A new class of weakly K-analytic Banach spaces

S. Mercourakis, E. Stamati

Abstract. In this paper we define and investigate a new subclass of those Banach spaces which are K-analytic in their weak topology; we call them strongly weakly K-analytic (SWKA) Banach spaces. The class of SWKA Banach spaces extends the known class of strongly weakly compactly generated (SWCG) Banach spaces (and their subspaces) and it is related to that in the same way as the familiar classes of weakly K-analytic (WKA) and weakly compactly generated (WCG) Banach spaces are related.

We show that: (i) not every separable Banach space is SWKA; (ii) every separable SWKA Banach space not containing ℓ^1 is Polish; (iii) we answer in the negative a question posed in [S-W] by constructing a subspace X of the SWCG space $L^1[0,1]$ which is not SWCG.

Keywords: WKA, SWKA Banach spaces, K-analytic space, Baire space, Polish space

Classification: Primary 46B20, 54H05, 03E75; Secondary 46B26

Introduction

The purpose of the present paper is to introduce and study a new class of Banach spaces which are K-analytic in their weak topology ([T], [N]), that extends the class of strongly weakly compactly generated (SWCG) Banach spaces (and their subspaces) introduced by Schlüchtermann and Wheeler in [S-W]. We call them strongly weakly K-analytic-SWKA Banach spaces (see Definition 1.4).

Our results show that the classes of SWCG and SWKA Banach spaces are related in the same way as the known classes of weakly compactly generated (WCG) and weakly K-analytic (WKA) Banach spaces.

In Section 1, we study the general properties of SWKA Banach spaces. We show in particular, that every subspace of a SWCG Banach space is SWKA (Proposition 1.5) and also, that a Banach space with separable dual is SWKA iff it is Polish (Proposition 1.9). So we get as an easy consequence that not every separable Banach space is SWKA (Corollary 1.10). We also give a useful characterization of SWKA Banach spaces by using the pointwise order of the Baire space $\Sigma = \mathbb{N}^{\mathbb{N}}$ (Proposition 1.7), which is similar to a corresponding characterization of WKA Banach spaces due to Talagrand ([T, Proposition 6.13]).

In the same section we strengthen the classical K-analytic property for topological spaces by introducing the notion of a strongly K-analytic topological space (Definition 1.11), so that a Banach space X is SWKA iff X is a strongly K-analytic topological space in its weak topology. Then we characterize strongly

K-analytic topological spaces as those topological spaces X for which the space $\mathcal{K}(X)$ of non empty compact subspaces of X endowed with Vietoris topology is a (strongly) K-analytic topological space (Theorem 1.12).

In Section 2, we show, by using results of Stegall (Theorem 2.3) and Stern (Theorem 2.2), that every separable SWKA Banach space not containing ℓ^1 is Polish (Theorem 2.6). This result generalizes a result from [S-W, Proposition 2.10]: Every (subspace of a) SWCG space not containing ℓ^1 is reflexive. We also give an example of a separable dual space X_0 with an unconditional basis (hence X_0 is weakly sequentially complete) which is not SWKA (Example 2.9); we mention, in contrast, that every (subspace of a) SWCG space is weakly sequentially complete ([S-W, Theorem 2.5]).

The third section of the paper is devoted to the construction of a closed linear subspace X of the SWCG space $L^1[0,1]$ (thus X is SWKA) which is not a SWCG space. This construction is closely related to the classical construction of Rosenthal [R1] of a non WCG subspace of the space $L^1[0,1]^{\mathfrak{c}}$ ($\mathfrak{c}$ = the cardinality of the continuum). The present example of the space X answers a question posed in [S-W].

We would like to thank S. Argyros for suggesting us a result of J. Stern (Theorem 2.2) that allowed us to avoid the previous use of Martin's axiom in the proof of Theorem 2.6 and to give a ZFC proof of that.

Preliminaries and notation

We denote by Σ the set $\mathbb{N}^{\mathbb{N}}$ of infinite sequences of positive integers, endowed with the cartesian product topology, which makes Σ (usually called the "Baire space") a Polish space (i.e., homeomorphic to a complete separable metric space).

