \text{ is finite}\}. \end{aligned}$$

As is easily observed, $F(X)$ is a dense subsemilattice of $\mathrm{LSC}(X)$. Moreover, it should be noted that $F(X) \cap \mathrm{LSC_B}(X) = \emptyset$ if X is infinite.

Lemma 3.4. *For every second countable locally compact Hausdorff space X, $F(X)$ is homotopy dense in* $\mathrm{LSC}(X)$.

PROOF: By Proposition 3.2, it suffices to show that $F(X)$ is relatively LC^0 in $\mathrm{LSC}(X)$. To this end, let $f \in \mathrm{LSC}(X)$ and W an open neighborhood of f in $\mathrm{LSC}(X)$. Since $\mathrm{LSC}(X)$ is a Lawson semilattice, we may assume that W is a subsemilattice of $\mathrm{LSC}(X)$. For each $f_1, f_2 \in W \cap F(X)$, we can define a path $h : \mathbf{I} \to F(X)$ as follows:

$$h(t)(x) = \begin{cases} f_1(x) & \text{if } f_1(x) \le f_2(x), \\ \theta^{-1}((1-t)\theta(f_1(x)) + t\theta(f_2(x))) & \text{if } f_2(x) < f_1(x), \end{cases}$$

where $\theta : [-\infty, \infty] \to \mathbf{I}$ is the homeomorphism defined in §1. It is easy to see that h is a path in $W \cap F(X)$ connecting $h(0) = f_1$ and $h(1) = f_1 \cup f_2$. Similarly, f_2 can be connected to $f_1 \cup f_2$ by a path in $W \cap F(X)$. Then, f_1 and f_2 are connected by a path in $W \cap F(X)$. Therefore, $F(X)$ is relatively LC^0 in $\mathrm{LSC}(X)$. □

Since $F(X) \subset \mathrm{LSC}(X) \setminus \mathrm{LSC_B}(X)$, the following follows from Lemma 3.4:

Proposition 3.5. *For every infinite second countable locally compact Hausdorff space X, $\mathrm{LSC}(X) \setminus \mathrm{LSC}_{\mathrm{B}}(X)$ is homotopy dense in $\mathrm{LSC}(X)$.* $\square$

A closed subset $A \subset Y$ is called a *Z-set* in Y if for each open cover $\mathcal{U}$, there exists a map[4] $f : Y \to Y \setminus A$ which is $\mathcal{U}$-close to the identity.[5] A countable union of Z-sets is called a *Z_σ-set*. One should note that a closed set (resp. an F_σ-set) $A \subset Y$ is a Z-set (resp. a Z_σ-set) if the complement $Y \setminus A$ is homotopy dense in Y.

Lemma 3.6. *Let X be a second countable locally compact Hausdorff space.*

(1) *The space $\mathrm{L}_{-\infty}(X)$ is an AR.*
(2) *If X is compact then $\mathrm{L}_{-\infty}(X)$ is a compact Z-set in $\mathrm{L}(X)$.*
(3) *If X is non-compact then $\mathrm{L}_{-\infty}(X)$ is homotopy dense in $\mathrm{L}(X)$.*

Proof: (1) Take $f_1, f_2 \in \mathrm{L}_{-\infty}(X)$. All the same as in the proof of Lemma 3.4, we can obtain a path $h : \mathbf{I} \to \mathrm{L}_{-\infty}(X)$ from f_1 to f_2, hence $\mathrm{L}_{-\infty}(X)$ is path-connected. Recall that $\mathrm{L}_{-\infty}(X)$ is a Lawson semilattice. If both f_1 and f_2 are in some open subsemilattice W of $\mathrm{L}_{-\infty}(X)$, then h is a path in W. Hence, $\mathrm{L}_{-\infty}(X)$ is LC^0. Thus, $\mathrm{L}_{-\infty}(X)$ is an AR by Proposition 3.2.

