The conjugate of a product of linear relations

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Abstract. Let X, Y and Z be normed linear spaces with $T(X \to Y)$ and $S(Y \to Z)$ linear relations, i.e. setvalued maps. We seek necessary and sufficient conditions that would ensure that $(ST)' = T'S'$. First, we cast the concepts of relative boundedness and co-continuity in the set valued case and establish a duality. This duality turns out to be similar to the one that exists for densely defined linear operators and is then used to establish the necessary and sufficient conditions. These conditions are similar to those for the single valued case. In the process, the well known characterisation of relativeboundedness for closed linear operators by Sz.-Nagy is extended to the multi-valued linear maps and we compare our results to other known necessary and sufficient conditions.

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Introduction

Let X and Y be normed linear spaces. By a linear relation (cf. Cross [Cro98]) we shall mean a setvalued map $T : D(T) \subset X \to 2^Y$ which is linear in the sense that $Tx + Ty = T(x + y)$ and for $\alpha \neq 0$ we have $T(\alpha x) = \alpha Tx$. We use the convention that the domain of T is $D(T) = \{x \in X : Tx \neq \emptyset\}$. For a linear relation $T \in LR(X, Y)$, its conjugate (or adjoint) T' is defined by its graph (cf. [Cro98, III.1.1]) $G(T') = G(-T^{-1})^{\perp}$, so by [Cro98, III.1.2] we have

$$G(T') = \left\{(y', x') \in (Y', X') : y'y - x'x = 0 \ \text{ for all } \ (x, y) \in G(T)\right\}.$$

Note that this definition coincides with the classical one when T is a densely defined operator and it allows one to define the conjugate of a nondensely defined operator. In this article we seek necessary and sufficient conditions that would guarantee that the conjugate of the product of linear relations coincides with the product of the conjugates.

The equivalent problem for linear operators between Banach spaces received attention in the literature earlier. In particular, M.A. Kascic in [Kas68] considered it in the context of polynomials of linear operators. A useful set of necessary and sufficient conditions were reported by Förster and Liebetrau (cf. [FL77]) in 1977. In the process of establishing their result, they used the concept of a core of an operator which was introduced by T. Kato in [Kat66] and the duality between co-continuity and relative boundedness for operators. The duality of co-continuity and relative boundedness rests upon the relative boundedness characterisation of B. Sz.-Nagy (cf. [SN51]) for closed operators in Banach spaces.

The technical concepts and results used by [FL77] we first recast for linear relations, i.e. the multivalued context. Note that the core of a linear relation as well as the concept of relative boundedness of linear relations appeared earlier in [Cro98]. We extend the co-continuity concept to linear relations and establish the duality with relative boundedness. The validity of the Sz.-Nagy characterisation of relative boundedness is then also established.

2. Relative boundedness and co-continuity

According to Kato (in [Kat66]), $M \subset X$ is a core of $T(X \to Y)$ if $G(T) \subset \overline{G(T_M)}$, where $G(T_M) = \{(x,y) \in G(T) : x \in M\}$. T_M is called the restriction of T to M. The following is a useful characterisation of the core of an operator in terms of its conjugate. It is a well known result for linear operators and was also recently established for linear relations (cf. [Cro98, IV.4.4]), and we include a direct proof for completeness, which is independent of the Bipolar theorem.

Theorem 2.1. *Let $T \in LR(X,Y)$. Then $E \subset D(T)$ is a core of T if and only if $T'_E = T'$.*

Proof: Suppose E is a core of T. Since $G(T') \subset G(T'_E)$ it would suffice to show that $G(T'_E) \subset G(T')$. To this end, let $(y', x') \in G(T'_E)$. Then $y'y - x'x = 0$ for all $(x,y) \in G(T_E)$. To show that $(y',x') \in G(T')$ we have to establish that $y'y - x'x = 0$ for every $(x,y) \in G(T)$. So let $(x,y) \in G(T) \subset \overline{G(T_E)}$. Then there exists a sequence $(x_n) \in E$ with $(x_n, Tx_n) \to (x,y)$. So

$$\lim_{n\to\infty}(y'Tx_n - x'x_n) = y'(\lim_{n\to\infty} Tx_n) - x'(\lim_{n\to\infty} x_n) = y'y - x'x.$$

But as $(y'Tx_n - x'x_n) = 0$ for every $n \in \mathbb{N}$ we have that $y'y - x'x = 0$ showing that $(y'x') \in G(T')$.

