A characterization of polynomially Riesz strongly continuous semigroups

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Abstract. In this paper we characterize the class of polynomially Riesz strongly continuous semigroups on a Banach space X. Our main results assert, in particular, that the generators of such semigroups are either polynomially Riesz (then bounded) or there exist two closed infinite dimensional invariant subspaces X_0 and X_1 of X with $X = X_0 \oplus X_1$ such that the part of the generator in X_0 is unbounded with resolvent of Riesz type while its part in X_1 is a polynomially Riesz operator.

Keywords: strongly continuous semigroups, Riesz operators, polynomially Riesz operators

Classification: 47B06, 47D03

1. Introduction and preliminaries

Let X be a Banach space over the complex field and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators on X. The subset of all compact operators of $\mathcal{L}(X)$ is designated by $\mathcal{K}(X)$. For $A \in \mathcal{L}(X)$, we let $\sigma(A)$, $\rho(A)$, $R(\lambda, A)$, $N(A)$ and $R(A)$ denote, respectively, the spectrum, the resolvent set, the resolvent operator, the null space and the range of A. The nullity of A, $\alpha(A)$, is defined as the dimension of $N(A)$ and the deficiency of A, $\beta(A)$, is defined as the codimension of $R(A)$ in X.

Write

$$\Phi_+(X) = \{A \in \mathcal{L}(X) : \alpha(A) < \infty \ \text{ and } R(A) \text{ is closed in } X\},$$
$$\Phi_-(X) = \{A \in \mathcal{L}(X) : \beta(A) < \infty \ \text{ (then } R(A) \text{ is closed in } X)\}.$$

Operators in $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators, while $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ denotes the set of Fredholm operators of $\mathcal{L}(X)$. If $A \in \Phi_\pm(X)$, the number $i(A) = \alpha(A) - \beta(A) \in \mathbb{Z} \cup \{-\infty, +\infty\}$, is the index of A. We say that a complex number λ belongs to the Fredholm domain of A, Φ_A, if $\lambda - A$ belongs to $\Phi(X)$. The subset $\sigma_e(A)$ of $\sigma(A)$ defined by $\sigma_e(A) := \mathbb{C} \backslash \Phi_A$ is called the Fredholm essential spectrum of A. For the properties of these sets we refer to [5], [6], [9] or [16].

An operator $T \in \mathcal{L}(X)$ is called a Riesz operator if $\lambda - T \in \Phi(X)$ for all scalars $\lambda \neq 0$. Let $\mathcal{R}(X)$ denote the class of all Riesz operators. It is worth noticing that there are many characterizations of Riesz operators. Ruston has characterized $\mathcal{R}(X)$ as the class of asymptotically quasi-compact operators, i.e., those $T \in \mathcal{L}(X)$ for which

$$\lim_{n\to\infty} \left\{ \inf_{K\in\mathcal{K}(X)} \|T^n - K\| \right\}^{\frac{1}{n}} = 0, \tag{1.1}$$

(cf. [17], [1], [2]). We recall that Riesz operators satisfy the Riesz-Schauder theory of compact operators and $\mathcal{R}(X)$ is not an ideal of $\mathcal{L}(X)$ ([2]). Moreover, using (1.1) we get the following result established independently by Caradus [1] and West [17].

Proposition 1.1. *Let X be a Banach space and let T and S be two commuting operators of $\mathcal{L}(X)$.*

(1) *If $T \in \mathcal{R}(X)$, then $TS \in \mathcal{R}(X)$.*
(2) *If T and S are in $\mathcal{R}(X)$, then $T + S \in \mathcal{R}(X)$.*

Let $F \in \mathcal{L}(X)$, F is called a Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$. The set of Fredholm perturbations is denoted by $\mathcal{F}(X)$. We recall that $\mathcal{F}(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in $\mathcal{R}(X)$ see [2], [16].

In [3] Cuthbert considered a class of C_0-semigroups $(T(t))_{t\geq 0}$ having the property of being near the identity, in the sense that, for some value of t, $T(t) - I \in \mathcal{K}(X)$. His result asserts that if $(T(t))_{t\geq 0}$ is a C_0-semigroup with infinitesimal generator A and if $\mathcal{O} = \{t > 0 : T(t) - I \text{ compact}\}$, then the following conditions are equivalent:

(a) $\mathcal{O} =]0, \infty[$,
(b) A is compact,
(c) $\lambda R(\lambda, A) - I$ is compact for some (and then for all) $\lambda > \omega$,

where ω denotes the type of $(T(t))_{t\geq 0}$. Cuthbert's result was extended by many authors to other strongly continuous families of operators such as cosine or resolvent families of operators (see [7], [12], [13]). In the paper [10], it was established that the assertions (a), (b) and (c) remain equivalent for strongly continuous semigroups $(T(t))_{t\geq 0}$ near the identity in the sense that there exists $t_0 > 0$ such that $T(t_0) - I \in \mathcal{J}(X)$ where $\mathcal{J}(X)$ is any arbitrary closed proper two-sided ideal of the algebra $\mathcal{L}(X)$ contained in the set of Fredholm perturbations. Note that in all these works the generator A is compact or belongs to an ideal of $\mathcal{L}(X)$ contained in $\mathcal{F}(X)$. The general case where A is a Riesz operator, not necessarily belonging to $\mathcal{F}(X)$, was considered in [8].

Let $\mathcal{J}(X)$ be a non trivial two-sided ideal of $\mathcal{L}(X)$. We say that an operator $A \in \mathcal{L}(X)$ belongs to $P\mathcal{J}(X)$ if there is a nonzero complex polynomial $p(\cdot)$ such that the operator $p(A) \in \mathcal{J}(X)$. In a recent work [11], the results obtained in [3], [8] and [10] were extended to semigroups for which there exists a non trivial polynomial $p(\cdot) \in \mathbb{C}[z]$ such that, for some $t > 0$, $p(T(t)) \in \mathcal{J}(X)$ where $\mathcal{J}(X)$ is an arbitrary proper two-sided ideal of $\mathcal{L}(X)$ contained in the set of Fredholm perturbations. In contrast to the previous results, in this case the infinitesimal generator of the semigroup is not necessarily a Riesz operator.

We say that $A \in \mathcal{L}(X)$ is polynomially Riesz if there exists a nonzero complex polynomial $p(\cdot)$ such that the operator $p(A) \in \mathcal{R}(X)$. The set of polynomially Riesz operators will be denoted by $P\mathcal{R}(X)$.