(2) By Lemma 2.6, $\mathrm{L}_{-\infty}(X)$ is compact. Since $\mathrm{L}_{-\infty}(X) \cap \mathrm{LSC}_{\mathrm{B}}(X) = \emptyset$ and $\mathrm{LSC}_{\mathrm{B}}(X)$ is homotopy dense in $\mathrm{L}(X)$ by Proposition 3.1, it follows that $\mathrm{L}_{-\infty}(X)$ is a Z-set in $\mathrm{L}(X)$.

(3) When X is non-compact, it is easy to see that $\mathrm{L}_{-\infty}(X)$ is dense in $\mathrm{L}(X)$. Similarly to Lemma 3.4, we can prove that $\mathrm{L}_{-\infty}(X)$ is homotopy dense in $\mathrm{L}(X)$. $\square$

Proposition 3.7. *Let X be a second countable locally compact Hausdorff space. Then, $\mathrm{L}(X)$, $\mathrm{LSC}(X)$, $\mathrm{LSC}_{\mathrm{B}}(X)$ and $\mathrm{LSC}(X) \setminus \mathrm{LSC}_{\mathrm{B}}(X)$ are AR's.*

Proof: We can define a map $\lambda : \mathrm{LSC}_{\mathrm{B}}(X)^2 \times \mathbf{I} \to \mathrm{LSC}_{\mathrm{B}}(X)$ as follows:

$$\lambda(f, g, t)(x) = (1-t)f(x) + tg(x) \quad \text{for each} \quad (f, g, t) \in \mathrm{LSC}_{\mathrm{B}}(X)^2 \times \mathbf{I}.$$

Then, $\lambda(f, g, 0) = f$, $\lambda(f, g, 1) = g$ and $\lambda(f, f, t) = f$, namely $\mathrm{LSC}_{\mathrm{B}}(X)$ is equi-connected, so $\mathrm{LSC}_{\mathrm{B}}(X)$ is path-connected and locally path-connected. Note that $\mathrm{LSC}_{\mathrm{B}}(X)$ is a Lawson semilattice as a subsemilattice of the Lawson semilattice $\mathrm{Cld}^*_F(X \times \mathbb{R})$ (Proposition 3.3). Therefore, $\mathrm{LSC}_{\mathrm{B}}(X)$ is an AR by Proposition 3.2. Since $\mathrm{LSC}_{\mathrm{B}}(X)$ is homotopy dense in $\mathrm{L}(X)$ by Proposition 3.1, it follows that $\mathrm{L}(X)$ and $\mathrm{LSC}(X)$ are AR's. Moreover, since $\mathrm{LSC}(X) \setminus \mathrm{LSC}_{\mathrm{B}}(X)$ is homotopy dense in $\mathrm{L}(X)$ by Proposition 3.5, $\mathrm{LSC}(X) \setminus \mathrm{LSC}_{\mathrm{B}}(X)$ is also an AR. $\square$

[4] Here, a map is a continuous function

[5] Two maps $f, g : X \to Y$ are *$\mathcal{U}$-close* if each $\{f(x), g(x)\}$ is contained in some $U \in \mathcal{U}$.

4. Proof of Theorems

The following property is called the *disjoint cells property*.

- For each $n \in \mathbb{N}$, and each open cover $\mathcal{U}$ of X, every maps $f, g : \mathbf{I}^n \to X$ are $\mathcal{U}$-close to maps $f', g' : \mathbf{I}^n \to X$ such that $f'(\mathbf{I}^n) \cap g'(\mathbf{I}^n) = \emptyset$.

To prove Theorem 1.1, we apply the following Toruńczyk's characterization of the Hilbert cube [10] ([8, Corollary 7.8.4]).

Theorem 4.1. *In order that $X \approx Q$, it is necessary and sufficient that X is a compact AR with the disjoint cells property.* □

Using this characterization of Q, we shall show Theorem 1.1.