We denote by S the set $\bigcup_{n=0}^{\infty} \mathbb{N}^n$ ($\mathbb{N}^0 = \{\emptyset\}$) of finite sequences of positive integers. We give the following partial order in S: for $s = (s_1, \dots, s_n), t = (t_1, \dots, t_m)$ members of S we define $s \le t$ iff $n \le m$ and $s_i = t_i$ for $i = 1, \dots, n$. If $s = (s_1, \dots, s_k) \in S$, $\sigma = (n_1, n_2, \dots, n_i, \dots) \in \Sigma$ and $m \in \mathbb{N}$ then we write,

(i) $s < \sigma$ iff $s_i = n_i$ for $i = 1, \dots, k$ and
(ii) $\sigma \,|\, m$ for the finite sequence $(n_1, n_2, \dots, n_m)$.

For every $s \in S$, we set $V_s = \{\sigma \in \Sigma : s < \sigma\}$; it is easy to see that the countable family $\{V_s : s \in S\}$ is a base for the topology of Σ consisting of open and closed sets.

For a (Hausdorff) topological space X, $\mathcal{K}(X)$ is the set of non empty compact subsets of X. A map $F : \Sigma \to \mathcal{K}(X)$ is said to be upper semicontinuous iff for every $\sigma \in \Sigma$ and every open subset U of X with $F(\sigma) \subseteq U$ there exists $s \in S$ with $s < \sigma$ so that $F(V_s) \subseteq U$.

Let X be a (real) Banach space and X^* its dual. Then B_X, B_{X^*} will denote the closed unit balls of X and X^* respectively. The Mackey topology $\tau = \tau(X^*, X)$ is the finest locally convex topology on X^* whose dual space is X. The Mackey-

Arens theorem characterizes τ as the topology on X^* of uniform convergence on weakly compact (absolutely convex) subsets of X (see [H-H-Z, pp. 163–165]).

1

Definition 1.1. Let X be a Banach space and $K \subseteq X$. We shall say that K *strongly generates* X if for every weakly compact subset L of X and every $\varepsilon > 0$ there exists $\lambda > 0$ such that

$$L \subseteq \lambda K + \varepsilon B_X. \tag{1}$$

Remark 1.2. If a set K strongly generates a Banach space X then clearly K is total in X. In the converse direction we observe that, if the convex symmetric set K is total in X then condition (1) of the previous definition holds for every norm-compact subset of X. Indeed, if $L \subseteq X$ is norm-compact and $\varepsilon > 0$, pick $x_1, \dots, x_n \in L$ such that $L \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon/2)$. Let now $\lambda > 0$ and $w_1, \dots, w_n \in K$ with $\|x_i - \lambda w_i\| < \varepsilon/2$ for $i = 1, 2, \dots, n$, then it is easy to see that $L \subseteq \lambda K + \varepsilon B_X$.

A Banach space X is said to be *strongly weakly compactly generated* (SWCG) if it is strongly generated by a weakly compact set K ([S-W]). We recall that a Banach space X is called weakly compactly generated (WCG), if it contains a weakly compact set K which is total in X since we may assume by Krein's theorem that K in addition is convex and symmetric, it follows from Remark 1.2 that every SWCG Banach space is WCG. The two notions are distinct. Indeed, every separable Banach space is WCG (let $\{x_n : n \in \mathbb{N}\}$ be a bounded total subset of X, then the set $\{\frac{x_n}{n} : n \in \mathbb{N}\} \cup \{0\}$ is a norm-compact total subset of X). On the other hand, every SWCG Banach space is both WCG and weakly sequentially complete, so the space c_0 is not SWCG (see [S-W, Theorem 2.5]). It follows in particular that not every separable Banach space is SWCG.

Example 1.3 (Examples of SWCG spaces [S-W]).

(i) Reflexive spaces (obvious).

(ii) Separable Schur spaces (let K be a norm compact convex symmetric set which is total in X, since by definition weakly compact subsets of X are norm compact, the conclusion follows from Remark 1.2).

(iii) The space $L^1(\mu)$, for a σ-finite measure μ. If μ is finite then it is proved, using Dunfford-Pettis' criterion for weakly compact subsets of $L^1(\mu)$, that the closed unit ball K of $L^\infty(\mu)$ considered as a subset of $L^1(\mu)$, is a weakly compact set that strongly generates $L^1(\mu)$. It then follows from standard results that $L^1(\mu)$ is SWCG for any σ-finite measure μ.

For further examples and results about SWCG property we refer the reader to [S-W].