If A belongs $P\mathcal{R}(X)$, then there exists a nonzero polynomial $p(\cdot)$ such that $p(A) \in \mathcal{R}(X)$. So, $\sigma(p(A))$ must be finite or countable with zero as the only possible accumulation point. Moreover, the nonzero points of $\sigma(p(A))$ are isolated and the corresponding spectral projections are all finite dimensional. According to the spectral mapping theorem, the only possible accumulation points of $\sigma(A)$ are contained in the set of roots of $p(\cdot)$. Let λ_i be a root of $p(\cdot)$ and assume that $\lambda_i \notin \sigma(A)$. Set $q(z) = (z - \lambda_i)^{-1} p(z)$. Obviously, $\deg(q) < \deg(p)$. Moreover, since $q(A) = (A-\lambda_i)^{-1} p(A) = p(A)(A-\lambda_i)^{-1}$, applying Proposition 1.1 one sees that $q(A) \in P\mathcal{R}(X)$. These observations show that if $A \in P\mathcal{R}(X)$, then $\sigma_e(A)$ is necessarily finite, say $\{\lambda_1, \dots , \lambda_n\}$, and $p(z) = (z-\lambda_1)\dots(z-\lambda_n)$ is the nonzero polynomial of least degree and leading coefficient 1 such that $p(A) \in \mathcal{R}(X)$. It will be called the minimal polynomial of A. Conversely, let $p(z) = (z-\lambda_1)\dots(z-\lambda_n)$ is the minimal polynomial (in the sense defined above) of A. Arguing as above one sees that each $\lambda_i \in \sigma_e(A)$, and so $\sigma_e(A) = \{\lambda_1, \dots , \lambda_n\}$. This leads to the following characterization of the set of polynomially Riesz operators.

Proposition 1.2. *Let X be a Banach space. An operator $A \in \mathcal{L}(X)$ belongs to $P\mathcal{R}(X)$ if and only if $\sigma_e(A)$ is finite, say, $\sigma_e(A) = \{\lambda_1, \dots , \lambda_n\}$. Moreover, the minimal polynomial of A can be written in the form $p(z) = (z - \lambda_1)\dots(z - \lambda_n)$.*

Remark 1.1. Note that in contrast to the case where A is polynomially compact (or, more generally, if $A \in P\mathcal{J}(X)$ where $\mathcal{J}(X)$ is an arbitrary proper two sided ideal of $\mathcal{L}(X)$ [11]), the minimal polynomial of a polynomially Riesz operator has only simple roots. (Evidently, for $A \in P\mathcal{J}(X)$, the definition of the minimal polynomial is taken in the ideal sense.) $\square$

Let n be an integer and let φ be a function defined from its domain into $\mathbb{C}^n$, that is, $\varphi : \mathcal{D}(\varphi) \subseteq \mathbb{R} \to \mathbb{C}^n$, $t \to (\varphi_1(t), \dots , \varphi_n(t))$. Set $\mathcal{D}^+(\varphi) := \mathcal{D}(\varphi)\cap]0, \infty[$ and introduce the assumption

$$(\mathcal{A}1) \qquad \begin{cases} n \geq 1,\ \varphi(\cdot) \text{ is continuous and for all } t \in \mathcal{D}^+(\varphi), \\ \varphi_i(t) \neq 0 \text{ and } \prod_{i=1}^n (T(t) - \varphi_i(t)) \in \mathcal{R}(X). \end{cases}$$

We are now ready to state our first result.

Theorem 1.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with infinitesimal generator A and assume that the assumption $(\mathcal{A}1)$ holds true. Then the following conditions are equivalent.*

(1) *There are two constants a, $b \in]0,\infty[$, $a < b$ such that $]a,b[\subseteq \mathcal{D}^+(\varphi)$.*
(2) *$\mathcal{D}^+(\varphi) =]0,+\infty[$.*
(3) *The operator A belongs to $P\mathcal{R}(X)$.*
(4) *$(\lambda R(\lambda, A) - I)$ belongs to $P\mathcal{R}(X)$ for every $\lambda \in \rho(A)$.*
(5) *$(\lambda R(\lambda, A) - I)$ belongs to $P\mathcal{R}(X)$ for some $\lambda \in \rho(A)$.*

It is worth noticing that in the assumption $(\mathcal{A}1)$, the component functions of φ do not attain the value zero. This means that, for each $t > 0$, the roots of the minimal polynomial $p_t(\cdot)$ of $T(t)$ are all different from zero. Accordingly, $(T(t))_{t\geq 0}$ can always be embedded in a C_0-group on X (cf. Lemma 2.4). This assumption was motivated by Phillips's result, Lemma 2.5, which is basic in the proof of Lemma 2.6 and then Theorem 1.1. This gives rise to the following question: what can be said about the semigroup and its generator if there exists at least one $t_0 \in \mathcal{D}^+(\varphi)$ such that $\varphi_i(t_0) = 0$ for some $i \in \{1,\dots,n\}$? This question will be discussed in the theorem below. Clearly, if $\varphi_i(t_0) = 0$ for some $i \in \{1,\dots,n\}$, then according to Lemma 2.8(1), $\varphi_i(t) = 0$ for all $t \in \mathcal{D}^+(\varphi)$ which shows that, for $t \in \mathcal{D}^+(\varphi)$, 0 is a root of the minimal polynomial of $T(t)$. Therefore the result of Theorem 1.1 ceases to be true. Actually we will prove that the space X has the following decomposition $X = X_0 \oplus X_1$, where X_0 and X_1 are closed $T(t)$-invariant subspaces of X such that the restriction of $(T(t))_{t\geq 0}$ to X_0 is a Riesz semigroup with unbounded generator while Theorem 1.1 holds true for $(T(t)_{|X_1})_{t\geq 0}$. To state our second result we need the following assumption.

Let n be an integer and let φ be a function defined from its domain into $\mathbb{C}^n$, that is, $\varphi : \mathcal{D}(\varphi) \subseteq \mathbb{R} \to \mathbb{C}^n$, $t \to (\varphi_1(t),\dots,\varphi_n(t))$. Let us now introduce the following assumption

$$(\mathcal{A}2) \qquad \begin{cases} n \geq 2,\ \varphi(\cdot) \ \text{ is continuous, there exist } \ i_0 \in \{1,\dots,n\} \\ \text{and } \ t_0 \in \mathcal{D}^+(\varphi) \ \text{ such that } \ \varphi_{i_0}(t_0) = 0 \ \text{ and,} \\ \text{for all } \ t \in \mathcal{D}^+(\varphi),\ \prod_{i=1}^n (T(t) - \varphi_i(t)) \in \mathcal{R}(X). \end{cases}$$

Remark 1.2. Assume that there are $i_1,\dots,i_\alpha \in \{1,\dots,n\}$ with $\alpha \leq n-1$ and $t_1,\dots,t_\alpha \in \mathcal{D}^+(\varphi)$ such that $\varphi_{i_1}(t_1) = \ldots = \varphi_{i_\alpha}(t_\alpha) = 0$. Then applying Lemma 2.8(1) one obtains $\varphi_{i_1}(t) = \ldots = \varphi_{i_\alpha}(t) = 0$ for all $t \in \mathcal{D}^+(\varphi)$. So, 0 is a root of $\prod_{i=1}^n (T(t) - \varphi_i(t))$ of multiplicity α. Therefore, the use of Proposition 1.2 and Remark 1.1 shows that the minimal polynomial of $T(t)$ is given by $T(t)\prod_{j\neq i_1,\dots,i_\alpha}(T(t) - \varphi_j(t))$. □