Proof of Theorem 1.1: The "necessity" follows from Corollary 2.5 and Fact. We prove the "sufficiency". By Corollary 2.4 and Proposition 3.7, $\mathrm{L}(X)$ is a compact AR. Since both $\mathrm{LSC_B}(X)$ and $\mathrm{L}(X) \setminus \mathrm{LSC_B}(X)$ are homotopy dense in $\mathrm{L}(X)$ by Propositions 3.1 and 3.5, $\mathrm{L}(X)$ has the disjoint cells property. Thus, we have $\mathrm{L}(X) \approx Q$ by Theorem 4.1. □

In [1], introducing the notion of cap-sets characterizing subsets $M \subset Q$ such that $(Q, M) \approx (Q, \Sigma)$, R. Anderson proved that $(Q, \Sigma) \approx (Q, Q \setminus s)$ (cf. [3]). The following is a combination of Lemmas 4.2 and 4.4 in [3].

Lemma 4.2. *Suppose that $(Q, M) \approx (Q, \Sigma)$. If L is a Z_σ-set in Q and K is a Z-set in Q then $(Q, (M \cup L) \setminus K) \approx (Q, \Sigma)$.* □

The following is the combination of Lemmas 4.3 and 4.4 in [3].

Lemma 4.3. *Suppose that $(Q, M) \approx (Q, N) \approx (Q, \Sigma)$ and K is a Z-set in Q with $K \cap M = K \cap N$. Then, for each $\varepsilon > 0$, there is a homeomorphism $h : Q \to Q$ such that $h(M) = N$, $h|K = \mathrm{id}$ and h is ε-close to id. Moreover if $M \cup N \subset s$ then h also satisfies $h(Q \setminus s) = Q \setminus s$, that is, $h(s) = s$.* □

A tower $(M_i)_{i \in \mathbb{N}}$ of closed sets in X has the *deformation property* in X if there is a homotopy $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}$ and, for each $t > 0$, $h(X \times [t, 1])$ is contained in some M_i. We apply the following Curtis' result ([4, Corollary 4.9]:

Lemma 4.4. *Let $M = \bigcup_{i \in \mathbb{N}} M_i \subset Q$, where $M_1 \subset M_2 \subset \cdots$ satisfy the following conditions:*

(1) *$M_i \approx Q$ for each $i \in \mathbb{N}$;*
(2) *each M_i is a Z-set in M_{i+1};*
(3) *$(M_i)_{i \in \mathbb{N}}$ has the deformation property in Q.*

Then, $(Q, M) \approx (Q, \Sigma)$. □

Before proving Theorems 1.2 and 1.3, we show the following:

Theorem 4.5. *For a Hausdorff space X, $(\mathrm{L}(X), \mathrm{LSC_B}(X)) \approx (Q, \Sigma)$ if and only if X is locally compact, second countable and infinite.*

Proof: The "only if" part follows from Theorem 1.1. To see the "if" part, assume that X is locally compact and second countable. For each $n \in \mathbb{N}$, let

$$B_n = \{f \in \mathrm{L}(X) \mid f(X) \subset [-n, n]\} \ \text{ and}$$
$$F_n = \{f \in B_n \mid f(x) = n \ \text{ except for finitely many } x \in X\}.$$

Then, as is easily observed, $(B_n, F_n) \approx (\mathrm{L}(X), F(X))$, hence we have $B_n \approx Q$ by Theorem 1.1 and F_n is homotopy dense in B_n by Lemma 3.4. Since $B_n \cap F_{n+1} = \emptyset$ and F_{n+1} is homotopy dense in B_{n+1}, it follows that B_n is a Z-set in B_{n+1}. Let $\xi : \mathrm{L}(X) \times \mathbf{I} \to \mathrm{L}(X)$ be the homotopy defined in §3. For each $t > 0$, choose $n \in \mathbb{N}$ so that $n \geq 1/t$. Then, $\xi(\mathrm{L}(X) \times [t, 1]) \subset B_n$. Thus, $(B_n)_{n\in\mathbb{N}}$ has the deformation property in $\mathrm{L}(X)$. Since $\mathrm{LSC_B}(X) = \bigcup_{n\in\mathbb{N}} B_n$, we have $(\mathrm{L}(X), \mathrm{LSC_B}(X)) \approx (Q, \Sigma)$ by Lemma 4.4. □