We shall introduce now a new class of Banach spaces; we call it the class of strongly weakly K-analytic (SWKA) Banach spaces, which is related to the class of SWCG as WCG spaces are related to weakly K-analytic (WKA) spaces ([T], [N], [M-N], [D-G-Z]). We recall that a Banach space X is said to be WKA if there exists an upper semicontinuous map $F : \Sigma = \mathbb{N}^{\mathbb{N}} \to K(X)$, where $\mathcal{K}(X)$ is the set of weakly compact non empty subsets of X, so that $F(\Sigma) = X$ (i.e., $\bigcup_{\sigma\in\Sigma} F(\sigma) = X$).

Definition 1.4. A Banach space will be called *strongly weakly K-analytic* (SWKA) if there exists an upper semicontinuous map $F : \Sigma \to \mathcal{K}(X)$ so that ($F(\Sigma) = X$ and with the further property), for every weakly compact subset L of X there exists $\sigma \in \Sigma$ with $L \subseteq F(\sigma)$.

It clearly follows from the above that every SWKA space is WKA and also that every closed linear subspace of a SWKA space is SWKA itself. It is well known from a result of Preiss and Talagrand that every (subspace of a) WCG space is WKA ([T], [H-H-Z, Proposition 288]). The method of the proof of this result gives the similar result for the relation of SWCG and SWKA properties.

Proposition 1.5. *Every closed linear subspace of a SWCG space X is SWKA.*

Proof: It follows from the above remarks that it suffices to prove the assertion for a SWCG space X. Let K be a weakly compact (convex and symmetric) set that strongly generates X. We consider the map $F : \Sigma \to K(X)$ defined by the rule

$$F(\sigma) = \bigcap_{n=1}^{\infty} \Big(\sigma(n)K + \frac{1}{n}B_{X^{**}}\Big),$$

where $\sigma \in \Sigma$ and $B_{X^{**}}$ is the closed unit ball of X^{**} endowed with weak* topology. It is easy to verify that F is upper semicontinuous and that $F(\Sigma) = X$, in particular X is WKA (see the proof of the result of Preiss-Talagrand). Let now $L \subseteq X$ be a weakly compact set; since K strongly generates X, there exists for every $n \in \mathbb{N}$, $m_n \in \mathbb{N}$ so that $L \subseteq m_n K + \frac{1}{n}B_X$. Set $\sigma = (m_1, m_2, \dots, m_n, \dots)$, then it is obvious that $L \subseteq F(\sigma)$. $\square$

Note. In fact the method of the proof gives the further property

$$X = \bigcap_{n=1}^{\infty} \Big(\bigcup_{m=1}^{\infty} (mK + \frac{1}{n}B_{X^{**}})\Big),$$

that is, X is a $K_{\sigma\delta}$ subset of (X^{**}, ω^*) (see [Ta, Theorem 3.2]).

Remark 1.6. (1) H. Rosenthal has constructed a non WCG subspace X of the space $(L^1[0,1]^{\mathfrak{c}}, \mu)$, where μ the Lebesgue product measure ([R1]). Therefore the SWCG property is not preserved by closed linear subspaces;

this space X is obviously SWKA from Proposition 1.5 above. We shall present later (in Section 3) a refinement of the construction of Rosenthal which yields a non SWCG subspace of $L^1[0,1]$; this example answers in the negative a question posed in [S-W].

(2) If, for a Banach space X, there exists an upper semicontinuous map $F : \Sigma \to \mathcal{K}(B_X)$ such that (i) $F(\Sigma) = B_X$ and (ii) for every $L \subseteq B_X$ weakly compact there is $\sigma \in \Sigma$ with $L \subseteq F(\sigma)$, then X is SWKA. In order to make this clear, define a map $\Phi : \Sigma \times \mathbb{N} \to \mathcal{K}(X)$ by the rule $\Phi(\sigma, n) = nF(\sigma)$. It is easy to check that Φ makes X a SWKA space.

The following proposition gives a simple criterion to check the SWKA property; its analogue for WKA has been proved by Talagrand ([T, Proposition 6.13]). In the statement of our proposition we need the following partial order of the Baire space Σ: For $\sigma, \tau \in \Sigma$ we let $\sigma \leq \tau$ iff $\sigma(n) \leq \tau(n)$ for all $n \in \mathbb{N}$. We notice that every set of the form $\Sigma(\sigma) = \{\tau \in \Sigma : \tau \leq \sigma\}$ is compact in Σ and also that the family $\{\Sigma(\sigma) : \sigma \in \Sigma\}$ dominates the compact sets in Σ.