Theorem 1.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with infinitesimal generator A and let $(\mathcal{A}2)$ be satisfied. Then there exist two closed* *subspaces of* X, X_0 *and* X_1, *such that* $X = X_0 \oplus X_1$ *and, for all* $t \geq 0$, $T(t)X_i \subseteq X_i$, $i = 0, 1$. *Moreover, the following statements hold true.*

(1) $(T(t)_{|X_0})_{t\geq 0}$ *is a* C_0*-semigroup of Riesz type on* X_0, *i.e.* $T(t)_{|X_0} \in \mathcal{R}(X_0)$ *for all* $t > 0$. *Its generator* $A_{|X_0}$ *is unbounded on* X_0 *and, for any* $\lambda \in \rho(A_{|X_0})$, $(\lambda - A_{|X_0})^{-1} \in \mathcal{R}(X_0)$.

(2) $(T(t)_{|X_1})_{t\geq 0}$ *is a* C_0*-semigroup on* X_1 *which can be embedded in a* C_0*-group. Moreover,* $(T(t)_{|X_1})_{t\geq 0}$ *and its generator* $A_{|X_1}$ *satisfy the assertions* (1)–(5) *of Theorem* 1.1.

The remainder of this paper is organized as follows. In Section 2, we define the Browder essential spectrum and gather auxiliary results required later. The proofs of Theorems 1.1. and 1.2 are the topic of Section 3.

We close this introduction by noticing that, as indicated above, the Cuthbert theorem was generalized to bounded cosine operator functions ([7], [13]) and resolvent families of bounded operators ([12]). Do Theorems 1.1 and 1.2 extend to these two families of strongly continuous operators?

2. Some lemmas

Let X be a Banach space and let $A \in \mathcal{L}(X)$. Set $\Phi_A^0 := \{\lambda \in \Phi_A : i(\lambda - A) = 0\}$ and define the set

$$\sigma_b(A) := \mathbb{C}\backslash\rho_b(A)$$

where

$$\rho_b(A) := \{\lambda \in \Phi_A^0 \text{ such that all scalars near } \lambda \text{ are in } \rho(A)\}.$$

Following [6] and [15], $\sigma_b(\cdot)$ is called the Browder essential spectrum. It is well known that

$$\sigma_e(A) \subseteq \sigma_b(A) \text{ and } \sigma_b(A) = \sigma(A)\backslash\Pi(A) \tag{2.1}$$

where $\Pi(A)$ stands for the set of all isolated eigenvalues of A with finite algebraic multiplicity.

Let us notice that if $A \in P\mathcal{R}(X)$, then $p(A) \in \mathcal{R}(X)$ (where $p(\cdot)$ is the minimal polynomial of A) and therefore $\sigma_b(p(A)) = \{0\}$. On the other hand, if $\dim(X) = \infty$, one has also $\emptyset \neq \sigma_e(p(A)) \subseteq \sigma_b(p(A)) = \{0\}$. Consequently, $\sigma_e(p(A)) = \sigma_b(p(A)) = \{0\}$. Also, by the spectral mapping theorem for $\sigma_e(\cdot)$ and $\sigma_b(\cdot)$ ([6]) we have

$$\sigma_e(p(A)) = p(\sigma_e(A)) = \{0\} \text{ and } \sigma_b(p(A)) = p(\sigma_b(A)) = \{0\}. \tag{2.2}$$

Lemma 2.1. *Let $A \in \mathcal{L}(X)$. If $A \in P\mathcal{R}(X)$, then, except a finite set, the spectrum of A consists of isolated points which are eigenvalues with finite algebraic multiplicity.*

Proof: Since $A \in P\mathcal{R}(X)$, there exists a polynomial $p(\cdot) \neq 0$ such that $p(A) \in \mathcal{R}(X)$. So, it suffices to prove that $\sigma_b(A) = \sigma_e(A) \subseteq \{\lambda \in \mathbb{C}$ such that $p(\lambda) = 0\}$. To do so, let us first observe that (2.1) and (2.2) imply $\sigma_e(A) \subseteq \sigma_b(A) \subseteq \{\lambda : p(\lambda) = 0\}$. Consequently, $\sigma_b(A)$ is a finite set and all elements in $\sigma(A)\backslash\sigma_b(A)$ are isolated points in $\sigma(A)$. It remains to show that $\sigma_b(A) \subseteq \sigma_e(A)$. Indeed, let $\lambda_0 \in \sigma_b(A)$, we can write $\sigma(A) = \sigma_0 \cup \sigma_1$ where σ_0, σ_1 are clopen subsets of $\sigma(A)$ and $\sigma_b(A) \cap \sigma_0 = \{\lambda_0\}$. Then we have a decomposition of A according to the decomposition $X = X_0 \oplus X_1$ of the space in such a way that the spectra of the parts of A in X_0 and X_1, i.e. A_0 and A_1, coincide with σ_0 and σ_1, respectively. Therefore, for any $\lambda \in \sigma_0$, $\lambda - A_1 \in \Phi(X_1)$ and by [11, Lemma 2.1] $\sigma_e(A_0) = \sigma_0 \cap \sigma_e(A) \subseteq \sigma_b(A) \cap \sigma_0 = \{\lambda_0\}$. Since X_0 is not finite dimensional, $\sigma_e(A_0) \neq \emptyset$ and therefore $\sigma_e(A_0) = \sigma_e(A) \cap \sigma_0 = \{\lambda_0\}$. Accordingly $\lambda_0 \in \sigma_e(A)$ and then $\sigma_e(A) = \sigma_b(A)$ which ends the proof. □

Lemma 2.2. *Let $A \in \mathcal{L}(X)$, assume that $\Omega \neq \emptyset$ is a connected open subset of $\mathbb{C}$ such that $\sigma(A) \subseteq \Omega$ and let $f : \Omega \to \mathbb{C}$, $f \neq 0$ be an analytic function. If $f(A) \in \mathcal{R}(X)$, then $A \in P\mathcal{R}(X)$.*

