On existence and regularity of solutions to a class of generalized stationary Stokes problem

Nguyen Duc Huy, Jana Stará

Abstract. We investigate the existence of weak solutions and their smoothness properties for a generalized Stokes problem. The generalization is twofold: the Laplace operator is replaced by a general second order linear elliptic operator in divergence form and the "pressure" gradient ∇p is replaced by a linear operator of first order.

Keywords: generalized Stokes problem, weak solutions, regularity up to the boundary

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with boundary $\partial\Omega$. We study the following generalization of the linear Stokes problem: For given $f = (f_1, \dots, f_d) : \Omega \longrightarrow \mathbb{R}^d$, $g : \Omega \longrightarrow \mathbb{R}$, $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^{d} : \Omega \longrightarrow \mathbb{R}^{d^2 \times d^2}$ and a $d \times d$ matrix $B = \left(B_{ij}\right)_{i,j=1}^{d}$ we look for $u = (u_1, \dots, u_d) : \Omega \longrightarrow \mathbb{R}^d$ and $p : \Omega \longrightarrow \mathbb{R}$ solving

$$\begin{aligned} - \operatorname{div}(A\nabla u) + B\nabla p &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= g \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}$$

The generalization of the classical Stokes problem consists in two points: instead of the Laplace operator we consider a general second order elliptic operator in divergence form and instead of the gradient of p we consider a class of general first order linear operators. The new feature of system (1.1) compared with classical Stokes system lies in the fact that operators $\operatorname{div} u$ and $B\nabla p$ (for $B \neq E$) do not act as adjoint operators in suitable Banach spaces. While existence of weak solutions to (1.1) with $B = E$ was extensively studied (see e.g. [4], [5], [9] and references given there), both existence and smoothness properties of solutions to system (1.1) with a general B — as far as we know — have not been investigated yet.

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Our motivation to consider system (1.1) began with the study of smoothness of flows of incompressible fluids with viscosities that depend on the shear rate and the pressure. The typical example we have in mind is

$$
\begin{aligned}
u_t - \operatorname{div} T(Du,p) + (u\cdot\nabla)u + \nabla p = f \quad &\text{in} \quad I\times\Omega,\\
\operatorname{div} u = 0 \quad &\text{in} \quad I\times\Omega
\end{aligned}
\tag{1.2}
$$

accompanied by initial and boundary conditions. Here u stands for the velocity, $Du = \frac{1}{2}(\nabla u + \nabla^T u)$, p for pressure, f for external body forces and $T(Du,p)$ for the Cauchy stress tensor.

We assume that

(1a) T is continuously differentiable on $\mathbb{R}^{d^2+1}$ and

$$T_{ij}(\xi,\tau) = \nu(|\xi|,\tau)\xi_{ij}, \ i,j = 1,\dots,d;$$

(1b) there are $m\in(1,2]$, λ_0, λ_1, $\nu_0>0$ such that for any $\tau\in\mathbb{R}$ and symmetric matrix $d\times d$ ξ, it holds

$$\lambda_0(1+|\xi|^2)^{\frac{m-2}{2}} \le \sum_{i,j,k,l=1}^{d} \frac{\partial T_{ij}}{\partial \xi_{kl}}(\xi,\tau)\xi_{ij}\xi_{kl} \le \lambda_1(1+|\xi|^2)^{\frac{m-2}{2}},$$

$$\sum_{i,j=1}^{d} \frac{\partial T_{ij}}{\partial\tau}(\xi,\tau) \le \nu_0(1+|\xi|^2)^{\frac{m-2}{4}}.$$

Then if f, the boundary $\partial\Omega$ and initial data satisfy natural conditions, $m\in(\frac{3d}{d+2},2]$ and ν_0 is small enough with respect to λ_0, there is a pair

$$u\in L^m(I,W^{1,m}(\Omega))\cap L^\infty(I,L^2(\Omega)); p\in L^m(I\times\Omega)$$

satisfying (1.2) (see [11], [12] and [13]). The smoothness of u and p is a more delicate problem even in the stationary case (for which the existence was proved in [13]). As we deal with a system of nonlinear elliptic PDEs we cannot expect full regularity in space dimensions $d\ge 3$. When proving partial regularity results for such models we come to the so-called "blow up" system of (1.2) which has the form (1.1) with

$$
\begin{aligned}
A^{ij}_{kl} &= \frac{1}{2}\Big(\frac{\partial T_{ij}}{\partial\xi_{kl}}(a,b) + \frac{\partial T_{il}}{\partial\xi_{kj}}(a,b)\Big), \ i,j,k,l = 1,\dots,d;\\
B_{ij} &= \delta_{ij} - \frac{\partial T_{ij}}{\partial\tau}(a,b), \ i,j=1,\dots,d,
\end{aligned}
$$

where $a = \lim\limits_{R\to 0+}\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} Du\,dx$, $b = \lim\limits_{R\to 0+}\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} p\,dx$.

Saying differently, behaviour of solutions to (1.1) with such A and B predicts behaviour of solutions to (1.2) in regular points x_0.

The arrangement of the paper is as follows. In Section 2 we introduce notation, definitions and recall some results used later. In the next section we present existence and uniqueness results for a constant matrix B. In addition, we illustrate the type of this generalized linear Stokes system by several examples. In Section 4 we show the regularity of solutions u, p in $W^{k,2}(\Omega)$ under natural conditions on f, g, A, B, Ω.

2. Preliminaries

In this section, we introduce notation, definitions and also recall some well-known results that will be used later.

Let Ω be a domain with Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^d$ $(d \geq 2)$. For $1 \leq q \leq \infty$, $k \in \mathbb{N}$; $L^q(\Omega)$ and $W^{k,q}(\Omega)$ denote the usual Lebesgue and Sobolev spaces. The norm of $u \in L^q(\Omega)$ is denoted by

$$\|u\|_q = \|u\|_{q,\Omega} := \Big(\int_\Omega |u|^q \, dx\Big)^{1/q}.$$

The norm of $u \in W^{k,q}(\Omega)$ is defined as

$$\|u\|_{k,q} = \|u\|_{k,q;\Omega} := \Big(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^q \, dx\Big)^{1/q}.$$

As usual, $W_0^{k,q}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{k,q}(\Omega)$. We denote by $W^{-1,q'}(\Omega)$ the dual space to $W_0^{1,q}(\Omega)$ where $\frac{1}{q'} + \frac{1}{q} = 1$. If $f \in W^{-1,q'}(\Omega)$, $v \in W_0^{1,q}(\Omega)$ we use the notation $[f, v]$ for the value of the functional f at v.

Set $W^{k,q}(\Omega)^m := W^{k,q}(\Omega, \mathbb{R}^m) = [W^{k,q}(\Omega)]^m$ with norm

$$\|u\|_{k,q} = \|u\|_{k,q;\Omega} = \|(u_1, \ldots, u_m)\|_{k,q;\Omega} := \Big(\sum_{j=1}^m \|u_j\|_{k,q}^q\Big)^{1/q}.$$

In a similar way we obtain vector valued Banach spaces $W_0^{1,q}(\Omega)^m$, $L^q(\Omega)^m$ and $W^{-1,q'}(\Omega)^m$ (which denotes the dual space to $W_0^{1,q}(\Omega)^m$). We will also use the symbol $\|u\|_{-1,q'} = \|u\|_{-1,q';\Omega}$ to denote the norm of $u \in W^{-1,q'}(\Omega)$ or $u \in W^{-1,q'}(\Omega)^m$.

The space $W_{0,\mathrm{div}}^{1,2}(\Omega)$ is determined by the condition

$$W_{0,\mathrm{div}}^{1,2}(\Omega) := \Big\{u \in W_0^{1,2}(\Omega)^d;\ \operatorname{div} u = 0\Big\}$$

where the equation $\operatorname{div} u = 0$ is satisfied in distributional sense. $W^{1,2}_{0,\mathrm{div}}(\Omega)$ is a closed subspace of $W^{1,2}_0(\Omega)^d$ and thus it is a Hilbert space with scalar product induced from $W^{1,2}_0(\Omega)^d$.

Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $f \in W^{k,q}(\Omega)$, $u = (u_1, \dots, u_d) \in W^{k,q}(\Omega)^d$. We introduce notation

$$D_j f := \frac{\partial f}{\partial x_j},\ D_j^2 f := \frac{\partial^2 f}{\partial x_j^2},\ \nabla f := (D_j f)_{j=1}^d,\ \nabla^2 f := (D_j D_l f)_{j,l=1}^d,$$

$$x = (x', x_d),\ x' = (x_1, \dots, x_{d-1}),\ u = (u', u_d),\ u' = (u_1, \dots, u_{d-1}),$$

$$\nabla = (\nabla', D_d),\ \nabla' = (D_1, \dots, D_{d-1}),\ D_j u := (D_j u_1 \dots, D_j u_d).$$

$$B_r(x_0) := \left\{ x \in \mathbb{R}^d; |x - x_0| < r \right\},\ x_0 \in \mathbb{R}^d,\ r > 0;$$

$$B'_r := \{ y' \in \mathbb{R}^{d-1};\ |y'| < r \},\ r > 0.$$

If $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, we use the notation

$$x \cdot y := x_1 y_1 + \dots + x_m y_m$$

for the scalar product of x and y. For M an $d^2 \times d^2$ matrix and x, y being $d \times d$ matrices we write

$$Mx : y = \sum_{\alpha,\beta,i,j=1}^{d} M_{ij}^{\alpha\beta} x_i^\alpha y_j^\beta.$$

For points $x \in \mathbb{R}^d$ as well as for matrices $M = (M_{ij})_{i,j=1}^m$ we write

$$|x| = \Big(\sum_{i=1}^d |x_i|^2 \Big)^{\frac{1}{2}}, \quad |M| := \Big(\sum_{i,j=1}^m |M_{ij}|^2 \Big)^{\frac{1}{2}}.$$

Next, we recall the local description of the boundary $\partial\Omega$ which allows us to define domains with smooth boundary (see [1], [7]).
Given $x_0 \in \mathbb{R}^d$, $r > 0$, $\beta > 0$, a local coordinate system centered in x_0 with coordinates $y = (y', y_d)$ and a real continuous function $h : B'_r \longmapsto \mathbb{R}$ we denote

$$U_{r,\beta,h}(x_0) := \left\{ (y', y_d) \in \mathbb{R}^d; h(y') - \beta < y_d < h(y') + \beta, |y'| < r \right\}.$$

A domain (i.e. open, connected set) $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is called a Lipschitz domain, iff for each $x_0 \in \partial\Omega$, there exist constants $r > 0$, $\beta > 0$, a local coordinate system centered in x_0, and a Lipschitz continuous function $h : B'_r \longmapsto \mathbb{R}$ with the following properties

$$U_{r,\beta,h}(x_0) \cap \partial\Omega = \left\{(y', y_d); y_d = h(y'), |y'| < r\right\}, \tag{2.1}$$

$$U_{r,\beta,h}(x_0) \cap \Omega = \left\{(y', y_d); h(y') - \beta < y_d < h(y'), |y'| < r\right\}, \tag{2.2}$$

$$U_{r,\beta,h}(x_0) \cap (\mathbb{R}^d \setminus \Omega) = \left\{(y', y_d); h(y') < y_d < h(y') + \beta, |y'| < r\right\}. \tag{2.3}$$

For $k \in \mathbb{N}$ the domain Ω is called a C^k-domain, iff for each $x_0 \in \partial\Omega$, the function h describing the boundary in (2.1), (2.2), (2.3) belongs to $C^k(\overline{B'_r})$.

If Ω is a bounded C^k-domain then for all $\gamma > 0$ we find $x_1, \dots, x_m \in \partial\Omega$, $h_j := h_{x_j}$, $r_j := r_{x_j}$, $B'_j = B'_{r_j}$, $U_j := U_{r_j,\beta_j,h_j}(x_j)$, $j = 1, \dots, m$ with the properties (2.1), (2.2), (2.3) such that $\partial\Omega \subset \bigcup_{j=1}^m U_j$. Moreover, $h_j \in C^k(\overline{B'_j})$ and $\|h_j\|_{C^k(\overline{B'_j})} \le \gamma$ for $j = 1, \dots, m$.

Partition of unity gives existence of functions $\varphi_j \in C_0^\infty(\mathbb{R}^d)$, $j = 1, \dots, m$; a sequence of open balls $B_k \subset\subset \Omega$, $k = 1, \dots, l$ and a sequence of functions $\psi_k \in C_0^\infty(\mathbb{R}^d)$, $k = 1, \dots, l$ with the following properties

$$\operatorname{supp} \varphi_j \subset U_j,\ 0 \le \varphi_j \le 1,\ j = 1, \dots, m;$$
$$\operatorname{supp} \psi_k \subset B_k,\ 0 \le \psi_k \le 1,\ k = 1, \dots, l;$$
$$\overline{\Omega} \subset (\bigcup_{k=1}^{l} B_k) \cup (\bigcup_{j=1}^{m} U_j);\ \sum_{k=1}^{l} \psi_k(x) + \sum_{j=1}^{m} \varphi_j(x) = 1 \quad \text{for all} \quad x \in \overline{\Omega}.$$

We conclude this section by recalling some results on solvability of equations $\operatorname{div} v = g$ and $\nabla p = f$.

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and let Ω_0 be a nonempty subdomain of Ω. Let $1 < q < \infty$, $q' = \frac{q}{q-1}$. Then it holds:*

(a) *there is a constant $C = C(q, \Omega) > 0$ such that for each $g \in L^q(\Omega)$ with $\int_\Omega g\, dx = 0$ there exists at least one $v \in W_0^{1,q}(\Omega)^d$ satisfying*

$$\operatorname{div} v = g \quad in \quad \Omega,\ \|\nabla v\|_q \le C\|g\|_q.$$

(b) *there is a constant $C = C(q, \Omega, \Omega_0) > 0$ such that for each $f \in W^{-1,q}(\Omega)^d$ satisfying condition $[f, v] = 0$ for all $v \in W^{1,q'}_{0,\mathrm{div}}(\Omega)$ there exists a unique $p \in L^q(\Omega)$ satisfying*

$$\nabla p = f \quad in \quad \Omega,\ \int_{\Omega_0} p\, dx = 0 \quad and \quad \|p\|_q \le C\|f\|_{-1,q}.$$

Proof: See [8, Chapter 2, Lemma 2.1.1, 2.2.2]. □

3. Existence of solutions

Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded Lipschitz domain with boundary $\partial\Omega$. We will prove the existence and uniqueness of solutions to the generalized linear Stokes system (1.1) with $g = 0$. Thus, we consider the system

$$\begin{aligned} -\operatorname{div}(A\nabla u) + B\nabla p &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.1}$$

Here $f = (f_1, \dots, f_d) : \Omega \to \mathbb{R}^d$, $A = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^d : \Omega \to \mathbb{R}^{d^2\times d^2}$, $B = \left(B_{ij}\right)_{i,j=1}^d \in \mathbb{R}^{d\times d}$ are given quantities and $u = (u_1, \dots, u_d) : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ are unknown functions.

Definiton 3.1. Let $f \in W^{-1,2}(\Omega)^d$. Then a pair $(u,p) \in W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega)$ is called a weak solution to system (3.1) if and only if

$$-\operatorname{div}(A\nabla u) + B\nabla p = f \quad \text{in} \quad \Omega \tag{3.2}$$

holds in the sense of distributions, i.e.,

$$\sum_{\alpha,\beta,i,j=1}^{d} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_j D_\alpha v_i \; dx - \sum_{i,j=1}^{d} \int_\Omega p B_{ij} D_j v_i \; dx = [f, v] \tag{3.3}$$

holds for all $v \in W_0^{1,2}(\Omega)^d$.

Remark. If B is a regular matrix then $(u,p) \in W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega)$ is a weak solution to equations (3.1) in the sense of distribution iff (u,p) is a weak solution to equations

$$-\operatorname{div}(B^{-1}A\nabla u) + \nabla p = B^{-1} f \tag{3.4}$$

where we have denoted by $B^{-1}A$ a $d^2\times d^2$ matrix C with $C_{ij}^{\alpha\beta} = \sum_{k=1}^d (B^{-1})_{ik} A_{kj}^{\alpha\beta}$ for $i, j, \alpha, \beta = 1, \dots, d$, i.e.,

$$\sum_{\alpha,\beta,i,j=1}^{d} \int_\Omega \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} D_\beta u_j D_\alpha v_i \; dx - \sum_{i=1}^{d} \int_\Omega p \; D_i v_i \; dx = [B^{-1} f, v] \tag{3.5}$$

holds for all $v \in W_0^{1,2}(\Omega)^d$.

We assume throughout this section that A, B satisfy the following conditions:

(3a) B is constant regular matrix,

(3b) $A_{ij}^{\alpha\beta}$ belongs to $L^\infty(\Omega)$ and there is a positive Λ_A such that

$$\text{ess sup}\,|A_{ij}^{\alpha\beta}| \leq \Lambda_A \quad \text{for all} \quad i, j, \alpha, \beta = 1, \dots, d,$$

(3c) $B^{-1}A$ generates elliptic (generally nonsymetric) bilinear form a on $W_0^{1,2}(\Omega)^d$ where

$$a(u, v) = \int_\Omega (B^{-1} A \nabla u) : \nabla v \, dx = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} \Big(\sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} \Big) \, D^\alpha u_i \, D^\beta v_j \, dx$$

for $u, v \in W_{0,\mathrm{div}}^{1,2}(\Omega)$ and there exists a $\lambda > 0$ such that

$$a(v, v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} \, D^\alpha v_i \, D^\beta v_j \, dx \geq \lambda \|\nabla v\|_2^2$$

for all $v \in W_0^{1,2}(\Omega)^d$.
Under the above assumptions, we prove the existence and uniqueness of a weak solution (u, p) of system (3.1) for every right hand side $f \in W^{-1,2}(\Omega)^d$.