Conversely suppose that $T'_E = T'$. By the definition of conjugate, it is clear that E is dense in $D(T)$. Suppose that $z' \in Y' \times X'$ is such that $z'(G(T_E)) = 0$. Then there are some $x' \in X'$ and $y' \in Y'$ such that $z'(x, Tx) = x'x + y'y$ for every $x \in E$. But then $(y', -x') \in G(T'_E) = G(T')$ and so $y'Tx + x'x = 0$ for every $x \in D(T)$. Hence $z'(G(T)) = 0$ which shows by the Hahn-Banach theorem that $G(T) \subset \overline{G(T_E)}$. Consequently E is a core of T. □

Let $T(X \to Y)$ and $S(X \to Y)$ be two linear relations. We shall say that S is an extension of T if $D(T) \subset D(S)$ and $Tx = Sx$ for each $x \in D(T)$. The linear relation $\overline{T}$ has the graph $G(\overline{T}) = \overline{G(T)}$ and is called the closure of T. According to Cross [Cro98], we shall call a linear relation closable if $\overline{T}$ is an extension of T, that is $\overline{T}x = Tx$ for every $x \in D(T)$.

Corollary 2.2. *Let $T_1, T \in LR(X,Y)$ be closable linear relations. Then $\overline{T} = \overline{T_1}$ if and only if $T' = T_1'$.*

PROOF: If T is closable, then $T' = \overline{T}'$. Since we have $\overline{T} = \overline{T_1}$, we have that $T' = \overline{T}' = \overline{T_1}' = T_1'$.

Conversely suppose that $T' = T_1'$; then $\overline{T}' = \overline{T_1}'$. For the result to follow it would suffice to show that $G(\overline{T}) = G(\overline{T_1})$. Since $G(-\overline{T}^{-1})^{\perp} = G(\overline{T}') = G(\overline{T_1}') = G(-\overline{T_1}^{-1})^{\perp}$, we have $G(-\overline{T}^{-1}) = G(-\overline{T}^{-1})^{\perp\top} = G(-\overline{T_1}^{-1})^{\perp\top} = G(-\overline{T_1}^{-1})$ from which it follows that $G(\overline{T}) = G(\overline{T_1})$. □

We shall denote by B_X the closed unit ball in X, i.e. $B_X = \{x \in X : \|x\| \leq 1\}$.

Definition 2.3. In this definition, we extend the concept of co-continuity, as defined by Förster in [For74], to linear relations. Let $T(X \to Y)$ and $S(Z \to Y)$ be linear relations. Then T is S-co-continuous if there exist constants $\alpha, \beta > 0$ such that

$$TB_X \subset \alpha SB_Z + \beta B_Y + T(0).$$

Remark 2.4. This seems a useful extension since for T single valued we have $T(0) = 0$ and then we get the classical definition (see for example [FL77]), namely there are $\alpha, \beta > 0$ with

$$TB_X \subset \alpha SB_Z + \beta B_Y,$$

and if T is a continuous linear relation we note that there is an $\alpha > 0$ with

$$TB_X \subset \alpha B_Y + T(0)$$

(see for example [Cro98, II.1.10]).

The next two results concern products of linear relations and sufficient conditions for a core of any of the factors to be a core of the product.