Proof: Obviously f has only a finite number of zeros on $\sigma(A)$, say $\lambda_1, \dots, \lambda_m$, hence $f(z) = (\prod_{i=1}^m (z - \lambda_i)^{\alpha_i})\, \zeta(z)$, where α_i is the order of the zero λ_i and $\zeta(z) \neq 0$ is an analytic function on a neighborhood of $\sigma(A)$. Set $\vartheta(z) = 1/\zeta(z)$. Clearly ϑ is analytic on a neighborhood of $\sigma(A)$ and $\prod_{i=1}^m (z-\lambda_i)^{\alpha_i} = f(z)\vartheta(z) = \vartheta(z) f(z)$. Therefore $\prod_{i=1}^m (A - \lambda_i)^{\alpha_i} = f(A)\vartheta(A) = \vartheta(A) f(A)$. Since $f(A) \in \mathcal{R}(X)$, according to Proposition 1.1, $\prod_{i=1}^m (A-\lambda_i)^{\alpha_i}$ belongs to $\mathcal{R}(X)$ which ends the proof. □

In what follows $\mathcal{A}(X)$ will denote the subset of $P\mathcal{R}(X)$ defined by

$$\mathcal{A}(X) := \Big\{F \in P\mathcal{R}(X) \text{ such that the minimal polynomial } p(\cdot) \text{ of } F \text{ satisfies } p(-1) \neq 0\Big\}.$$

Lemma 2.3. *If $F \in \mathcal{A}(X)$, then $I + F \in \Phi(X)$ and $i(I + F) = 0$.*

Proof: Let $p(\cdot)$ denote the minimal polynomial of F. Since $p(F) \in \mathcal{R}(X)$, $\sigma_b(p(F)) = \{0\}$. By hypothesis $p(-1) \neq 0$, hence $p(-1) \notin \sigma_b(p(F))$. Next, making use of the spectral mapping theorem for the Browder essential spectrum [6, Theorem 4] we conclude that $-1 \notin \sigma_b(F)$, i.e., $-1 \in \rho_b(F)$. This completes the proof. □

The following lemma improves Proposition 4.1 in [10].

Lemma 2.4. *Let $(T(t))_{t\geq 0}$ be a* C_0*-semigroup on a Banach space X with infinitesimal generator A and assume that the hypothesis $(\mathcal{A}1)$ is satisfied. If $\mathcal{D}^+(\varphi)\neq\emptyset$, then $(T(t))_{t\geq 0}$ can be embedded in a C_0-group on X.*

Proof: By hypothesis there exists $t_0>0$ such that $T(t_0)\in P\mathcal{R}(X)$. Let $p_{t_0}(z)=\prod_{i=1}^n(z-\varphi_i(t_0))$ be the minimal polynomial of $T(t_0)$. Thus $p_{t_0}(T(t_0))=\prod_{i=1}^n(T(t_0)-\varphi_i(t_0)\,I)\in\mathcal{R}(X)$. Writing $T(t_0)$ in the form $T(t_0)=I+(T(t_0)-I)$ one sees that

$$p_{t_0}(T(t_0))=\prod_{i=1}^n((T(t_0)-I)-(\varphi_i(t_0)-1)I)=\bar{p}_{t_0}(T(t_0)-I)$$

where $\bar{p}_{t_0}(z)=\prod_{i=1}^n(z-(\varphi_i(t_0)-1))$. Clearly, $\bar{p}_{t_0}(-1)=\prod_{i=1}^n(-\varphi_i(t_0))\neq 0$, therefore $T(t_0)-I\in\mathcal{A}(X)$. It follows from Lemma 2.3 that $T(t_0)=I+(T(t_0)-I)$ is a Fredholm operator and $i(T(t_0))=0$. Now the use of Theorem 2.1 in [10] gives the desired result. □

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with infinitesimal generator A and assume that the condition $(\mathcal{A}1)$ is satisfied. Then there exists a continuous function $\varphi:\mathcal{D}(\varphi)\subseteq\mathbb{R}\to\mathbb{C}^n$, $t\to(\varphi_1(t),\dots,\varphi_n(t))$ such that for all $t\in\mathcal{D}^+(\varphi)$, $\varphi_i(t)\neq 0$ and $\prod_{i=1}^n(T(t)-\varphi_i(t))\in\mathcal{R}(X)$, i.e., for all $t\in\mathcal{D}^+(\varphi)$, $T(t)\in P\mathcal{R}(X)$. By Lemma 2.1, the spectrum of $T(t)$ consists of eigenvalues with finite algebraic multiplicity possibly accumulating at the points $\varphi_i(t)$, $i=1,\dots,n$. So, the spectral mapping theorem for the point spectrum (cf. [4, Equation (3.16), p. 277]) yields $\sigma(T(t))\backslash\{\varphi_i(t),\, i=1,\dots,n\}=\{e^{\eta t}:\eta\in\sigma_p(A)\}$ ($\sigma_p(A)$ denotes the point spectrum of A).

Set

$$\mathcal{I}:=\{\operatorname{Im}\lambda:\lambda\in\sigma_p(A)\}.$$

Lemma 2.5. *Let $(T(t))_{t\geq 0}$ be a* C_0*-semigroup on a Banach space X with infinitesimal generator A and assume that the hypothesis $(\mathcal{A}1)$ is satisfied. If $\mathcal{D}^+(\varphi)\neq\emptyset$, then the set $\mathcal{I}$ is bounded.*

Proof: To prove this we will proceed by contradiction. If $\mathcal{I}$ is unbounded, then there exists in $\mathcal{I}$ a sequence $(a_k)_{k\in\mathbb{N}}$ such that $a_{k+1}>a_k$ and $\lim_{k\to+\infty}a_k=+\infty$ or $a_{k+1}<a_k$ and $\lim_{k\to+\infty}a_k=-\infty$. The treatment of these two cases is the same, so we restrict ourselves to the first one. Thus, there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that $\lambda_k\in\sigma_p(A)$ and $a_k=\operatorname{Im}\lambda_k$. So $e^{t\lambda_k}\in\sigma_p(T(t))$ and $\arg(e^{t\lambda_k})=ta_k+2m\pi$ for some $m\in\mathbb{Z}$. Since $\varphi_i(t)$, $i=1,\dots,n$ are the only possible accumulation points of $\sigma_p(T(t))$, then, for all $\alpha>0$, $\{k\in\mathbb{N}:\sup_{1\leq i\leq n}|e^{t\lambda_k}-\varphi_i(t)|>\alpha\}$ is finite. So, for all ε satisfying $0<\varepsilon<1/2$,

$$\Big\{k\in\mathbb{N}:\sup_{1\leq i\leq n,\, m\in\mathbb{Z}}|ta_k-\arg(\varphi_i(t))+2m\pi|>\frac{\varepsilon}{2}\Big\}$$

is finite, i.e.