Theorem 3.1. *Let the assumptions* (3a), (3b), (3c) *be in force and* Ω *be a bounded Lipschitz domain, let* Ω_0 *be a nonempty subdomain of* Ω. *Suppose that* $f \in W^{-1,2}(\Omega)^d$. *Then there exists a unique pair* $(u, p) \in W_{0,\mathrm{div}}^{1,2}(\Omega) \times L^2(\Omega)$ *satisfying* $\int_{\Omega_0} p \, dx = 0$ *and solving system* (3.1).
Moreover, the inequality

$$\|u\|_{1,2} + \|p\|_2 \leq C \|f\|_{-1,2} \tag{3.6}$$

holds with a constant $C = C(A, B, \Omega, \Omega_0) > 0$.

Proof: It is obvious that $a(u, v)$ is a bilinear form on $W_{0,\mathrm{div}}^{1,2}(\Omega)$ and there is a constant $C = C(A, B, \Omega) > 0$ such that for all $u, v \in W_{0,\mathrm{div}}^{1,2}(\Omega)$

$$|a(u, v)| \leq C \|\nabla u\|_2 \|\nabla v\|_2 \leq C \|u\|_{1,2} \|v\|_{1,2}.$$

By the assumption (3c) and Poincaré's inequality we have

$$a(u, u) \geq \lambda \|\nabla u\|_2^2 \geq \frac{\lambda}{C} \|u\|_{1,2}^2$$

for all $u \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ with constant $C = C(\Omega) > 0$.
Applying the Lax-Milgram theorem, we conclude the existence and uniqueness of $u \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ satisfying

$$\int_\Omega (B^{-1}A)\nabla u : \nabla v \, dx = [B^{-1}f, v] \quad \text{for all} \quad v \in W^{1,2}_{0,\mathrm{div}}(\Omega). \tag{3.7}$$

By (3c), we obtain

$$\lambda \|\nabla u\|_2^2 \le \int_\Omega (B^{-1}A)\nabla u : \nabla v \, dx = [B^{-1}f, v] \le C\|f\|_{-1,2}\|u\|_{1,2},$$

so that

$$\|u\|_{1,2} \le C\|f\|_{-1,2} \tag{3.8}$$

with $C = C(A, B, \Omega) > 0$.

Now we focus on the existence of pressure p. Consider a functional $G : W^{1,2}_0(\Omega)^d \longrightarrow \mathbb{R}$ defined by

$$[G, v] := [B^{-1}f + \mathrm{div}(B^{-1}A\nabla u), v] = [B^{-1}f, v] - \int_\Omega (B^{-1}A)\nabla u : \nabla v \, dx.$$

From (3.7) we have $[G, v] = 0$ for all $v \in W^{1,2}_{0,\mathrm{div}}(\Omega)$.
Due to (3.8), it is easily seen that for all $v \in W^{1,2}_0(\Omega)$

$$|[G, v]| \le \|B^{-1}f\|_{-1,2}\|v\|_{1,2} + \Lambda_A|B^{-1}|\|u\|_{1,2}\|v\|_{1,2} \le C\|f\|_{-1,2}\|v\|_{1,2},$$

where a constant $C > 0$ depends on A, B and Ω. Therefore, Lemma 2.1 guarantees existence and uniqueness of $p \in L^2(\Omega)$ with $\nabla p = G$ and $\int_{\Omega_0} p \, dx = 0$. It implies that (u, p) is a weak solution of system (3.1). Moreover, we have

$$\|p\|_2 \le C\|G\|_{-1,2} \le C\|f\|_{-1,2} \tag{3.9}$$

with a constant $C = C(A, B, \Omega, \Omega_0) > 0$. From (3.8), (3.9), the inequality (3.6) follows.

To prove the uniqueness of (u, p), we suppose that $(\tilde{u}, \tilde{p}) \in W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega)$ is another pair solving (3.1). We see that

$$\int_\Omega (B^{-1}A)\nabla(u - \tilde{u}) : \nabla v \, dx = 0 \quad \text{for all} \quad v \in W^{1,2}_{0,\mathrm{div}}(\Omega).$$

Setting $v := u - \tilde{u}$, we obtain

$$0 = \int_\Omega (B^{-1}A)\nabla(u-\tilde{u}) : \nabla(u-\tilde{u})\,dx \geq \lambda\|\nabla(u-\tilde{u})\|_2.$$

It implies $\|\nabla(u-\tilde{u})\|_2 = 0$ and, as $u, \tilde{u} \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, also $u = \tilde{u}$.
Of course, the uniqueness of p follows from the above proof, when applying Lemma 2.1. □

Next, we will use Theorem 3.1 and solve a more general system

$$\begin{aligned} -\operatorname{div}(A\nabla u) + B\nabla p &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= g \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.10}$$

Theorem 3.2. *Let assumptions* (3a), (3b), (3c) *be in force and* Ω *be a bounded Lipschitz domain, let* Ω_0 *be a nonempty subdomain of* Ω. *Suppose that* $f \in W^{-1,2}(\Omega)^d$, $g \in L^2(\Omega)$ *such that* $\int_\Omega g\,dx = 0$. *Then there exists unique pair* $(u,p) \in W^{1,2}_0(\Omega)^d \times L^2(\Omega)$ *that solves the system* (3.10) *satisfying condition* $\int_{\Omega_0} p\,dx = 0$.
Moreover, (u,p) *satisfies the inequality*

$$\|u\|_{1,2} + \|p\|_2 \leq C\left(\|f\|_{-1,2} + \|g\|_2\right) \tag{3.11}$$

with a constant $C = C(A,B,\Omega,\Omega_0) > 0$.

Remark. We show that u is of the form $u = u_0 + u_1$ with $u_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ and $u_1 \in W^{1,2}_0(\Omega)^d$, $\operatorname{div} u_1 = g$ in Ω.

Proof: According to Lemma 2.1, we can choose $u_1 \in W^{1,2}_0(\Omega)^d$ satisfying $\operatorname{div} u_1 = g$ and

$$\|\nabla u_1\|_2 \;\leq\; C\|g\|_2. \tag{3.12}$$

Then using Theorem 3.1, we find a unique pair $(u_0,p) \in W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega)$ satisfying $\int_{\Omega_0} p\,dx = 0$ and $-\operatorname{div}(A\nabla u_0) + B\nabla p = f + \operatorname{div}(A\nabla u_1)$. If we set $u := u_0 + u_1$, then the pair (u,p) solves the system (3.6).
From Theorem 3.1, the inequalities (3.8) and (3.12) we have the estimate

$$\begin{aligned} \|u\|_{1,2} + \|p\|_2 &\leq C\left(\|\nabla u_0\|_2 + \|p\|_2 + \|\nabla u_1\|_2\right) \\ &\leq C\left(\|f\|_{-1,2} + \Lambda_A\|u_1\|_{1,2}\right) + \|u_1\|_{1,2} \\ &\leq C\left(\|f\|_{-1,2} + \|g\|_2\right) \end{aligned} \tag{3.13}$$

with a constant $C = C(A, B, \Omega, \Omega_0) > 0$.

To prove uniqueness we suppose that $(\tilde{u}, \tilde{p})$ is another pair solving system (3.10) and $\tilde{u}$ has a decomposition $\tilde{u} = \tilde{u}_0 + \tilde{u}_1$ where $\tilde{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, $\tilde{u}_1 \in W^{1,2}_0(\Omega)$, $\operatorname{div} \tilde{u}_1 = g$; $\int_{\Omega_0} \tilde{p}\ dx = 0$. Then $u_1 - \tilde{u}_1 \in W^{1,2}_0(\Omega)^d$ and $\operatorname{div}(u_1 - \tilde{u}_1) = 0$. Therefore $(u - \tilde{u}, p - \tilde{p})$ is a solution to (3.1) as $\operatorname{div}(u - \tilde{u}) = 0$, $f = 0$, $\int_{\Omega_0} (p - \tilde{p})\ dx = 0$. The uniqueness result established in Theorem 3.1 implies that $u = \tilde{u}$, $p = \tilde{p}$. □

Examples. To illustrate the type of systems we have in mind we show some examples that satisfy conditions (3a), (3b), (3c).

