Another proof of Derriennic's reverse maximal inequality for the supremum of ergodic ratios

Ryotaro Sato

Abstract. Using the ratio ergodic theorem for a measure preserving transformation in a σ-finite measure space we give a straightforward proof of Derriennic's reverse maximal inequality for the supremum of ergodic ratios.

Keywords: σ-finite measure space, measure preserving transformation, conservative, ergodic, supremum of ergodic ratios, maximal and reverse maximal inequalities

Classification: Primary 28D05, 47A35

1. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T be a measure preserving transformation in $(X, \mathcal{F}, \mu)$. Given two measurable functions f and g on X such that $0 \leq f$, $g \leq \infty$ on X and $0 < \int_X g\, d\mu \leq \infty$, let

$$s(f,g)(x) = \sup_{n \geq 0} \frac{\sum_{i=0}^{n} f(T^i x)}{\sum_{i=0}^{n} g(T^i x)}.$$

(Throughout this note we define $a/\infty = 0$ and $a/0 = \infty$ for any a, with $0 \leq a \leq \infty$.) In this note we use the ratio ergodic theorem to give a straightforward proof of the following reverse maximal inequality due to Derriennic [1] (cf. also Ornstein [5]). It is interesting to note that the author was inspired by reading Ephremidze's paper [3].

Theorem. *Suppose that T is conservative and ergodic, and that $\int_X f\, d\mu < \infty$. If $\alpha > \int_X f\, d\mu / \int_X g\, d\mu$, then, letting $E(\alpha) = \{x \mid s(f,g)(x) > \alpha\}$, we have*

$$\int_{E(\alpha)} f\, d\mu \leq \alpha \int_{E(\alpha) \cup T^{-1}E(\alpha)} g\, d\mu.$$

Proof: We may assume that $\mu(E(\alpha)) > 0$. For $x \in X$, let $K(x) = \{n \geq 0 \mid T^n x \in E(\alpha)\}$ and $L(x) = \{0, 1, \dots\} \setminus K(x)$. Since T is conservative and ergodic, $K(x)$ is infinite for a.a. $x \in X$. To see that $L(x)$ is also infinite for a.a. $x \in X$, suppose there exists $k \geq 0$ such that $i \in K(x)$ for all $i \geq k$. Then clearly we have

$$\limsup_{l \to \infty} \frac{\sum_{i=k}^{l} f(T^i x)}{\sum_{i=k}^{l} g(T^i x)} \geq \alpha. \tag{1}$$

But this is a contradiction, since

$$\lim_{l\to\infty} \frac{\sum_{i=k}^{l} f(T^i x)}{\sum_{i=k}^{l} g(T^i x)} = \frac{\int_X f\,d\mu}{\int_X g\,d\mu} < \alpha \tag{2}$$

for a.a. $x \in X$ by the ratio ergodic theorem (cf. Theorem 3.3.4 in [4]).

Since $K(x)$ and $L(x)$ are infinite for a.a. $x \in X$, we can write $K(x) = \bigcup_{n=1}^{\infty} I_n$ (disjoint union), where $I_n = [k_n,\ l_n]$ $(= \{i \mid k_n \le i \le l_n\})$ and $0 \le k_n \le l_n < l_n + 2 \le k_{n+1}$ for each $n \ge 1$. Hence the set $J(x) = \{n \ge 0 \mid T^n x \in E(\alpha) \cup T^{-1}E(\alpha)\}$ has the form

$$J(x) = \begin{cases} [0,\ l_1] \cup \bigcup_{n=2}^{\infty} [k_n - 1,\ l_n] & \text{if } k_1 = 0, \\ \bigcup_{n=1}^{\infty} [k_n - 1,\ l_n] & \text{if } k_1 \ge 1. \end{cases}$$

Since $T^{k_n - 1} x \notin E(\alpha)$ for $n \ge 2$, we have

$$\frac{\sum_{i=k_n-1}^{l_n} f(T^i x)}{\sum_{i=k_n-1}^{l_n} g(T^i x)} \le \alpha \qquad (n \ge 2). \tag{3}$$

On the other hand, if h is a function in $L_1(\mu)$ such that $\int_X h\,d\mu = 1$ and $0 < h < \infty$ on X, then, by the ratio ergodic theorem,