$$\Big\{k \in \mathbb{N} : ta_k \notin \bigcup_{1\leq i\leq n,\, m\in\mathbb{Z}} \Big[\arg(\varphi_i(t)) + 2m\pi - \frac{\varepsilon}{2}, \arg(\varphi_i(t)) + 2m\pi + \frac{\varepsilon}{2}\Big]\Big\}$$

is finite. Let $t_0 > 0$ be a fixed point such that in the set

$$G_\varepsilon = \bigcup_{1\leq i\leq n,\, m\in\mathbb{Z}} \Big[\arg(\varphi_i(t_0)) + 2m\pi - \frac{\varepsilon}{2}, \arg(\varphi_i(t_0)) + 2m\pi + \frac{\varepsilon}{2}\Big]$$

the intervals are disjoint or identical. (We also choose a determination of $\arg(\varphi_i(t_0))$ such that none of $\varphi_i(t_0)$ are on the half axis of discontinuity of $\arg(\cdot)$. This is possible because there are in finite number and different from zero.) Clearly, the complement of G_ε in $\mathbb{R}$ is a reunion of open intervals, hence G_ε is closed. Let $\delta > 0$ be such that $|t - t_0| < \delta$ implies $|\arg(\varphi_i(t)) - \arg(\varphi_i(t_0))| < \frac{\varepsilon}{2}$ for $i = 1, 2, \dots, n$ (use the continuity of $\varphi_i(\cdot)$ and $\arg(\cdot)$ at $\varphi_i(t_0)$). Hence $\forall t \in]t_0 - \delta, t_0 + \delta[$, $\{k \in \mathbb{N},\ ta_k \notin G_\varepsilon\}$ is finite, so $\forall t \in]t_0 - \delta, t_0 + \delta[$, $\exists N_t \in \mathbb{N}$ such that $k \geq N_t$ implies $ta_k \in G_\varepsilon$, (or $t \in \frac{1}{a_k}G_\varepsilon$) and

$$]t_0 - \delta, t_0 + \delta[\subseteq \bigcup_{N\in\mathbb{N}} \Big(\bigcap_{k\geq N} \frac{1}{a_k}G_\varepsilon\Big).$$

By the Baire category theorem we conclude that there exists $N \in \mathbb{N}$ such that $\bigcap_{k\geq N} \frac{1}{a_k}G_\varepsilon$ has a non empty interior. Accordingly, there are a and b in $]0, \infty[$ with $a < b$ and $]a, b[\subseteq (\bigcap_{k\geq N} \frac{1}{a_k}G_\varepsilon)$. But $]a, b[$ would be contained in one of the connected components of $\bigcap_{k\geq N} \frac{1}{a_k}G_\varepsilon$. Consequently, $b - a \leq \frac{2\varepsilon}{a_k}$ for every $k \in \mathbb{N}$, so we come to a contradiction because $a_k \to \infty$ as $k \to \infty$. This completes the proof. □

Lemma 2.6. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with infinitesimal generator A and assume that the hypothesis $(\mathcal{A}1)$ is satisfied. If $\mathcal{D}^+(\varphi)$ contains a set with nonempty interior, then A is bounded on X.*

For completeness we recall the following result owing to R.S. Phillips (cf. [14, Corollary 4.1]) required in the proof of Lemma 2.6.

Lemma 2.7. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Suppose that there exists an unbounded open connected set $\Omega \subseteq \mathbb{C}$ such that $0 \in \Omega$ and $\sigma(T(t)) \cap \Omega = \emptyset$ for t belonging to some interval $]a, b[$ with $0 \leq a < b$. Then the infinitesimal generator of $(T(t))_{t\geq 0}$ is bounded.*

Proof of Lemma 2.6: By the hypotheses there exists a continuous function $\varphi : \mathcal{D}(\varphi) \subseteq \mathbb{R} \to \mathbb{C}^n$, $t \to (\varphi_1(t), \dots, \varphi_n(t))$ (with $n \geq 2$) such that for all $t \in \mathcal{D}^+(\varphi)$, $\varphi_i(t) \neq 0$ and $\prod_{i=1}^n (T(t) - \varphi_i(t))^{\alpha_i} \in \mathcal{R}(X)$, i.e., for all $t \in \mathcal{D}^+(\varphi)$,

$T(t) \in \mathcal{PR}(X)$. By Lemma 2.1, the spectrum of $T(t)$ consists of eigenvalues with finite algebraic multiplicity possibly accumulating at the points $\varphi_i(t)$, $i = 1, \dots, n$. So, by the spectral mapping theorem for the point spectrum (cf. [4, Equation (3.16), p. 277]), $\sigma(T(t)) \backslash \{\varphi_i(t),\, i = 1, \dots, n\} = \{e^{\eta t} : \eta \in \sigma_p(A)\}$.

We know by Lemma 2.5 that $\mathcal{I} := \{\operatorname{Im} \lambda : \lambda \in \sigma_p(A)\}$ is bounded. Set

$$M := \sup\{|\operatorname{Im}(\lambda)| : \lambda \in \sigma_p(A)\}.$$

We will construct an open set satisfying the hypotheses of Lemma 2.7. To do so, let us first observe that, if $\lambda \in \sigma_p(A)$ and $t \in [0, \frac{\pi}{2M}]$ then $|\arg(e^{\lambda t})| \leq \frac{\pi}{2}$. So, for $t \in [0, \frac{\pi}{2M}]$, one has

$$e^{t\sigma_p(A)} = \sigma_p(T(t)) \subseteq \{z : \operatorname{Re}(z) \geq 0\}.$$

On the other hand, according to Lemma 2.4, the semigroup $(T(t))_{t\geq 0}$ can be embedded in a C_0-group, that is,

$$\widetilde{T}(t) = \begin{cases} T(t) & \text{if } t \geq 0, \\ T(-t) & \text{if } t \leq 0. \end{cases}$$

Hence, for each $x \in X$, the map $t \in [0, \frac{\pi}{2M}] \to T(-t)x$ is continuous. So, there exists $M_x \geq 0$ such that $\|T(-t)x\| \leq M_x$ for all $t \in [0, \frac{\pi}{2M}]$. Therefore, by the Banach-Steinhaus theorem, there exists $M' \geq 0$ such that

$$\|T(-t)\| \leq M' \quad \text{for all} \quad t \in \left[0, \frac{\pi}{2M}\right].$$

Since $T(-t) = T(t)^{-1}$, it follows that $\|T(t)^{-1}x\| \leq M'\|x\|$ and then

$$\|T(t)x\| \geq \frac{1}{M'}\,\|x\| \quad \text{for all} \quad x \in X. \tag{2.3}$$

Let λ be such that $|\lambda| < \frac{1}{M'}$. It follows from (2.3) that, if $t \in [0, \frac{\pi}{2M}]$, then

$$\|(T(t) - \lambda)x\| \geq (\frac{1}{M'} - |\lambda|)\,\|x\|$$

and consequently $\lambda \notin \sigma_p(T(t))$. So, for $t \in [0, \frac{\pi}{2M}]$,

$$\sigma_p(T(t)) \subseteq \left\{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0 \quad \text{and} \quad |z| \geq \frac{1}{M'}\right\}.$$