Proposition 3.1 (A elliptic, B near to identity)**.** *Suppose that*

- $A^{\alpha\beta}_{ij}$ *belong to* $L^\infty(\Omega)$ *and there is a positive* Λ_A *such that*

$$\text{ess sup}\, |A^{\alpha\beta}_{ij}| \le \Lambda_A \quad \textit{for all} \quad i, j, \alpha, \beta = 1, \dots, d,$$

- A *generates an elliptic bilinear form* a *on* $W^{1,2}_0(\Omega)^d$ *i.e. there is a positive constant* λ_A *such that*

$$a(v, v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A^{\alpha\beta}_{ij}\ D^\alpha v_i\ D^\beta v_j\ dx \ge \lambda_A \|\nabla v\|_2^2 \quad \textit{for all} \quad v \in W^{1,2}_0(\Omega)^d,$$

- B *is a constant* $d \times d$ *matrix such that*

$$\zeta = |B - E| < \frac{\lambda_A}{\lambda_A + d^4 \Lambda_A}\,, \tag{3.14}$$

  where E *is the identity* $d \times d$ *matrix.*

Then conditions (3a), (3b) *and* (3c) *hold.*

Proof: We need only to check condition (3c).
We have $B = E - (E - B)$ and because of the assumption (3.14), B is regular and $B^{-1} = \sum_{l=0}^\infty (E - B)^l$. Thus, the conditions (3a), (3b) are satisfied. We have for all $v \in W^{1,2}_0(\Omega)^d$

$$\begin{aligned} &\int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A^{\alpha\beta}_{kj}\ D^\alpha v_i\ D^\beta v_j\ dx \\ &\quad = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A^{\alpha\beta}_{ij}\ \ D^\alpha v_i\ D^\beta v_j\ \ dx + \sum_{l=1}^{\infty} \int_\Omega ((E - B)^l A)\nabla v : \nabla v\ dx \end{aligned}$$

$$\begin{aligned}
&\geq \lambda_A \|\nabla v\|_2^2 - \sum_{l=1}^{\infty} |(B-E)^l| \|A\|_\infty \|\nabla v\|_2^2 \\
&\geq \|\nabla v\|_2^2 (\lambda_A - \frac{\zeta}{1-\zeta} \Lambda_A d^4) \\
&\geq \epsilon \|\nabla v\|_2^2,
\end{aligned}$$

where positive ϵ is so small that $\zeta < \frac{\lambda_A - \epsilon}{\Lambda_A d^4 + \lambda_A - \epsilon} < \frac{\lambda_A}{\Lambda_A d^4 + \lambda_A}$. The condition (3c) is satisfied with $\lambda = \epsilon$.

Remark. Note that A is elliptic for example if

$$a(u,v) = \int_\Omega \sum_{\alpha,\beta,i,j=1}^{d} A_{ij}^{\alpha\beta} \, D^\alpha u_i \, D^\beta v_j \, dx$$

and there is a positive λ_A such that

$$\sum_{\alpha,\beta,i,j=1}^{d} A_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \geq \lambda_A |\xi|^2 \ \text{ for all } \ \xi \in R^{d\times d},$$

or

$$a(u,v) = \int_\Omega \sum_{i,j,k,l=1}^{d} A_{ij}^{kl} \, D^{ij} u \, D^{kl} v \, dx$$

where $D^{ij} u = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the symmetric part of ∇u and there is a positive λ_A such that

$$\sum_{i,j,k,l=1}^{d} A_{ij}^{kl} \eta_{ij} \eta_{kl} \geq \lambda_A |\eta|^2 \ \text{ for all symmetric } \ \eta \in R^{d\times d}.$$

$\square$

Proposition 3.2 (A Laplace operator on the diagonal, B positive definite)**.** *Suppose that* $\operatorname{div}(A\nabla v)$ *is Laplace operator on* v_j *in the* j*-th equation,* $j = 1, \ldots, d$*, i.e.,*

$$A_{ij}^{\alpha\beta} = \delta_{\alpha\beta}\delta_{ij} \ \textit{ for all } \ i,j,\alpha,\beta = 1,\ldots,d,$$

and B *is constant, self adjoint and positive definite matrix.*
Then conditions (3a), (3b), (3c) *are satisfied.*

Remark. Under the assumptions of Proposition 3.2, the system (3.10) takes the form

$$\begin{aligned} -\triangle u + B\nabla p &= f \ \text{ in } \ \Omega, \\ \operatorname{div} u &= g \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega. \end{aligned} \tag{3.15}$$

Proof: It is easy to check the validity of the conditions (3a), (3b). We prove that the condition (3c) is satisfied as well.

As B is self adjoint and positive definite then B^{-1} is also self adjoint positive definite, i.e., there exists constant $\lambda_{B^{-1}} > 0$ such that

$$\sum_{i,j} (B^{-1})_{ij} \xi_i \xi_j \geq \lambda_{B^{-1}} |\xi|^2 \ \text{ for all } \ \xi \in \mathbb{R}^d.$$

Hence we have

$$\begin{aligned} \sum_{i,j,\alpha,\beta}^{d} \left[\sum_{k=1}^{d} (B^{-1})_{ik} \, A_{kj}^{\alpha\beta} \right] \xi_\alpha^i \, \xi_\beta^j &= \sum_{i,j,\alpha,\beta}^{d} \left[\sum_{k=1}^{d} (B^{-1})_{ik} \, \delta_{kj} \, \delta_{\alpha\beta} \right] \xi_\alpha^i \, \xi_\beta^j \\ &= \sum_{i,j,\alpha,\beta}^{d} \left[(B^{-1})_{ij} \, \delta_{\alpha\beta} \right] \xi_\alpha^i \, \xi_\beta^j \\ &= \sum_{i,j,\alpha}^{d} \left[(B^{-1})_{ij} \, \right] \xi_\alpha^i \, \xi_\alpha^j \\ &\geq \lambda_{B^{-1}} |\xi|^2 \ \text{ for all } \ \xi \in \mathbb{R}^d. \end{aligned}$$

Thus (3c) is satisfied and it completes the proof. □

Counterexample 3.3. If B is not regular it is easily seen that system (3.10) need not have in general any solution u, reason being that the system is overdetermined. If, for example, A is the Laplace operator on the diagonal, $B = 0$, $d = 2$, $\Omega := (0, \pi) \times (0, \pi)$ and $f = (2 \sin x_1 \sin x_2, 0)$, the system (3.1) reduces to

$$\begin{aligned} -\triangle u &= f \ \text{ in } \ \Omega, \\ \operatorname{div} u &= 0 \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega. \end{aligned} \tag{3.16}$$

By elementary calculation, the system

$$\begin{aligned} -\triangle u &= f \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

has a unique solution $u = (\sin x_1 \sin x_2, 0)$. This solution does not satisfy the equation $\operatorname{div} u = 0$ in Ω ($\operatorname{div} u = \cos x_1 \sin x_2$). Consequently, the system (3.16) has no solution.

4. Regularity of (u,p) in $W^{k,2}(\Omega)^d \times W^{k-1,2}(\Omega)$

Our purpose is to investigate regularity of solutions to a generalized Stokes system

$$\begin{aligned} -\operatorname{div}(A\nabla u) + B\nabla p &= f \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= g \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{4.1}$$

where A is a $d^2 \times d^2$ matrix and B is a $d \times d$ matrix of sufficiently smooth functions. We assume throughout this section that A and B satisfy the following conditions:

(4a) B is regular,

(4b) $B^{-1}A$ satisfies uniformly the strong ellipticity condition, i.e. there exists a positive λ so that

$$\sum_{\alpha,\beta,i,j=1}^{d} \sum_{k=1}^{d} (B^{-1})_{ik} A_{kj}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \ \geq \lambda |\xi|^2 \quad \text{in} \quad \Omega \quad \text{for all} \quad \xi \in R^{d\times d}.$$

Under the assumption (4a) and assuming that $A_{ij}^{\alpha\beta},\ B_{ij} \in C^{0,1}(\overline{\Omega})$ for all $i, j, \alpha, \beta = 1, \dots, d$; the system (4.1) can be transformed to

$$\begin{aligned} -\operatorname{div}(\bar{A}\nabla u) + \nabla p &= \bar{f} \quad \text{in} \quad \Omega, \\ \operatorname{div} u &= g \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{4.2}$$

where $\bar{f} = (\bar{f})_{i=1}^d := \big((B^{-1}f)_i - (\sum_{\alpha,\beta,j,k=1}^d D_\alpha (B^{-1})_{ik} A_{kj}^{\alpha\beta} D_\beta u_j)\big)_{i=1}^d$, $\bar{A} := B^{-1}A$.

We show that any solution pair $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ under natural conditions on f, g, A, B, Ω satisfies $u \in W^{k+2,2}(\Omega)^d$ and $p \in W^{k+1,2}(\Omega)$, $(k \in \mathbb{N})$.