Proposition 2.5. *Let $T(X \to Y)$ and $S(Y \to Z)$ be linear relations. Suppose that E is a core of T with $TE \subset D(S)$. If S is $(ST)_E$-co-continuous then E is a core of ST.*

PROOF: By Theorem 2.1 we need only show that $(ST_E)' = (ST)'$, and since $(ST)_E$ is a restriction of ST it would suffice to show that $D((ST)'_E) \subset D((ST)')$. Suppose that $z' \in D((ST)'_E)$. Then $z'ST$ is single-valued and continuous on E. But then $z'S$ is single valued. Since S is $(ST)_E$-co-continuous, we have $SB_Y \subset \alpha B_Z + \beta STB_E + S(0)$ for some α and β. So if $\|y\| \leq 1$, $\left|z'Sy\right| = \left|\alpha z'z + \beta z'STe + z'S(0)\right| \leq \left\|\alpha z'\right\| + \left\|\beta z'ST\right\|$ which are constants independent of y and so $z' \in D(S')$. Since $z'(ST)$ is continuous on E we have $S'z' \subset D(T'_E) = D(T')$. But then $z'(ST)$ is continuous on $D(T)$ from which the result follows. □

Proposition 2.6. *Let $T(X \to Y)$ and $S(Y \to Z)$ be linear relations with $R(T) \subset D(S)$ and suppose that E is a core of S. If T is continuous then $M = T^{-1}E$ is a core of ST.*

PROOF: Suppose that M is not a core of ST. Then by Theorem 2.1 there is a $y' \in Y'$ with $y'ST$ continuous on M but not continuous on $D(ST)$. In that case $y'S$ is not continuous on $D(S)$ and so $y' \notin D(S')$. Since $y'STz = y'S(Tz)$, we have that $y'S$ is continuous on $E = TM$. Since E is a core of S, $y'S$ is continuous on $D(S)$ and so $y' \in D(S')$ — a contradiction. Hence M is a core of ST. $\square$

Definition 2.7. Let $T(X \to Y)$ and $S(X \to Z)$ be linear relations. Then T is S bounded if $D(S) \subset D(T)$ and there exist constants $\alpha, \beta > 0$ such that for all $x \in D(S)$ we have
$$\|Tx\| \leq \alpha \|Sx\| + \beta \|x\| .$$

Remark 2.8. For a linear relation $T \in LR(X,Y)$ we have $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)$ (cf. [Cro98, II.1.4]).

Proposition 2.9. *Suppose $y' \in D(T')$. Let x'_0 denote the extension of $x' = y'T$ to $\overline{D(T)}$ and let $\|x'_0\| = \sup\{|y'Tx| : \|x\| \leq 1, x \in D(T)\}$. Then $\|y'T\| = \|T'y'\|$.*

PROOF: Note that $\|T'y'\| = d(x', T'(0)) = \inf\{\|x' - z'\| : z' \in T'(0)\}$ where $x' \in T'y'$. Let $x' \in T'y'$, then x' is an extension of $y'T$, and suppose $z' \in T'(0)$. Then
$$\begin{aligned} \|x'_0\| &= \sup\{|x'x| : \|x\| \leq 1, x \in D(T)\} \\ &= \sup\{|(x' - z')x| : \|x\| \leq 1, x \in D(T)\} \\ &\leq \|x' - z'\| . \end{aligned}$$

But then $\|x'_0\| \leq \inf\{\|x' - z'\| : z' \in T'(0)\} = \|T'y'\|$.

On the other hand, let $x' \in T'y'$ with $x'x = 0$ when $x \notin \overline{D(T)}$. Then
$$\begin{aligned} \|T'y'\| &= d(x', T'(0)) \\ &= \inf\{\|x' - z'\| : z' \in T'(0)\} \\ &\leq \|x' - 0\| \\ &= \|x'\| \\ &= \sup\{|x'x| : \|x\| \leq 1, x \in D(T)\} \\ &= \|x'_0\| . \end{aligned}$$

$\square$

From the above it follows now that T' is S' bounded if and only if $D(S') \subset D(T')$ and there are constants $\alpha, \beta > 0$ such that for all $y' \in D(S')$ we have

$$\|y'T\| \leq \alpha \|y'S\| + \beta \|y'\|.$$

We are now ready to deal with the duality between relative boundedness and co-continuity.