Recall that, by hypothesis, for $t > 0$ (use Lemma 2.1),

$$\sigma(T(t)) = \sigma_p(T(t)) \cup \sigma_e(T(t)) \subseteq \sigma_p(T(t)) \cup \{\varphi_1(t), \dots, \varphi_n(t)\}.$$

Let $t_0 \in]0, \frac{\pi}{2M}[$ and $\varepsilon > 0$ be given. Then there exists $\delta >$ such that $]t_0-\delta, t_0+\delta[\subset]0, \frac{\pi}{2M}[$ and $|t-t_0| < \delta$ imply $|\varphi_i(t)-\varphi_i(t_0)| \le \varepsilon$ $\forall i \in \{1, \dots, n\}$. So, $|t-t_0| < \delta$ implies $\sigma(T(t)) \subseteq \mathcal{S}(t_0, \varepsilon)$ where

$$\mathcal{S}(t_0,\varepsilon) := \Big\{ z \in \mathbb{C} : \operatorname{Re}(z) \ge 0 \ \text{ and } \ |z| \ge \frac{1}{M'} \Big\} \cup \\ \cup \Big(\bigcup_{1 \le i \le n} \{ z \in \mathbb{C} : |z - \varphi_i(t_0)| \le \varepsilon \} \Big).$$

This shows that, for $\varepsilon > 0$ small enough, the complement in $\mathbb{C}$ of the set $\mathcal{S}(t_0, \varepsilon)$ is an unbounded open connected set Ω with $0 \in \Omega$. Now applying Lemma 2.7 we conclude that $A \in \mathcal{L}(X)$ which ends the proof. $\square$

Lemma 2.8. *Let $(T(t))_{t\ge 0}$ be a C_0-semigroup on a Banach space X and let $t_0 > 0$.*

(1) *If $0 \in \sigma_e(T(t_0))$, then $0 \in \sigma_e(T(t))$ for all $t > 0$.*
(2) *If $T(t_0) \in \mathcal{R}(X)$, then $(T(t))_{t\ge 0}$ is a C_0-semigroup of Riesz type.*

Let $\mathcal{J}(X)$ be a proper ideal (one-sided ideal suffices) of $\mathcal{L}(X)$ and $(T(t))_{t\ge 0}$ a strongly continuous semigroup. If, for some $t_0 > 0$, $T(t_0)$ belongs to $\mathcal{J}(X)$, then $T(t) \in \mathcal{J}(X)$ for all $t \ge t_0$. However, for $t < t_0$, $T(t)$ does not necessarily belong to $\mathcal{J}(X)$. The item (2) of the preceding lemma shows that when dealing with $\mathcal{R}(X)$, C_0-semigroups possess a more regular behaviour in the sense that if, for some $t_0 > 0$, $T(t_0) \in \mathcal{R}(X)$, then $T(t) \in \mathcal{R}(X)$ for all $t > 0$. A similar behaviour is provided by Proposition 2.2 in [10] which asserts that if, for some $t_0 > 0$, $T(t_0)$ is a semi-Fredholm (resp. Fredholm) operator, then $(T(t))_{t\ge 0}$ is a semi-Fredholm (resp. Fredholm) C_0-semigroup.

Proof of Lemma 2.8: (1) Let $t_0 > 0$ be such that $0 \in \sigma_e(T(t_0))$. If, for some $t > 0$, $0 \notin \sigma_e(T(t))$, then $T(t)$ is a Fredholm operator. Hence, by Proposition 2.2 in [10], we infer that $T(t)$ is invertible for all $t \ge 0$. This contradicts the fact that $0 \in \sigma_e(T(t_0))$. Therefore, for all $t > 0$, $0 \in \sigma_e(T(t))$.

(2) Let $t > 0$. If $t > t_0$, then $T(t) = T(t_0)T(t-t_0)$. Since $T(t_0)$ and $T(t-t_0)$ commute, the use of Proposition 1.1 implies that $T(t) \in \mathcal{R}(X)$. Assume now that $t < t_0$. There exists $n \in \mathbb{N}$ such that $\frac{t_0}{n} < t$. Hence $T(t) = T(\frac{t_0}{n})T(t - \frac{t_0}{n})$. Since $T(t_0) \in \mathcal{R}(X)$ and $T(t_0) = [T(\frac{t_0}{n})]^n$, the spectral mapping theorem shows that $T(\frac{t_0}{n})$ is also a Riesz operator. Now using the fact that $T(\frac{t_0}{n})$ and $T(t - \frac{t_0}{n})$ commute together with Proposition 1.1 one sees that $T(t) \in \mathcal{R}(X)$. $\square$

Lemma 2.9. *Let X be a Banach space and assume that $X = X_1 \oplus X_2$, $A \in \mathcal{L}(X)$ with $A(X_1) \subset X_1$, $A(X_2) \subseteq X_2$. Let $A_1 = A_{|X_1} \in \mathcal{L}(X_1)$ and $A_2 = A_{|X_2} \in \mathcal{L}(X_2)$. Then:*

(1) $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$,

(2) $\sigma_e(A) = \sigma_e(A_1) \cup \sigma_e(A_2)$,
(3) *if* λ *is an isolated point of* $\sigma(A_1)$ *such that* $\lambda \notin \sigma(A_2)$, *then* $P_\lambda(A)(X) = P_\lambda(A_1)(X_1)$, *where* $P_\lambda(A)$ *(resp.* $P_\lambda(A_1)$*) denotes the spectral projection associated to* $\{\lambda\}$ *for* A *(resp.* A_1*) in* $\mathcal{L}(X)$ *(resp.* $\mathcal{L}(X_1)$*).*

Proof: The statements (1) and (2) are well known so their proofs are dropped.

(3) Let $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$. Then there exists a spectral decomposition of the space X_1, that is, $X_1 = Y_1 \oplus Y_2$ and $Y_1 = P_\lambda(A_1)X_1$ (where $P_\lambda(A_1)$ denotes the spectral projection associated to the spectral set $\{\lambda\}$). Accordingly, $\sigma(A_{1|Y_1}) = \{\lambda\}$, and $\sigma(A_{1|Y_2}) = \sigma(A_1)\backslash\{\lambda\}$. Note also that

$$X = Y_1 \oplus (Y_2 \oplus X_2),\ AY_1 = A_1Y_1 \subseteq Y_1 \text{ and } A(Y_2 \oplus X_2) \subseteq Y_2 \oplus X_2.$$

Now applying the first assertion to the operators $A_{|Y_2 \oplus X_2}$ and $A_{|Y_2}$ we get $\sigma(A_{|Y_2 \oplus X_2}) = \sigma(A_2) \cup \sigma(A_{|Y_2})$. But $\sigma(A_{|Y_2}) = \sigma(A_{1|Y_2}) = \sigma(A_1)\backslash\{\lambda\}$ and $\sigma(A) = \sigma(A_2) \cup \sigma(A_1)$, so $\sigma(A_{|Y_2 \oplus X_2}) = \sigma(A)\backslash\{\lambda\}$. Accordingly, $\sigma(A_{|Y_1}) = \sigma(A_{1|Y_1}) = \{\lambda\}$. This leads to $Y_1 = R(P_\lambda(A))$ (the range of the spectral projection associated to the spectral set $\{\lambda\}$ of $\sigma(A)$) which completes the proof. □

The item (3) of the preceding lemma will be used in the proof of the implication (3) ⇒ (2) of Theorem 1.1 in the following form.