Theorem 4.1. *Let $k \in \mathbb{N}$, Ω be a bounded C^{k+2}-domain in R^d, $(d \geq 2)$. Suppose that $f \in W^{k,2}(\Omega)^d$, $g \in W^{k+1,2}(\Omega)$, $A, B \in C^{k,1}(\overline{\Omega})$ fulfilling* (4a) *and* (4b), *and $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ be a weak solution of system* (4.1). *Then we have*

$$u \in W^{k+2,2}(\Omega)^d,\ p \in W^{k+1,2}(\Omega), \tag{4.3}$$

and the inequality

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \leq C\left(\|f\|_{k,2} + \|g\|_{k+1,2} + \|u\|_{1,2} + \|p\|_2\right) \tag{4.4}$$

holds with a constant $C = C(A,B,\Omega) > 0$.

Proof: We shall prove Theorem 4.1 for $k = 0$ and indicate how the proof can be continued by induction for $k \in \mathbb{N}$.

Let $k = 0$. In Lemmas 4.1–4.3 we prove the assertion under auxiliary assumptions on supports of u and p. Using the decomposition of Ω, partition of unity and these results we shall complete the proof of Theorem 4.1 for Ω.

Lemma 4.1. *Let Ω be a bounded domain in $\mathbb{R}^d$, $(d \geq 2)$, $R_0 > 0$, $x_0 \in \Omega$. Suppose that*

$$f \in L^2(\Omega)^d,\ g \in W^{1,2}(\Omega),\ A, B \in C^{0,1}(\overline{\Omega}).$$

Let $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ be a weak solution to the system (4.1) and $\operatorname{supp} u$, $\operatorname{supp} p \subset B_{R_0}(x_0) \subset\subset \Omega$. *Then*

$$u \in W^{2,2}(\Omega)^d,\ p \in W^{1,2}(\Omega), \tag{4.5}$$

and

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\left(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2\right) \tag{4.6}$$

with a constant $C = C(A, B, \Omega) > 0$.

Proof: By assumptions of Lemma 4.1, it is easily seen that $\bar{A}$ defines an elliptic bilinear form, $\bar{A} \in C^{0,1}(\overline{\Omega})$, $\bar{f} \in L^2(\Omega)$ and we have

$$\|\bar{f}\|_2 \leq C(\|f\|_2 + \|\nabla u\|_2). \tag{4.7}$$

Let e_s denote the unit vector in the x_s direction ($s = 1, \dots, d$). For $0 < \delta < \operatorname{dist}(\partial\Omega, B_{R_0}(x_0))$, $s = 1, \dots, d$, the difference quotient in the x_s direction is denoted through $\triangle_{\delta,s} u = \frac{1}{\delta}[u(x + \delta e_s) - u(x)]$.
Since (u,p) solve the system (4.1) we have

$$\begin{aligned} - \operatorname{div}[\bar{A}(x + \delta e_s)\nabla u(x + \delta e_s)] + \nabla p(x + \delta e_s) &= \bar{f}(x + \delta e_s), \\ \operatorname{div} u(x + \delta e_s) &= g(x + \delta e_s) \end{aligned}$$

on $B_{R_0}(x_0)$. Subtracting (4.2) from those equations and then dividing by δ, we obtain

$$\begin{aligned} &- \operatorname{div}\left[\bar{A}(x)\nabla(\triangle_{\delta,s} u(x))\right] + \nabla(\triangle_{\delta,s} p(x)) \\ &\quad = \triangle_{\delta,s}\bar{f}(x) + \operatorname{div}[\triangle_{\delta,s}(\bar{A}(x))\nabla u(x + \delta e_s)] \quad \text{on} \quad B_{R_0}(x_0), \\ &\operatorname{div}(\triangle_{\delta,s} u(x)) = \triangle_{\delta,s} g(x) \quad \text{on} \quad B_{R_0}(x_0). \end{aligned} \tag{4.8}$$

On the other hand, we observe that (thanks to Nirenberg's lemma — see Exercise 2.10, II.2 in [6] and Lemma 2.1) $\triangle_{\delta,s}\bar{f}$, $\triangle_{\delta,s} g(x)$ can be written correspondingly in the form $\operatorname{div} F_{\delta,s}$, $\operatorname{div} G_{\delta,s}$ with some $F_{\delta,s} \in L^2(\Omega)^{d^2}$, $G_{\delta,s} \in W_0^{1,2}(\Omega)^d$ such that $\|F_{\delta,s}\|_2 \leq C\|\bar{f}\|_2$, $\|\nabla G_{\delta,s}\|_2 \leq C\|\triangle_{\delta,s} g\|_2$ with C independent of δ and s.

Using definition of weak solution to the system (4.8) and taking $(\triangle_{\delta,s}u - G_{\delta,s}) \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ as the test function, we obtain

$$\begin{aligned}\int_\Omega \bar{A}(x)\left[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)\right] : \left[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)\right]\, dx \\ + \int_\Omega \bar{A}(x)[\nabla G_{\delta,s}(x)] : \left[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)\right]\, dx \\ = -\int_\Omega F_{\delta,s}.[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\, dx \\ - \int_\Omega \triangle_{\delta,s}\bar{A}(x)[\nabla u(x+\delta e_s)] : [\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)]\, dx.\end{aligned}$$

Assumptions of Lemma 4.1 and (4b) then lead to

$$\begin{aligned}&\|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2^2 \\ &\le \frac{1}{\lambda}\int_\Omega \bar{A}(x)\left[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)\right] : \left[\nabla(\triangle_{\delta,s}u(x)) - \nabla G_{\delta,s}(x)\right]\, dx \\ &\le \frac{C}{\lambda}(\|\bar{A}\|_{C^0(\overline{\Omega})}\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}\|\nabla u\|_2)\|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2,\end{aligned}$$

where we estimated $\triangle_{\delta,s}\bar{A}$ by $\|\bar{A}\|_{C^{0,1}(\overline{\Omega})}$. It implies that

$$\|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2 \le \frac{C}{\lambda}\left(\|\bar{A}\|_{C^0(\overline{\Omega})}\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}\|\nabla u\|_2\right)$$

and

$$\begin{aligned}\|\nabla\triangle_{\delta,s}u\|_2 &\le \|\nabla\triangle_{\delta,s}u - \nabla G_{\delta,s}\|_2 + \|\nabla G_{\delta,s}\|_2 \\ &\le C[(\|\bar{A}\|_{C^0(\overline{\Omega})} + 1)\|\triangle_{\delta,s}g\|_2 + \|\bar{f}\|_2 + \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}\|\nabla u\|_2]\end{aligned} \tag{4.9}$$

with a constant $C = C(\lambda, \Omega) > 0$ not depending on $\delta,\ s$.
As $\operatorname{supp} p \subset B_{R_0}(x_0)$ and δ is small, we have $\int_\Omega \triangle_{\delta,s}p\ dx = 0$. Applying Lemma 2.1 to system (4.8) and using the inequality (4.9) we conclude that $\triangle_{\delta,s}p \in L^2(\Omega)$ for all $s = 1,\dots,d$ and we have the estimate

$$\begin{aligned}\|\triangle_{\delta,s}p\|_2 \le C[(\|\bar{A}\|^2_{C^0(\overline{\Omega})} + \|\bar{A}\|_{C^0(\overline{\Omega})})\|\triangle_{\delta,s}g\|_2 + (\|\bar{A}\|_{C^0(\overline{\Omega})}) + 1)\|\bar{f}\|_2 \\ + \|\bar{A}\|_{C^{0,1}(\overline{\Omega})}(\|\bar{A}\|_{C^0(\overline{\Omega})}) + 1)\|\nabla u\|_2]\end{aligned} \tag{4.10}$$

with a constant $C = C(\lambda, \Omega) > 0$ not depending on δ, s.
If we let $\delta \to 0$ in inequalities (4.9), (4.10), we deduce that $D_s\nabla u \in L^2(\Omega)^{d^2}$, $D_s p \in L^2(\Omega)$ for all $s = 1,\dots,d$ and we have estimate

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \le C\left(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2\right)$$

with a constant $C = C(A, B, \Omega)$. Lemma 4.1 is proved. $\square$

Remarks. Lemma 4.1 holds under weaker ellipticity assumptions satisfied by examples presented in the previous section.

The next lemma deals with estimates near the flat boundary. It is given here to explain the main ideas of the proof and will not be explicitly used later.