To prove Theorem 1.2, we use the following:

Lemma 4.6. *For every second countable compact infinite Hausdorff space X, $\mathrm{L}_{-\infty}(X) \approx Q$.*

Proof: By Lemma 3.6, $\mathrm{L}_{-\infty}(X)$ is a compact AR. Let $\eta : \mathrm{L}(X) \times \mathbf{I} \to \mathrm{L}(X)$ be the homotopy defined in §3. Observe that $\eta(\mathrm{L}_{-\infty}(X) \times \mathbf{I}) \subset \mathrm{L}_{-\infty}(X)$. Since X is infinite, it follows that

$$\eta_t(\mathrm{L}_{-\infty}(X)) \subset \mathrm{L}_{-\infty}(X) \setminus F(X) \ \text{ for } \ t > 0,$$

whence $\mathrm{L}_{-\infty}(X) \setminus F(X)$ is homotopy dense in $\mathrm{L}_{-\infty}(X)$. Moreover, by the same arguments as the proof of Lemma 3.4, it can be shown that $F(X) \cap \mathrm{L}_{-\infty}(X)$ is homotopy dense in $\mathrm{L}_{-\infty}(X)$. Hence, $\mathrm{L}_{-\infty}(X)$ has the disjoint cells property. By Theorem 4.1, we have $\mathrm{L}_{-\infty}(X) \approx Q$. □

Now, we shall prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2: The implication (c) ⇒ (b) is obvious. By Corollary 2.5, Proposition 2.7(3) and Fact, we have the implication (b) ⇒ (a).

(a) ⇒ (c): By Theorem 4.5 above, we have

$$(\mathrm{L}(X), \mathrm{LSC_B}(X)) \approx (Q, \Sigma) \approx (\mathrm{Cone}\, Q, \Sigma \times (0, 1)).$$

Since $\mathrm{L}_{-\infty}(X)$ is a Z-set in $\mathrm{L}(X)$ by Lemma 3.6(2) and $\mathrm{L}_{-\infty}(X) \approx Q$ by Lemma 4.6, we can apply the Z-set unknotting theorem to obtain a homeomorphism $g : \mathrm{L}(X) \to \mathrm{Cone}\, Q$ such that $g(\{\emptyset\}) = \{*\}$ and $g(\mathrm{L}_{-\infty}(X)) = Q \times \{0\}$. Note that

$$(Q \times \{0\} \cup \{*\}) \cap g(\mathrm{LSC_B}(X)) = \emptyset.$$

By Lemma 4.3, we have a homeomorphism $h : \operatorname{Cone} Q \to \operatorname{Cone} Q$ such that

$$hg(\mathrm{LSC_B}(X)) = \Sigma \times (0,1) \ \text{ and } \ h|Q \times \{0\} \cup \{*\} = \mathrm{id},$$

whence it follows that

$$\begin{aligned} hg(\mathrm{LSC}(X)) &= hg(\mathrm{L}(X) \setminus (\mathrm{L}_{-\infty}(X) \cup \{\emptyset\})) \\ &= \operatorname{Cone} Q \setminus (Q \times \{0\} \cup \{*\}) = Q \times (0,1). \end{aligned}$$

This completes the proof. □

Proof of Theorem 1.3: The implication (c) ⇒ (b) is obvious. The implication (b) ⇒ (a) follows from Corollary 2.5 and Proposition 2.7(2).