Proposition 1.7. *Let X be a Banach space. Then the following are equivalent.*

(i) *The space X is SWKA.*

(ii) *There exists a family $(W_\sigma)_{\sigma \in \Sigma}$ of weakly compact subsets of X (resp. of B_X) such that:*
   (a) $X = \bigcup_{\sigma \in \Sigma} W_\sigma$ *(resp.* $B_X = \bigcup_{\sigma \in \Sigma} W_\sigma$*);*
   (b) *for every $\sigma, \tau \in \Sigma$, if $\sigma \leq \tau$ then $W_\sigma \subseteq W_\tau$;*
   (c) *for every $L \subseteq X$ (resp. $L \subseteq B_X$) weakly compact there is $\sigma \in \Sigma$ with $L \subseteq W_\sigma$.*

Proof: (i) $\Rightarrow$ (ii). Let $F : \Sigma \to \mathcal{K}(X)$ be an upper semicontinuous map which makes X SWKA. We set $W_\sigma = F(\Sigma(\sigma))$ for $\sigma \in \Sigma$ (recall that $\Sigma(\sigma) = \{\tau \in \Sigma : \tau \leq \sigma\}$). It is easy to see that the family $(W_\sigma)_{\sigma \in \Sigma}$ satisfies assertion (ii) for X. It is obvious that the family $(W_\sigma \cap B_X)$ is the proper family for B_X.

(ii) $\Rightarrow$ (i). Let $(W_\sigma)_{\sigma \in \Sigma}$ be a family of weakly subsets of B_X that satisfies (a), (b) and (c) of assertion (ii). We set $W_s = \overline{\bigcup_{s<\tau} W_\tau}^{w}$, for $s \in S$ and then $F(\sigma) = \bigcap_{n=1}^{\infty} W_{\sigma\,|\,n}$. It is then easy to prove that F satisfies the requirements of Remark 1.6(2) and hence X is SWKA (cf. also the proof of Proposition 6.13 in [T]). □

At this point we state (for comparison with the above proposition but also because we shall use it) a nice characterization of Polish metric spaces due to Christensen ([Ch, Theorem 3.3, p. 58]).

Theorem 1.8 (Christensen)**.** *Let M be a metric space. Then M is Polish if and only if there exists a family $\{M_\sigma : \sigma \in \Sigma\}$ of compact subsets of M such that:* (a) *if $\sigma, \tau \in \Sigma$, $\sigma \leq \tau$ then $M_\sigma \subseteq M_\tau$ and* (b) *for every $L \subseteq M$ compact there exists $\sigma \in \Sigma$ with $L \subseteq M_\sigma$.*

Now we recall the concept of a Polish Banach space. A Banach space X is said to be Polish if its closed unit ball B_X endowed with the weak topology is a Polish space. Since (B_X, w) is then metrizable it follows in particular that a Polish Banach space has separable dual. Obvious examples of Polish Banach spaces are separable reflexive spaces. The predual JT_* of the James tree space JT is Polish (without being of course a reflexive space); the space c_0 is not Polish although it has separable dual ([E-W], [R2]).

Proposition 1.9. *Let X be a Banach space with separable dual. Then X is SWKA if and only if it is Polish.*

PROOF: Since X has separable dual its closed unit ball (B_X, w) is a separable metric space, therefore the result is an immediate consequence both of Proposition 1.7 and Theorem 1.8. □

It follows in particular from this proposition and previous remarks, that the space c_0 is not SWKA. This observation can be generalized as follows:

Corollary 1.10. *Let Ω be an infinite compact Hausdorff space. Then the Banach space $C(\Omega)$ is not SWKA.*

PROOF: Since Ω is infinite the space $C(\Omega)$ contains an isomorphic copy of c_0 which is not SWKA, but as we have noticed every subspace of a SWKA space is again SWKA, therefore $C(\Omega)$ is not SWKA. □

Remark 1.10.1. As it is proved in [S-W, Theorem 2.5] every SWCG Banach space is weakly sequentially complete, hence every subspace of a SWCG has the same property. So we get, from Rosenthal's ℓ^1-theorem [R0], that a subspace of a SWCG Banach space not containing ℓ^1 is reflexive. It follows in particular that a non reflexive Polish Banach space is not a subspace of a SWCG space.