For $R_0 > 0$, $\beta_0 > 0$, $x_0 = [x'_0, 0] \in \partial\Omega$ denote

$$\begin{aligned}\Gamma_{R_0} &= \left\{x = [x', 0] \in \mathbb{R}^d;\ x' \in B'_{R_0}\right\},\\ U^+_{R_0,\beta_0} &= \left\{x = [x', x_d] \in \mathbb{R}^d;\ x' \in B'_{R_0}; 0 < x_d < \beta_0\right\},\\ U_{R_0,\beta_0} &= \left\{x = [x', x_d] \in \mathbb{R}^d;\ x' \in B'_{R_0}; |x_d| < \beta_0\right\}.\end{aligned}$$

Lemma 4.2. *Let R_0, β_0 be positive; $\Gamma_{R_0} \subset \partial\Omega$, $U^+_{2R_0,2\beta_0} \subset \Omega$. Suppose that*

$$f \in L^2(\Omega)^d,\ g \in W^{1,2}(\Omega),\ A, B \in C^{0,1}(\overline{\Omega}).$$

Let $(u,p) \in W^{1,2}_0(\Omega)^d \times L^2(\Omega)$ be a weak solution of the system (4.1) such that

$$\operatorname{supp} u,\ \operatorname{supp} p \subset U^+_{R_0,\beta_0} \cup \Gamma_{R_0}.$$

Then it holds

$$u \in W^{2,2}(\Omega)^d,\ p \in W^{1,2}(\Omega), \tag{4.11}$$

and

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \le C\,(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2) \tag{4.12}$$

with a constant $C = C(A,B,\Omega) > 0$.

Proof: By the same way as in Lemma 4.1 we get

$$D_s \nabla u \in L^2(\Omega)^{d^2},\ D_s p \in L^2(\Omega)^d \ \text{ for all } \ s = 1, \dots, d-1,$$

and we have

$$\|\nabla' \nabla u\|_2 + \|\nabla' p\|_2 \le C\,(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2) \tag{4.13}$$

with a constant $C = C(A,B,\Omega) > 0$.

Using the structure of the system (4.2) we have

$$\sum_{j=1}^{d-1} \bar{A}_{ij}^{dd} D_d^2 u_j = G_i, \ i = 1, \dots, d-1, \tag{4.14}$$

$$\sum_{j=1}^{d} \bar{A}_{dj}^{dd} D_d^2 u_j = G_d + D_d p, \tag{4.15}$$

$$D_d^2 u_d = D_d g - \sum_{j=1}^{d-1} D_d D_j u_j, \tag{4.16}$$

where

$$\begin{aligned} G_i = D_i p - \bar{f}_i - \sum_{\alpha+\beta<2d} \bar{A}_{ij}^{\alpha\beta} D_\alpha D_\beta u_j \\ - \sum_{\alpha,\beta,j=1}^{d} (D_\alpha \bar{A}_{ij}^{\alpha\beta}) D_\beta u_j - \bar{A}_{id}^{dd} D_d^2 u_d, \ i = 1, \dots, d-1, \end{aligned} \tag{4.17}$$

$$G_d = -\bar{f}_d - \sum_{\alpha+\beta<2d} \bar{A}_{dj}^{\alpha\beta} D_\alpha D_\beta u_j - \sum_{\alpha,\beta,j=1}^{d} D_\alpha (\bar{A}_{dj}^{\alpha\beta}) D_\beta u_j. \tag{4.18}$$

From $D_j D_d u_j \in L^2(\Omega)$, $j = 1, \dots, d-1$, $g \in W^{1,2}(\Omega)$ and the equation (4.16), it follows $D_d^2 u_d \in L^2(\Omega)$. Since $f \in L^2(\Omega)^d$, $\nabla' p \in L^2(\Omega)^{d-1}$, $D' \nabla u \in L^2(\Omega)^{(d-1)d}$, (4.17), (4.18) shows that $G_i \in L^2(\Omega), \ i = 1, \dots, d$.

System (4.14) consists of $(d-1)$ linear equations, the matrix $\bar{A}_{ij}^{dd}$, $i, j = 1, \dots, d-1$ is regular and its inverse is bounded by $\frac{1}{\lambda}$. Therefore, the L^2-integrability of $D_d^2 u_j$, $j = 1, \dots, d-1$, follows from L^2-integrability of G_i, $i = 1, \dots, d-1$. Since $\bar{A} \in C^{0,1}(\overline{\Omega})$, we conclude that $D_d^2 u_j \in L^2(\Omega)$, for $j = 1, \dots, d-1$, and obtain that

$$\|D_d^2 u_j\|_2 \le C \sum_{i=1}^{d-1} \|G_i\|_2 \ \text{ for } \ j = 1, \dots, d-1. \tag{4.19}$$

Finally, since $\bar{f}_d$, $D_d^2 u_j \in L^2(\Omega)$, $j = 1 \dots, d$, $\bar{A} \in C^{0,1}(\overline{\Omega})$, it follows from the equation (4.13) that $D_d p \in L^2(\Omega)$ and (4.13), (4.16), (4.17), (4.18), (4.19), (4.15) imply the estimates

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \le C \left(\|f\|_2 + \|\nabla g\|_2 + \|\nabla u\|_2 \right)$$

with some constant $C = C(A, B, \Omega) > 0$. The lemma is proved. □

Lemma 4.3. *Let Ω be a bounded C^2-domain in $\mathbb{R}^d$, $R_0 > 0$, $\beta_0 > 0$, $x_0 \in \partial\Omega$. Suppose that*

$$f \in L^2(\Omega)^d,\ g \in W^{1,2}(\Omega),\ A, B \in C^{0,1}(\overline{\Omega}).$$

Let $(u,p) \in W_0^{1,2}(\Omega)^d \times L^2(\Omega)$ be a weak solution of the system (4.1), and $\operatorname{supp} u, \operatorname{supp} p \subset\ U_{R_0,\beta_0,h}(x_0) \cap \overline{\Omega}$.

Then there exists a constant $K > 0$ (given in (4.28)) so that for

$$\|h\|_{C^1(\overline{B'_{R_0}})} \leq K, \tag{4.20}$$

it holds

$$u \in W^{2,2}(\Omega)^d,\ p \in W^{1,2}(\Omega) \tag{4.21}$$

and

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \leq C\left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right). \tag{4.22}$$

Proof: In order to reduce the proof of Lemma 4.3 to previous case we use the transformation to new coordinates

$$y = \Phi(x) := (x', x_d - h(x')),\ x \in U_{R_0,\beta_0,h}(x_0). \tag{4.23}$$

We see that Φ is one-to-one mapping of $U_{R_0,\beta_0,h}(x_0)$ on U_{R_0,β_0}. Next, define $\hat{u}, \hat{p}, \hat{f}, \hat{g}, \hat{A}$ by

$$\begin{aligned}
&\hat{u}(y) := u(\Phi^{-1}(y)) = u(x),\ \hat{p}(y) := p(\Phi^{-1}(y)) = p(x),\\
&\hat{f}(y) := \bar{f}(\Phi^{-1}(y)) = \bar{f}(x),\ \hat{g}(y) := g(\Phi^{-1}(y)) = g(x),\\
&\hat{A}(y) := \bar{A}(\Phi^{-1}(y)) = \bar{A}(x).
\end{aligned} \tag{4.24}$$

We have also $u(x) = \hat{u}(\Phi(x))$ so that

$$D_\beta u(x) = D_\beta \hat{u}(y) - D_d \hat{u}(y) D_\beta h(y'),\ \beta = 1, \dots, d-1,\ D_d u(x) = D_d \hat{u}(y)$$

and correspondingly for $p, g, \bar{f}, \bar{A}$. An elementary calculation transforms (4.1) to a new system

$$\begin{aligned}
-\operatorname{div}(\tilde{A}\nabla\hat{u}) + \nabla\hat{p} &= \hat{f} + T - (D_d H_1)p + D_d(H_1 p),\\
\operatorname{div}\hat{u} &= \hat{g} + H_2,
\end{aligned} \tag{4.25}$$

where a $d^2 \times d^2$ matrix $\tilde{A} := (\tilde{A}_{ij}^{\alpha\beta})_{i,j,\alpha,\beta=1}^{d}$, a vector H_1 and a function H_2 are given by

$$\tilde{A}_{ij}^{\alpha\beta} := \hat{A}_{ij}^{\alpha\beta}, \ \text{ if } \ \alpha,\beta < d; \ \tilde{A}_{ij}^{\alpha d} := \hat{A}_{ij}^{\alpha d} - \sum_{\beta=1}^{d-1} \hat{A}_{ij}^{\alpha\beta} D_\beta h, \ \text{ if } \ \alpha < d;$$

$$\tilde{A}_{ij}^{d\beta} := \hat{A}_{ij}^{d\beta} - \sum_{\alpha=1}^{d-1} \hat{A}_{ij}^{\alpha\beta} D_\alpha h, \ \text{ if } \ \beta < d;$$

$$\tilde{A}_{ij}^{dd} := \hat{A}_{ij}^{dd} - \sum_{\alpha=1}^{d-1} \hat{A}_{ij}^{\alpha d} D_\alpha h - \sum_{\beta=1}^{d-1} \hat{A}_{ij}^{d\beta} D_\beta h + \sum_{\alpha,\beta=1}^{d-1} \hat{A}_{ij}^{\alpha\beta} D_\alpha h D_\beta h;$$

$$H_1 := (D_1 h, D_2 h, \ldots, D_{d-1} h, 0);$$

$$H_2 := \sum_{j=1}^{d-1} D_d \hat{u}_j D_j h;$$

$$T := (T_i)_{i=1}^{d} \text{ with } T_i := \sum_{\alpha,\beta,j=1}^{d} [D_\alpha \hat{A}_{ij}^{\alpha\beta} - (1-\delta_{\alpha d}) D_d \hat{A}_{ij}^{\alpha\beta} D_\alpha h][D_\beta \hat{u}_j - (1-\delta_{\beta d}) D_d \hat{u}_j D_\beta h] - \sum_{\alpha,\beta,j=1}^{d} D_\alpha \tilde{A}_{ij}^{\alpha\beta} D_\beta \hat{u}_j.$$