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} (\chi_{E(\alpha)\cup T^{-1}E(\alpha)} f)(T^i x)}{\sum_{i=0}^{n} h(T^i x)} = \int_{E(\alpha)\cup T^{-1}E(\alpha)} f\,d\mu \tag{4}$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} (\chi_{E(\alpha)\cup T^{-1}E(\alpha)} g)(T^i x)}{\sum_{i=0}^{n} h(T^i x)} = \int_{E(\alpha)\cup T^{-1}E(\alpha)} g\,d\mu \tag{5}$$

for a.a. $x \in X$. Since $\sum_{i=0}^{\infty} h(T^i x) = \infty$ for a.a. $x \in X$, combining (3), (4) and (5) yields

$$\int_{E(\alpha)\cup T^{-1}E(\alpha)} f\,d\mu \le \alpha \int_{E(\alpha)\cup T^{-1}E(\alpha)} g\,d\mu, \tag{6}$$

and this completes the proof, since $f \ge 0$ on X. □

2. Here we consider the case $g = 1$ on X. Then it follows that $s(f, 1) = f^*$, where $f^*(x) = \sup_{n\ge1} n^{-1} \sum_{i=0}^{n-1} f(T^i x)$. In this case we have the following reverse maximal inequality.

Proposition. *If* $\mu(X) = \infty$, T *is ergodic* (*but not necessarily conservative*), *and* f *satisfies* $\int_{\{f>t\}} f\,d\mu < \infty$ *for all* $t > 0$, *then we have* $\int_{\{f^*>\alpha\}} f\,d\mu \le 2\alpha\mu(\{f^* > \alpha\}) < \infty$ *for all* $\alpha > 0$.

PROOF: We first prove that $\mu(\{f^* > \alpha\}) < \infty$. To do this, let $f_1 = f\chi_{\{f\le\alpha/2\}}$ and $f_2 = f - f_1$. Then we have $f = f_1 + f_2$, $\|f_1\|_\infty \le \alpha/2$, and $\int_X f_2\,d\mu < \infty$. Since $f^* \le f_1^* + f_2^*$ and $\|f_1^*\|_\infty \le \alpha/2$, it follows that $\{f^* > \alpha\} \subset \{f_2^* > \alpha/2\}$, and by Hopf's maximal ergodic theorem (cf. Theorem 1.2.1 in [4])

$$\mu(\{f_2^* > \alpha/2\}) \le (2/\alpha) \int_{\{f_2^*>\alpha/2\}} f_2\,d\mu < \infty,$$

so that $\mu(\{f^* > \alpha\}) < \infty$. Putting $F = f - \alpha$, we then have $F^+ = (f-\alpha)^+ \in L_1(\mu)$ and $\{F^* > 0\} = \{f^* > \alpha\}$; furthermore $\int_X F\,d\mu = \int_X (f-\alpha)^+\,d\mu - \int_X (f-\alpha)^-\,d\mu = -\infty$ because $\mu(X) = \infty$. Hence by Theorem 1.4 in Ephremidze [2] we see that

$$\int_{\{f^*>\alpha\}\cup T^{-1}\{f^*>\alpha\}} (f-\alpha)\,d\mu \le 0.$$

Since $f \ge 0$ and $\mu(\{f^* > \alpha\}) < \infty$, we then have

$$\int_{\{f^*>\alpha\}} f\,d\mu \le \int_{\{f^*>\alpha\}\cup T^{-1}\{f^*>\alpha\}} f\,d\mu \le 2\alpha\mu(\{f^* > \alpha\}) < \infty,$$

completing the proof. □

Corollary. *If* $\mu(X) = \infty$, *and* T *is ergodic, then for any* $\beta \ge 0$ *we have*

$$\int_{\{f^*>t\}} f^* \left(\log \frac{f^*}{t}\right)^\beta d\mu < \infty \qquad \textit{for all } t > 0$$

if and only if

$$\int_{\{f>t\}} f \left(\log \frac{f}{t}\right)^{\beta+1} d\mu < \infty \qquad \textit{for all } t > 0.$$

PROOF: See the proof of Theorem 2 in [6]. □

(Of course, as is known, this holds when $\mu(X) < \infty$, by the Theorem.)

REFERENCES

[1] Derriennic Y., *On the integrability of the supremum of ergodic ratios*, Ann. Probability **1** (1973), 338–340.

[2] Ephremidze L., *On the distribution function of the majorant of ergodic means*, Studia Math. **103** (1992), 1–15.

[3] Ephremidze L., *A new proof of the ergodic maximal equality*, Real Anal. Exchange **29** (2003/04), 409–411.
[4] Krengel U., *Ergodic Theorems*, Walter de Gruyter, Berlin, 1985.
[5] Ornstein D., *A remark on the Birkhoff ergodic theorem*, Illinois J. Math. **15** (1971), 77–79.
[6] Sato R., *Maximal functions for a semiflow in an infinite measure space*, Pacific J. Math. **100** (1982), 437–443.

Department of Mathematics, Okayama University, Okayama, 700-8530 Japan
E-mail: satoryot@math.okayama-u.ac.jp

(*Received* May 2, 2005, *revised* November 15, 2005)