(a) ⇒ (c): We can write $X = \bigcup_{n\in\mathbb{N}} X_n$, where $\operatorname{int} X_1$ is infinite, each X_n is compact and $X_n \subsetneq \operatorname{int} X_{n+1}$. For each $n \in \mathbb{N}$, let

$$\begin{aligned} M_n &= \{f \in \mathrm{L}(X) \mid f(X \setminus \operatorname{int} X_n) = \{-\infty\}\} \ \text{ and} \\ N_n &= \{f \in M_n \mid f(\operatorname{int} X_n) \text{ is a bounded subset of } \mathbb{R}\}. \end{aligned}$$

Then, as is easily observed, we have

$$(M_n, N_n) \approx (\mathrm{L}(\operatorname{int} X_n), \mathrm{LSC_B}(\operatorname{int} X_n)),$$

whence $M_n \approx Q$ by Theorem 1.1 and N_n is homotopy dense in M_n by Proposition 3.1. Since $(X \setminus \operatorname{int} X_n) \cap \operatorname{int} X_{n+1} \neq \emptyset$, we have $M_n \cap N_{n+1} = \emptyset$, whence M_n is a Z-set in M_{n+1} because N_{n+1} is homotopy dense in M_{n+1}. We can define a homotopy $h : \mathrm{L}(X) \times \mathbf{I} \to \mathrm{L}(X)$ as follows: $h_0 = \mathrm{id}$,

$$h_{1/n}(f) = f \cup (X \setminus \operatorname{int} X_n) \times \mathbb{R},$$

and, for $1/(n+1) < t < 1/n$,

$$h_t(f) = h_{1/(n+1)}(f) \cup (X \setminus \operatorname{int} X_n) \times [\varphi_n(t), \infty),$$

where $\varphi_n : (1/(n+1), 1/n) \to \mathbb{R}$ is a continuous monotone function such that

$$\lim_{t\to 1/(n+1)} \varphi_n(t) = -\infty \ \text{ and } \ \lim_{t\to 1/n} \varphi_n(t) = \infty.$$

For each $t > 0$, choose $n \in \mathbb{N}$ so that $n \geq 1/t$. Then, $h(\mathrm{L}(X) \times [t,1]) \subset M_n$. Thus, $(M_n)_{n\in\mathbb{N}}$ has the deformation property in $\mathrm{L}(X)$. Let $M = \bigcup_{n\in\mathbb{N}} M_n$. We have $(\mathrm{L}(X), M) \approx (Q, \Sigma)$ by Lemma 4.4.

On the other hand, $\mathrm{LSC}(X)$ is a homotopy dense G_δ-set in $\mathrm{L}(X)$ by Propositions 2.7(1) and 3.1. Then,

$$\mathrm{L}_{-\infty}(X) \cup \{\emptyset\} = \mathrm{L}(X) \setminus \mathrm{LSC}(X)$$

is a Z_σ-set in $\mathrm{L}(X)$. Since $M \subset \mathrm{L}_{-\infty}(X)$, we apply Lemma 4.2 to have

$$(\mathrm{L}(X),\, \mathrm{L}_{-\infty}(X) \cup \{\emptyset\}) \approx (Q, \Sigma) \approx (Q,\, Q \setminus s),$$

hence $(\mathrm{L}(X),\, \mathrm{LSC}(X)) \approx (Q, s)$. Then, it follows from Lemma 4.3 that

$$(\mathrm{L}(X),\, \mathrm{LSC}(X),\, \mathrm{LSC_B}(X)) \approx (Q, s, \Sigma).$$

The proof is completed. □

Remark. In the proof above, we have $(\mathrm{L}(X),\, \mathrm{L}_{-\infty}(X)) \approx (Q, \Sigma)$ by the same reason as $\mathrm{L}_{-\infty}(X) \cup \{\emptyset\}$, that is,

Proposition 4.7. *For every second countable locally compact non-compact Hausdorff space X,* $(\mathrm{L}(X),\, \mathrm{L}_{-\infty}(X)) \approx (Q, \Sigma) \approx (Q,\, Q \setminus s)$. □

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