The assumption (4b) and the assumptions of Lemma 4.3 imply that there exists a constant $K_1 > 0$ such that if $\|h\|_{C^1(\overline{B'_{R_0}})} \le K_1$, then

(4c) $\sum_{\alpha,\beta,i,j=1}^{d} \tilde{A}_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \ \ge \frac{\lambda}{2} |\xi|^2$ for all $\xi \in \mathbb{R}^{d^2}$.
Thus

$$\Theta := \det \begin{pmatrix} \tilde{A}_{ij}^{dd} & D_i h \\ \tilde{A}_{dj}^{dd} & -1 \end{pmatrix}_{i,j=1}^{d-1} = -\det \left(\tilde{A}_{ij}^{dd}\right)_{i,j=1}^{d-1} + \sum_{k=1}^{d} (-1)^{k+d} D_k h \ \det \left(\tilde{A}_{ij}^{dd}\right)_{i\neq k}^{j\neq d}$$

$$\le -\det \left(\tilde{A}_{ij}^{dd}\right)_{i,j=1}^{d-1} + \|h\|_{C^1(\overline{B'_{R_0}})} C(\|\bar{A}\|_{C^0(\overline{\Omega})}, d)$$

with a constant $C(\|\bar{A}\|_{C^0(\overline{\Omega})}, d) > 0$.
If (4c) holds, then it is easy to check that there exists constant $C(\lambda, d) > 0$ such that $\det \left(\tilde{A}_{ij}^{dd}\right)_{i,j=1}^{d-1} > C(\lambda, d)$.
Therefore there exists constant $K_2 \in (0,1)$ such that (4c) holds, Θ is uniformly bounded away from zero and $\sum_{j=1}^{d-1} |D_j h| < 1$ for all $h \in C^1(\overline{B'_{R_0}})$ such that $\|h\|_{C^1(\overline{B'_{R_0}})} \le K_2$.

Using (4.23), (4.24) and the assumptions of Lemma 4.3 we have $\operatorname{supp}\hat{u}, \operatorname{supp}\hat{f} \subset U_{R_0,\beta_0} \cap \overline{\mathbb{R}}^d_+$. We set $Q := U_{R_0,\beta_0} \cap \mathbb{R}^d_+$.
Similarly as in Lemma 4.1, we obtain the estimate of $\triangle_{\delta,s}\nabla\hat{u}$, $\triangle_{\delta,s}\hat{p}$ for $s = 1, \dots, d-1$. Thus

$$\begin{aligned}(4.26)\quad \|\triangle_{\delta,s}\nabla\hat{u}\|_{2;Q} + \|\triangle_{\delta,s}\hat{p}\|_{2;Q} &\le C[(\|\tilde{A}\|^2_{C^0(\overline{Q})} + 1)(\|\triangle_{\delta,s}\hat{g}\|_{2;Q} \\ &+ \|h\|_{C^1(\overline{B'_{R_0}})}\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;Q} + \|h\|_{C^2(\overline{B'_{R_0}})}\|\nabla\hat{u}\|_{2;Q}) \\ + (\|\tilde{A}\|_{C^0(\overline{Q})} + 1)(\|\hat{f}\|_{2;Q} + \|T\|_{2;Q} &+ \|h\|_{C^1(\overline{B'_{R_0}})}\|\triangle_{\delta,s}\hat{p}\|_{2;Q} + \|h\|_{C^2(\overline{B'_{R_0}})}\|\hat{p}\|_{2;Q}) \\ &+ (\|\tilde{A}\|^2_{C^1(\overline{Q})} + 1)\|\nabla\hat{u}\|_{2;Q}]\end{aligned}$$

with a constant $C = C(\lambda, Q) > 0$.
Here we used the estimates

$$\begin{aligned}\|\triangle_{\delta,s}(D_i h p)\|_{2;Q} &\le \|h\|_{C^1(\overline{B'_{R_0}})}\|\triangle_{\delta,s}\hat{p}\|_{2;Q} + C\|h\|_{C^2(\overline{B'_{R_0}})}\|\hat{p}\|_{2;Q}, \\ \|\triangle_{\delta,s}(D_i h D_d \hat{u}_j)\|_{2;Q} &\le \|h\|_{C^1(\overline{B'_{R_0}})}\|\triangle_{\delta,s} D_d\hat{u}_j\|_{2;Q} + C\|h\|_{C^2(\overline{B'_{R_0}})}\|D_d\hat{u}_j\|_{2;Q}.\end{aligned}$$

As $h \in C^1(\overline{B'_{R_0}})$, there exists a constant $C > 0$ which does not depend on h, δ such that $\|\tilde{A}\|_{C^0(\overline{Q})} \le C\|\bar{A}\|_{C^0(\overline{\Omega})}$, therefore we have estimate

$$\begin{aligned}(4.27)\quad \|\triangle_{\delta,s}\nabla\hat{u}\|_{2;Q} + \|\triangle_{\delta,s}\hat{p}\|_{2;Q} &\le C[\|\triangle_{\delta,s}\hat{g}\|_{2;Q} + \|h\|_{C^2(\overline{B'_{R_0}})}\|\nabla\hat{u}\|_{2;Q} \\ + \|\hat{f}\|_{2;Q} + \|T\|_{2;Q} &+ \|h\|_{C^2(\overline{B'_{R_0}})}\|\hat{p}\|_{2;Q} + \|\tilde{A}\|_{C^{0,1}(\overline{Q})}\|\nabla\hat{u}\|_{2;Q} \\ &+ \|h\|_{C^1(\overline{B'_{R_0}})}(\|\triangle_{\delta,s}\nabla\hat{u}\|_{2;Q} + \|\triangle_{\delta,s}\hat{p}\|_{2;Q})]\end{aligned}$$

with a constant $C(\lambda, \|\bar{A}\|_{C^0(\overline{\Omega})}) > 0$.

Next, we can choose

$$(4.28)\qquad K = \min(K_2, \frac{C^{-1}}{2}).$$

Then we have

$$\begin{aligned}(4.29)\quad \|\triangle_{\delta,s}\nabla\hat{u}\|_{2;Q} + \|\triangle_{\delta,s}\hat{p}\|_{2;Q} &\le C(\|\triangle_{\delta,s}\hat{g}\|_{2;Q} + \|h\|_{C^2(\overline{B'_{R_0}})}\|\nabla\hat{u}\|_{2;Q} \\ &+ \|\hat{f}\|_{2;Q} + \|T\|_{2;Q} + \|h\|_{C^2(\overline{B'_{R_0}})}\|\hat{p}\|_{2;Q} + \|\tilde{A}\|_{C^{0,1}(\overline{Q})}\|\nabla\hat{u}\|_{2;Q})\end{aligned}$$

with a constant $C(\lambda, \|\bar{A}\|_{C^0(\overline{\Omega})}) > 0$ that does not depend on h, s and δ.

Letting $\delta \to 0$ in the inequality (4.29), we deduce that $D_s \nabla \hat{u} \in L^2(Q)^{d^2}$, $D_s \hat{p} \in L^2(Q)$ for all $s = 1, \dots, d-1$ and we have the estimate

$$\|\nabla' \nabla \hat{u}\|_{2;Q} + \|\nabla' \hat{p}\|_{2;Q} < C \left(\|\hat{f}\|_{2;Q} + \|\nabla \hat{g}\|_{2;Q} + \|\nabla \hat{u}\|_{2;Q} + \|\hat{p}\|_{2;Q} \right) \tag{4.30}$$

with a constant $C(\hat{A}, Q) > 0$.