Theorem 2.10. *Let $T \in LR(X,Y)$ and $S \in LR(Z,Y)$.*

(1) *If T is S-co-continuous then there are $\alpha, \beta > 0$ with*

$$\|y'T\| \leq \alpha \|y'S\| + \beta \|y'\| \qquad \forall y' \in D(S').$$

If furthermore $T(0) \subset S(0)$ then T' is S'-bounded.

(2) *Suppose there are $\alpha, \beta > 0$ with*

$$\|y'T\| \leq \alpha \|y'S\| + \beta \|y'\| \qquad \forall y' \in Y'.$$

Then T is S-co-continuous.

In particular, in the case when T and S are single valued we have that T is S-co-continuous if and only if T' is S'-bounded.

Proof: (1) Suppose that T is S-co-continuous. Then there exist nonnegative constants α, β with $TB_{D(T)} \subset \alpha SB_{D(S)} + \beta B_Y + T(0)$. Let $y' \in D(S')$ and $x \in D(T) \cap B_X$. For $y_x \in Tx$ there are $z \in B_Z$, $(z, y_z) \in G(S)$, $y_t \in T(0)$, $y_s \in S(0)$ and $y \in B_Y$ with

$$y_x = \alpha y_z + y_s + \beta y + y_t$$

or

$$y_x - y_t = \alpha y_z + y_s + \beta y.$$

As $y'y_s = 0$ for every $y_s \in S(0)$ we have

$$\begin{aligned} |y'(y_x - y_t)| &= |y'(\alpha y_z + \beta y)| \\ &\leq \alpha |y'y_z| + \beta |y'y| \\ &= \alpha |y'Sz| + \beta |y'y| \\ &\leq \alpha \|y'S\| + \beta \|y'\|. \end{aligned}$$

Thus $\|y'Tx\| = \inf \{|y'(y_x - y_0)| : y_0 \in T(0)\} \leq |y'(y_x - y_t)| \leq \alpha \|y'S\| + \beta \|y'\|$. Since this is true for every $x \in D(T) \cap B_X$, we have that $\|y'T\| \leq \alpha \|y'S\| + \beta \|y'\|$. If, furthermore, $T(0) \subset S(0)$, then $y'T$ is single valued and so $y' \in D(T')$. From this it follows that T' is S'-bounded.

(2) Suppose that there are $\alpha, \beta > 0$ with

$$\left\|y'T\right\| \leq \alpha \left\|y'S\right\| + \beta \left\|y'\right\| \forall y' \in Y'.$$

Let $\alpha_0 = \alpha + 1$ and $\beta_0 = \beta + 1$ and suppose $y' \in Y'$, $x \in B_X \cap D(T)$. Then there exist $z \in B_Z$ and $y \in B_Y$ with

$$\left\|y'Tx\right\| < \left\|\alpha_0 y'Sz\right\| + \beta_0 \left\|y'y\right\|.$$

Now let $y_x \in TB_X$. Then there are $y_t \in T(0)$, $y_z \in SB_Z$, $y_s \in S(0)$ and $y \in B_Y$ with

$$\left|y'(y_x - y_t)\right| < \alpha_0(y'(y_z - y_s)) + \beta_0 y'y.$$

Since this is true for every $y' \in Y'$, we have that $y_x - y_t \in \alpha_0 SB_Z + \beta_0 B_Y$ and as $y_t \in T(0)$ it follows that $y_x \in \alpha_0 SB_Z + \beta_0 B_Y + T(0)$. But then $TB_X \subset \alpha_0 SB_Z + \beta_0 B_Y + T(0)$ and so T is S-co-continuous.

In particular, if T and S are single valued then $T(0) = S(0) = 0$. Thus, if T is S-co-continuous then T' is S'-bounded by (1) above.