Corollary 2.1. *Let the hypotheses of Lemma* 2.9 *be satisfied. If, further,* $A_2 = 0$ *and* $\lambda \neq 0$ *is an isolated point of* $\sigma(A_1)$, *then*

$$P_\lambda(A_1)(X_1) = P_\lambda(A)(X).$$

3. Proofs

Proof of Theorem 1.1: (1) ⇒ (3). From Lemma 2.6 it follows that $A \in \mathcal{L}(X)$. So, it remains to check that A belongs to $P\mathcal{R}(X)$. To this end, let $t \in]a, b[$ and write $T(t) = e^{tA}$. Set $p_t(z) = \prod_{i=1}^n (z - \varphi_i(t))$. We know by the hypotheses that

$$p_t(e^{tA}) = \prod_{i=1}^n (e^{tA} - \varphi_i(t)) \in \mathcal{R}(X).$$

Accordingly, $f_t(A) \in \mathcal{R}(X)$ where $f_t(\cdot)$ is the entire function $z \to p_t(e^{tz})$. So, Lemma 2.2 implies that $A \in P\mathcal{R}(X)$.

(3) ⇒ (2). To establish this implication we will proceed in three steps.

First step : Let $t > 0$ and suppose that $A \in \mathcal{R}(X)$. Clearly the operator $T(t) - I$ may be written in the form

$$T(t) - I = e^{tA} - I = \sum_{k=1}^{\infty} \frac{t^k A^k}{k!} = A\, g(A),$$

where $g(\cdot)$ is the entire function $g(z) = \sum_{k=0}^{\infty} \frac{t^{k+1}z^k}{(k+1)!}$. Since A and $g(A)$ commute, $T(t) - I = A\,g(A) = g(A)\,A$. Since A is of Riesz type, the use of Proposition 1.1 implies that $T(t) - I \in \mathcal{R}(X)$.

Second step : Assume that $(A - \lambda I) \in \mathcal{R}(X)$ for some $\lambda \in \mathbb{C}$. Observe that the operator $T(t) - e^{\lambda t}$ can be written as

$$T(t) - e^{\lambda t} I = e^{\lambda t}(e^{-\lambda t} T(t) - I) = e^{\lambda t}(e^{-\lambda t} e^{tA} - I) = e^{\lambda t}(e^{t(A-\lambda I)} - I).$$

Then, since $(A - \lambda I) \in \mathcal{R}(X)$, using the result of the first step one obtains $(e^{t(A-\lambda I)} - I) \in \mathcal{R}(X)$ and therefore $T(t) - e^{\lambda t} I \in \mathcal{R}(X)$.

Third step : Assume now that $p(z) = (z-\lambda_1)\dots(z-\lambda_n)$ is the minimal polynomial of A, then $p(A) = (A - \lambda_1 I)\dots(A - \lambda_n I) \in \mathcal{R}(X)$. This implies that $\sigma_b(A) = \{\lambda_1, \dots, \lambda_n\}$, so we can write $\sigma(A) = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_n$ where σ_i for $1 \le i \le n$ are clopen sets in $\sigma(A)$ such that $\lambda_i \in \sigma_i$ for $i = 1, \dots, n$ and $\sigma_i \cap \sigma_j = \emptyset$ if $i \ne j$. Let $(X_i)_{1 \le i \le n}$ and $(A_i)_{1 \le i \le n}$ be the spectral subspaces and the restrictions of A associated to this decomposition, respectively. The fact that the sets X_i, $i = 1, \dots, n$, are stable by A implies that they are also invariant by e^{tA}. Let $e^{tA}{}_{|X_i}$ be the restriction of e^{tA} to X_i. Obviously $e^{tA}{}_{|X_i} \in \mathcal{L}(X_i)$ and $e^{tA}{}_{|X_i} = e^{tA_i}$.

Consider now the problem separately on each subspace X_i, $i = 1, \dots, n$. On X_i we can write $p(A) = (A_i - \lambda_i) \prod_{i \ne j}(A_i - \lambda_j)$. For each $j \ne i$, $\lambda_j \notin \sigma(A_i)$ and so the operator $\prod_{i \ne j}(A_i - \lambda_j)$ is invertible in $\mathcal{L}(X_i)$. Accordingly

$$(A_i - \lambda_i) = p(A_i)(\prod_{i \ne j}(A - \lambda_j))^{-1} \in \mathcal{L}(X_i).$$

On the other hand, the spectral mapping theorem leads to

$$\sigma(p(A_i)) = p(\sigma(A_i)) = \{p(\tau) : \tau \in \sigma(A_i)\}.$$

Therefore $\sigma(p(A_i))$ consists of eigenvalues with finite algebraic multiplicity accumulating to $p(\lambda_i) = 0$. So, $p(A_i) \in \mathcal{R}(X_i)$. This together with the fact that $p(A_i)$ and $(\prod_{i \ne j}(A - \lambda_j))^{-1}$ commute implies, thanks to Proposition 1.1, $(A_i - \lambda_i) \in \mathcal{R}(X_i)$.

On the other hand, for each $i \in \{1, \dots, n\}$, we have

$$\prod_{j=1}^{n} \left(e^{tA_i} - e^{\lambda_j t}\right) = \left(e^{tA_i} - e^{\lambda_i t}\right) \prod_{j \ne i} \left(e^{tA_i} - e^{\lambda_j t}\right).$$

Since $\prod_{j \ne i}(e^{tA_i} - e^{\lambda_j t})$ is invertible on X_i and $e^{tA_i} - e^{\lambda_i t} \in \mathcal{R}(X_i)$ (use the fact that $(A_i - \lambda_i) \in \mathcal{R}(X_i)$ and the second step), again by Proposition 1.1, we infer that

$$\prod_{j=1}^{n}\left(e^{tA}{}_{|X_i}-e^{\lambda_j t}\right)=\prod_{j=1}^{n}\left(e^{tA}-e^{\lambda_j t}\right)_{|X_i}\in\mathcal{R}(X_i).$$

Next, observe that the operator $\prod_{i=1}^{n}(e^{tA}-e^{\lambda_i t})$ writes in the form $\sum_{i=1}^{n}\mathcal{O}_i$ where

$$\mathcal{O}_i = J_i\Big[\prod_{j=1}^{n}\left(e^{tA}-e^{\lambda_j t}\right)_{|X_i}\Big]P_i,$$

with $J_i : X_i \to X$ is the canonical embedding and $P_i : X \to X_i$ denotes the spectral projection associated to the clopen subset σ_i. Clearly $\mathcal{O}_i\,\mathcal{O}_j = \mathcal{O}_j\,\mathcal{O}_i = 0$ for $i \neq j$. Moreover, using Lemma 2.9(3) (or Corollary 2.1) one sees that each $\mathcal{O}_i$, $i = 1,\dots,n$, belongs to $\mathcal{R}(X)$. Now applying Proposition 1.1 we get $]0,\infty[\subset \mathcal{D}(\varphi)$.