Adopting arguments of Lemma 4.1 we obtain that all second derivatives of $\hat{u}$ and first derivatives of $\hat{p}$ exist in Q and we have estimates of their L_2-norms in any strict subdomain. From the first part of this proof we have estimates of $\nabla' \nabla \hat{u}$ and $\nabla' \hat{p}$ up to the boundary of Q. For getting estimates of the remaining terms $D_d^2 \hat{u}_j$, $D_d \hat{p}$ up to the boundary we use system (4.25).
From the system (4.25), we have

$$\sum_{j=1}^{d-1} \tilde{A}_{ij}^{dd} D_d^2 \hat{u}_j + D_i h D_d \hat{p} = \hat{G}_i, \; i = 1, \dots, d-1; \tag{4.31}$$

$$\sum_{j=1}^{d-1} \tilde{A}_{dj}^{dd} D_d^2 \hat{u}_j - D_d \hat{p} = \hat{G}_d; \tag{4.32}$$

$$(1 - \sum_{j=1}^{d-1} D_j h) D_d^2 \hat{u}_d = D_d \hat{g} - \sum_{j=1}^{d-1} D_d D_j \hat{u}_j \tag{4.33}$$

where

$$\begin{aligned} \hat{G}_i = {} & D_i \hat{p} - \hat{f}_i - (T)_i - \sum_{\alpha+\beta<2d} \tilde{A}_{ij}^{\alpha\beta} D_\alpha D_\beta \hat{u}_j \\ & - \sum_{\alpha,\beta,j} D_\alpha \tilde{A}_{ij}^{\alpha\beta} D_\beta \hat{u}_j - \tilde{A}_{id}^{dd} D_d^2 \hat{u}_d, \; i = 1, \dots, d-1; \end{aligned} \tag{4.34}$$

$$\hat{G}_d = -\hat{f}_d - (T)_d - \sum_{\alpha+\beta<2d} \tilde{A}_{dj}^{\alpha\beta} D_\alpha D_\beta \hat{u}_j - \tilde{A}_{dd}^{dd} D_d^2 \hat{u}_d - \sum_{\alpha,\beta,j} D_\alpha \tilde{A}_{dj}^{\alpha\beta} D_\beta \hat{u}_j. \tag{4.35}$$

From $D_j D_d \hat{u}_j \in L^2(Q)$, $j = 1, \dots, d-1$, $\hat{g} \in W^{1,2}(Q)$ and equation (4.33), it follows $D_d^2 \hat{u}_d \in L^2(Q)$. Since $\hat{f} \in L^2(Q)^d$, $\nabla' \hat{p} \in L^2(Q)^{d-1}$, $D' \nabla \hat{u} \in L^2(Q)^{(d-1)d}$, $T \in L^2(Q)^d$, (4.34), (4.35) shows that $\hat{G}_i \in L^2(Q)$, $i = 1, \dots, d$.

The system (4.31)–(4.32) is a linear system of d equations, where the determinant of the corresponding matrix $\begin{pmatrix} \tilde{A}_{ij}^{dd} & D_i h \\ \tilde{A}_{dj}^{dd} & -1 \end{pmatrix}_{i,j=1}^{d-1}$ is bounded away from zero (because of (4.28)). Therefore, we can calculate $D_d^2 \hat{u}_j$, $j = 1, \dots, d-1$ and $D_d \hat{p}$

according to $\hat{G}_i$, $i = 1, \dots, d$. Since $\tilde{A} \in C^{0,1}(\overline{Q})$, we have $D_d^2 \hat{u}_j \in L^2(Q)$, for $j = 1, \dots, d-1$, $D_d \hat{p} \in L^2(Q)$ and we also have the following estimates

$$\|\nabla^2 \hat{u}\|_{2;Q} + \|\nabla \hat{p}\|_{2;Q} < C \left(\|\hat{f}\|_{2;Q} + \|\nabla \hat{g}\|_{2;Q} + \|\nabla \hat{u}\|_{2;Q} + \|\hat{p}\|_{2;Q} \right)$$

with a constant $C(\hat{A}, Q) > 0$.

Going back to the original x coordinates we obtain the inequality

$$\|\nabla^2 u\|_2 + \|\nabla p\|_2 \le C \left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2 \right)$$

with a constant $C = C(A, B, \Omega) > 0$.
It implies (4.22). The lemma is proved. □

Let us return to the proof of Theorem 4.1 on Ω. As Ω is a bounded C^2-domain in $\mathbb{R}^d$ we will use decomposition of Ω described in Section 2.

Clearly, it is enough to prove that $\|\varphi_{j_1} \nabla^2 u\|_2 < \infty$, $\|\varphi_{j_1} \nabla p\|_2 < \infty$, $j_1 = 1, \dots, m$; $\|\psi_{k_1} \nabla^2 u\|_2 < \infty$, $\|\psi_{k_1} \nabla p\|_2 < \infty$, $k_1 = 1, \dots, l$.
We multiply the both sides of the system (4.2) by φ_{j_1} and obtain

$$\begin{aligned} -\operatorname{div}[\bar{A} \nabla(\varphi_{j_1} u)] + \nabla(\varphi_{j_1} p) &= \tilde{f}, \\ \operatorname{div}(\varphi_{j_1} u) &= \tilde{g} \end{aligned} \tag{4.36}$$

where $\tilde{f}$, $\tilde{g}$ are given by

$$\begin{aligned} \tilde{f}_i := {} & \varphi_{j_1} \bar{f}_i + D_i \varphi_{j_1} p \\ & - \sum_{\alpha,\beta,j=1}^{d} \bar{A}_{ij}^{\alpha\beta} \left[D_\alpha \varphi_{j_1} D_\beta u_j + (D_\alpha D_\beta \varphi_{j_1}) u_j + D_\alpha u_j D_\beta \varphi_{j_1} \right] \\ & + \sum_{\alpha,\beta,j=1}^{d} (D_\alpha \bar{A}_{ij}^{\alpha\beta})(D_\beta \varphi_{j_1}) u_j, \ i = 1, \dots, d; \end{aligned} \tag{4.37}$$

$$\tilde{g} := \varphi_{j_1} g + (\nabla \varphi_{j_1}) u. \tag{4.38}$$

It follows from (4.37) and (4.38) that

$$\|\tilde{f}\|_2 \le \|\bar{f}\|_2 + C \left(\|p\|_2 + \|u\|_{1,2} \right) \quad \text{and} \quad \|\tilde{g}\|_{1,2} \le \|g\|_{1,2} + C\|u\|_{1,2}.$$

It is clear that $\operatorname{supp} \varphi_{j_1} u$ and $\operatorname{supp} \varphi_{j_1} p \subset U_{R_{j_1}, \beta_{j_1}, h_{j_1}}(x_{j_1}) \cap \overline{\Omega}$ and we can apply Lemma 4.3. We obtain

$$\begin{aligned} & \|\nabla^2(\varphi_{j_1} u)\|_2 + \|\nabla(\varphi_{j_1} p)\|_2 \le C \left(\|\tilde{f}\|_2 + \|\tilde{g}\|_{1,2} + \|u\|_{1,2} + \|p\|_2 \right) \\ & \le C \left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2 \right). \end{aligned}$$

In a similar way, applying Lemma 4.3 we conclude that

$$\|\psi_{k_1}\nabla^2 u\|_2 + \|\psi_{k_1}\nabla p\|_2 \le C\left(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2} + \|p\|_2\right) \tag{4.40}$$

for all ψ_{k_1}, $k_1 = 1, \ldots, l$ with a constant $C = C(A, B, \Omega) > 0$. This yields (4.4) and the theorem is proved for $k = 0$.

In the next step when $k = 1$ we realize that on any strictly embedded domain Ω', $(\triangle_{\delta,s}u, \triangle_{\delta,s}p)$ solve the system (4.8) for δ small enough and $s = 1, \ldots, d$. Thus an analogy of Lemma 4.1 gives the estimates independent of δ. Letting $\delta \to 0$ guarantees existence of all third derivatives of u and second derivatives of p with L^2 estimates in Ω'. An analogy of Lemma 4.2 can be proved then for $(\triangle_{\delta,s}u, \triangle_{\delta,s}p)$ with δ small enough and $s = 1, \ldots, d-1$ as these difference quotients solve system (4.8) and satisfy zero boundary condition on the flat part of the boundary. Letting $\delta \to 0$ we obtain the L^2-estimates of $\frac{\partial \nabla^2 u}{\partial x_s}$ for $s = 1, \ldots, d-1$. Using ellipticity condition we calculate from the equation differentiated with respect to x_d estimate of the last term $\frac{\partial^3 u}{\partial x_d^3}$. If we "flatten the boundary" as in Lemma 4.3 we can repeat the above proof and conclude Theorem 4.1 for $k = 1$. The general case of k follows by induction over k. $\square$

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Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic

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