Conversely, suppose now that T' is S'-bounded. Then there are $\alpha, \beta > 0$ such that we have for $y' \in D(S')$, $\left\|y'T\right\| \leq \alpha \left\|y'S\right\| + \beta \left\|y'\right\|$ with $D(S') \subset D(T')$. Note that if $y' \notin D(S')$ then $y'S$ is not a bounded functional and so we trivially have that $\left\|y'T\right\| \leq \alpha \left\|y'S\right\| + \beta \left\|y'\right\|$ for $y' \in D(T') \backslash D(S')$. Thus $\left\|y'T\right\| \leq \alpha \left\|y'S\right\| + \beta \left\|y'\right\|$ holds for every $y' \in Y'$. From (2) above it now follows that T is S-co-continuous. □

We now state the B. Sz.-Nagy characterisation of relative boundedness in the multivalued context. The proof follows similar lines as that one for linear operators that appeared in [Kaa64] and we use the Closed Graph Theorem for Linear Relations as appeared in [Cro98].

Lemma 2.11. *Suppose $T \in LR(X,Y)$ and $S \in LR(X,Z)$ are completely closed linear relations with $D(T) \subset D(S)$. Then S is T-bounded.*

Proof: Let $X_T = D(T)$ be the normed space with norm $\|x\|_T = \|x\| + \|Tx\|$. Since T is completely closed we have that X_T is a Banach space. Define $J : X_T \to Z$ by $(x, z) \in G(J)$ if $z \in Sx$. Then J is a linear relation with $J(0) = S(0)$. Since S is closed, we have that $J(0)$ is closed and so J is closable (see Cross [Cro98, II.5.5]). But as J is defined on all of X_T, J is closed. Since X_T is complete, J is bounded by the Closed Graph Theorem ([Cro98, III.4.2]). So there is an $\alpha > 0$ with $\|Jx\| \leq \alpha \|x\|_T$, or equivalently $\|Sx\| = \|Jx\| \leq \alpha \|x\|_T = \alpha \|x\| + \alpha \|Tx\|$, showing that S is T-bounded. □

3. The conjugate of a product of linear relations

Theorem 3.1. *Let $T \in LR(X,Y)$ and $S \in LR(Z,X)$ and suppose that $D(TS)$ is a core of S. Then $(TS)' = S'T'$ if and only if T' is $(TS)'$-bounded.*

Proof: Suppose that $(TS)' = S'T'$, then $D((TS)') \subset D(T')$. Since T' and $(TS)'$ are completely closed and $T(0) \subset TS(0)$, we have that T' is $(TS)'$-bounded.

Conversely suppose that T' is $(TS)'$-bounded. Then $D((TS)') \subset D(T')$. Since $S'T'$ is a restriction of $(TS)'$, it would suffice to show that $D((TS)') \subset D(S'T')$. To this end let $y' \in D((TS)')$, then $y' \in D(T')$. To show that $y' \in D(S'T')$ we need to establish the existence of an $x' \in D(S')$ with $(y', x') \in G(T')$. For this to hold, it would suffice to have $x'S$ bounded on $D(TS)$, which is a core of S. Note that $y' \in D(TS)'$ implies that $y'TS$ is single-valued, and $y' \in D(T')$ implies that $y'T$ is single-valued and so $x'S$ is single-valued for each $x' \in T'y'$.

Let $z \in B_Z \cap E$ then there exists an $x \in D(T)$ with $(z, x) \in G(S)$. So for $x \in Sz$ and $x' \in T'y'$ we have

$$\left|x'Sz\right| = \left|x'x\right| = \left|y'Tx\right| = \left|y'TSz\right| \leq \left\|y'TS\right\|$$

which shows that $x'S$ is bounded on $D(TS)$. □

In the rest of the section we compare our conditions to other known conditions.

Remark 3.2.

(1) Suppose that $T \in LR(X, Y)$ and $S \in LR(Z, X)$ with $D(TS)$ a core of S and T is S-co-continuous. Then $T(0) \subset TS(0)$ and so T' is $(TS)'$-bounded from which it follows that $(TS)' = S'T'$.

(2) Cross in [Cro98, II.1.6(a)] has concluded that $(TS)' = S'T'$ whenever
(a) $D(T') = X'$ and $R(S) \subset D(T)$ or
(b) $R(S') = Z'$ and $D(T) \subset R(S)$.

Note that if $D(T') = X'$ then T is continuous and surely it is TS-co-continuous. Furthermore $R(S) \subset D(T)$ ensures that $D(TS) = D(S)$ from which it follows that $D(TS)$ is a core of S. This result is thus a special case of the above theorem.