(2) $\Rightarrow$ (1) It is trivial.

(3) $\Rightarrow$ (4) Let us first observe that, by the spectral mapping theorem, for any $\lambda\in\rho(A)$, $-1\in\rho(\lambda R(\lambda,A)-I)$. Now consider the function f_λ defined by $f_\lambda : \mathbb{C}\backslash\{-1\}\to\mathbb{C}$, $z\to\lambda-\frac{\lambda}{z+1}$. Clearly $A = f_\lambda(\lambda R(\lambda,A)-I)$. By hypothesis there exists $p(\cdot)\in\mathbb{C}[z]\backslash\{0\}$ such that $p(A)\in\mathcal{R}(X)$. Therefore $(p\circ f_\lambda)(\lambda R(\lambda,A)-I)\in\mathcal{R}(X)$. Next applying Lemma 2.2 we conclude that $(\lambda R(\lambda,A)-I)\in P\mathcal{R}(X)$ for every $\lambda\in\rho(A)$.

(4) $\Rightarrow$ (5) It is trivial.

(5) $\Rightarrow$ (3) Let $\lambda\in\rho(A)$ be such that $(\lambda R(\lambda,A)-I)\in\mathcal{R}(X)$ and denote by g_λ the function defined by $g_\lambda : \mathbb{C}\backslash\{\lambda\}\to\mathbb{C}$, $z\to\frac{\lambda}{\lambda-z}-1$. Since $g_\lambda(A)=\lambda R(\lambda,A)-I$, the use of Lemma 2.2 gives $A\in P\mathcal{R}(X)$. $\square$

Proof of Theorem 1.2: The hypotheses say that there is a function $\varphi : \mathcal{D}(\varphi)\subset\mathbb{R}\to\mathbb{C}$, $t\to(\varphi_1(t),\dots,\varphi_n(t))$ such that $\forall\, t\in\mathcal{D}^+(\varphi)$ $\prod_{i=1}^{n}(T(t)-\varphi_i(t))\in\mathcal{R}(X)$. By the characterization of polynomially Riesz operators (Proposition 1.2) and Lemma 2.1, we have $\sigma_e(T(t))=\{\varphi_i(t),\, i=1,\dots,n\}$ and $\sigma(T(t))\backslash\sigma_e(T(t))$ consists in eigenvalues with finite algebraic multiplicity. Since $\varphi_{i_0}(t_0)=0$ for some $i_0\in\{1,\dots,n\}$, say $i_0=n$, it follows from Lemma 2.8(1) that $\varphi_n(t)=0$ for all $t>0$. This implies that $0\in\sigma_e(T(t))$ for all $t>0$. Consequently, $\sigma_e(T(t))=\{0,\varphi_1(t),\dots,\varphi_{n-1}(t)\}$ with $\varphi_i(t)\neq 0$ for all $t\in\mathcal{D}^+(\varphi)$ and $i=1,\dots,n-1$.

Let τ_0 and τ_1 be a partition of $\sigma(T(t_0))$ such that $\tau_0\cap\sigma_e(T(t_0))=\{0\}$. Hence there exist two closed subspaces X_0 and X_1 which reduce $T(t_0)$, that is, $X = X_0\oplus X_1$. Clearly, for all $t>0$, $T(t)$ commutes with $T(t_0)$ and therefore with the associated spectral projectors. So, $T(t)X_k\subseteq X_k$, $k=0,1$ and then $(T(t)_{|X_k})_{t\geq 0}$ is a C_0-semigroup on X_k, $A(D(A)\cap X_k)\subseteq X_k$ and so $A_{|X_k}$ is the generator of $(T(t)_{|X_k})_{t\geq 0}$. Since $0\notin\sigma_e(T(t_0)_{|X_1})$, it follows from

Lemma 2.8(1) that $0 \notin \sigma_e(T(t)_{|X_1})$ for all $t > 0$. Moreover, using Lemma 2.9(2), one sees that $\sigma_e(T(t)) = \sigma_e(T(t)_{|X_0}) \cup \sigma_e(T(t)_{|X_1})$ and therefore $\sigma_e(T(t)_{|X_1}) = \{\varphi_1(t), \dots, \varphi_{n-1}(t)\}$. This shows that $(T(t)_{|X_1})_{t\geq 0}$ satisfies the hypothesis $(\mathcal{A}1)$ and $]0, \infty[\subset \mathcal{D}(\varphi)$. Consequently, the statement (2) follows from Lemma 2.4 and Theorem 1.1.

To establish (1), let us note that since $0 \in \sigma_e(T(t_0)_{|X_0})$, it follows from Lemma 2.8(1) that $0 \in \sigma_e(T(t)_{|X_0})$ for all $t > 0$. To show that $(T(t)_{|X_0})_{t\geq 0}$ is a Riesz C_0-semigroup, it suffices to prove that $\sigma_e(T(t)_{|X_0}) = \{0\}$ for all $t > 0$. To see this, let $\mu \neq 0$ be such that $\mu \in \sigma_e(T(t)_{|X_0})$ for some $t > 0$. So we can write $X_0 = Z_1 \oplus Z_2$ with $\dim(Z_j) = \infty$, $j = 1, 2$ and $\sigma_e(T(t)_{|Z_2}) = \{\mu\}$. Obviously, $0 \notin \sigma_e(T(t)_{|Z_2})$, so by Lemma 2.8(1), $0 \notin \sigma_e(T(t_0)_{|Z_2})$. On the other hand, applying Lemma 2.9(2) one sees that $\sigma_e(T(t_0)_{|X_0}) = \sigma_e(T(t_0)_{|Z_1}) \cup \sigma_e(T(t_0)_{|Z_2}) = \{0\}$. Hence, $\sigma_e(T(t_0)_{|Z_2}) = \emptyset$ which contradicts the fact that $\dim Z_2 = \infty$. Consequently $\mu \notin \sigma_e(T(t)_{|X_0})$ which proves that $T(t)_{|X_0}$ is a Riesz operator for all $t > 0$. This ends the proof. □

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(*Received* January 18, 2005, *revised* January 13, 2006)