If $R(S') = Z'$ then S is bounded below, so there is a $c > 0$ with $\|Sz\| \geq c\,\|z\|$ for all $z \in D(S)$ (see example [Cro98, III.6.2(a)]). So let $y \in TB_X$, then there is an $x \in B_X$ with $(x, y) \in G(T)$. As $D(T) \subset R(S)$ and S is bounded below, there is a $z \in c^{-1}B_Z$ with $(z, x) \in G(S)$. But then $(z, y) \in G(TS)$ showing that $y \in c^{-1}TSB_Z$. But then $TB_X \subset c^{-1}TSB_Z$ from which it follows that T is TS-co-continuous. It does however, not follow from the hypothesis that $D(TS)$ is a core of S, but in the light of the next result, it would suffice if $D(T)$ was dense in $R(S)$.

The minimum modulus of a linear relation T is the quantity ([Cro98])

$$\gamma(T) = \sup\left\{\|Tx\| \geq \lambda d(x, N(T)) \text{ for } x \in D(T)\right\}.$$

Lemma 3.3. *Let $T \in LR(X,Y)$ with $\gamma(T) > 0$ and suppose that $M \subset Y$. Then $T^{-1}M$ is core of T if and only if $R(T) \cap M$ is dense in $R(T)$.*

Proof: Suppose that $R(T) \subset \overline{R(T) \cap M}$ and let $(x,y) \in G(T)$. Then there exists a sequence w_n in $D(T)$, $y_n \in Tw_n$ such that $y_n \to y$ and $y_n \in M$. Since $\gamma(T) > 0$, there is a sequence $k_n \in N(T)$ with $w_n + k_n \to x$. Put $x_n = w_n + k_n$ then $x_n \in T^{-1}M$ and $(x_n, y_n) \to (x,y)$. The converse is immediate. □

The next result extends those of van Casteren and Goldberg [CG70] to the multivalued case.

Theorem 3.4. *Let $S \in LR(Z,X)$ have a topologically complemented range with $\gamma(S) > 0$. If $T \in LR(X,Y)$ is densely defined then $D(TS)$ is a core of S. If furthermore T is continuous on a topological complement N of $R(S)$, then T is TS-co-continuous.*

Proof: Since $D(T)$ is dense in $X = R(S) \oplus N$, $D(T) \cap R(S)$ is dense in $R(S)$ and so $D(TS) = S^{-1}(D(T))$ is a core of S. Suppose now that T is continuous on N, where $R(S) \oplus N = Y$ and let P be the projection of X onto N along $R(S)$. We first show that there is a positive constant α such that

$$B_X \subset \alpha S B_Z + P B_X.$$

Let $x \in B_X$, then there is $(z,y) \in G(S)$ with $x = y + Px$ for some $z \in Z$. If $y \in S(0)$ then $x \in \alpha S B_Z + P B_X$ trivially. So suppose that $y \notin S(0)$. Then $d(z, N(S)) > 0$ and so there is a $z_o = z - n$ for some $n \in N(S)$ with $d(z_o, N(S)) \geq \frac{1}{2}\|z_o\|$. Now $\gamma(S) \leq \frac{\|Sz_o\|}{d(z_o, N(S))} \leq 2\frac{\|Sz_o\|}{\|z_o\|}$ and so

$$\|z_o\| \leq 2\frac{\|Sz_o\|}{\gamma(S)} = 2\frac{\|Sz\|}{\gamma(S)} \leq \frac{2}{\gamma(S)}\|I-P\|\,\|x\| \leq \frac{2}{\gamma(S)}\|I-P\|.$$

Thus taking $\alpha = \frac{2}{\gamma(S)}\|I-P\|$ it follows that $Sz = Sz_o \subset \alpha S B_Z$. Consequently $x = y + Px \in \alpha S B_Z + P B_X$. Now, since TP is continuous and $TB_X \subset \alpha TSB_Z + \|TP\|\, B_Y$ it follows that T is TS-co-continuous. □

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