The SWKA property for a Banach space is by definition a topological property of the weak topology of the space. There is no reason to restrict this concept only for the weak topology of Banach spaces; so we define the strong K-analyticity for every Hausdorff topological space.

Definition 1.11. A Hausdorff topological space X will be called *strongly K-analytic*, if there exists an upper semicontinuous map $F : \Sigma \to \mathcal{K}(X)$ such that

(a) $F(\Sigma) = X$ and
(b) for every $L \subseteq X$ compact there is $\sigma \in \Sigma$ with $L \subseteq F(\sigma)$.

Remark 1.11.1. (1) It is clear that every strongly K-analytic topological space is K-analytic in the classical sense; we shall see later that the converse is not true (see [J-R] and [T] for more information about K-analytic topological spaces). It is also obvious that SWKA Banach spaces are those Banach spaces which are strongly K-analytic in their weak topology according to the above definition.

(2) Every Polish metric space (M, d) is strongly K-analytic; indeed, it is known that the space $\mathcal{K}(M)$ of the compact non empty subsets of M with the Hausdorff metric is also Polish, (see [E, 4.5.22, p. 370]); so let $f : \Sigma \to \mathcal{K}(M)$ be a continuous onto map; then the map $F : \Sigma \to \mathcal{K}(M)$ defined by the rule $F(\sigma) = f(\Sigma(\sigma))$ has the desired properties. In the converse direction, we note that Theorem 1.8 easily implies that every strongly K-analytic metric space is Polish (see also Theorem 1.12). It is easily verified that every Čech-complete and Lindelöf topological space X (equivalently, X is homeomorphic to a closed subset of some product $M \times \Omega$ where M is a Polish metric space and Ω a compact Hausdorff space) is strongly K-analytic.

The following result says in particular that the class of strongly K-analytic topological spaces is natural and not "artificial". Before we state this result it is necessary to recall the definition of the Vietoris topology (otherwise, exponential or Hausdorff topology) on the space $K(X)$ of compact non empty subsets of a Hausdorff topological space X. The Vietoris topology τ_ν on $K(X)$ has as a basis consisting of subsets of $K(X)$ of the form:

$$\beta(V_1, \ldots, V_n) = \left\{ K \in K(X) : K \subseteq \bigcup_{i=1}^{n} V_i \text{ and } K \cap V_i \neq \emptyset \text{ for } i = 1, \ldots, n \right\}$$

where $n \in \mathbb{N}$ and $V_1, \ldots, V_n$ are open non empty subsets of X (see [E, 2.7.20, p. 162]).

In the proof of our result we shall make use of the following simple property of K-analytic topological spaces: If X is a Hausdorff and completely regular K-analytic topological space then there exists an upper semicontinuous map $F : \Sigma \to \mathcal{K}(X)$ such that

(i) $F(\Sigma) = X$ and
(ii) if $\sigma, \tau \in \Sigma$ and $\sigma \leq \tau$ then $F(\sigma) \subseteq F(\tau)$ (see [T, pp. 409–410]).

Theorem 1.12. *Let X be a Hausdorff and completely regular topological space. Then the following are equivalent:*

(i) *X is strongly K-analytic;*
(ii) *$(\mathcal{K}(X), \tau_\nu)$ is strongly K-analytic;*
(iii) *$(\mathcal{K}(X), \tau_\nu)$ is K-analytic.*

Proof: (i) $\Rightarrow$ (ii) Let $F : \Sigma \to \mathcal{K}(X)$ be an upper semicontinuous map with the further property that for every compact subset K of X there is $\sigma \in \Sigma$ with $K \subseteq F(\sigma)$. We define a set valued map $\Phi : \Sigma \to \mathcal{K}(\mathcal{K}(X))$ by the rule

$$\Phi(\sigma) = \mathcal{K}(F(\sigma)), \sigma \in \Sigma. \tag{1}$$

It then follows from standard properties of Vietoris topology that the set $\Phi(\sigma)$ is compact in $(\mathcal{K}(X), \tau_\nu)$ for all $\sigma \in \Sigma$ and hence Φ is well defined (see [E], [K] or [Ch]). Let $\Omega \subseteq (\mathcal{K}(X), \tau_\nu)$ be compact. Set $X_\Omega = \bigcup\{K : K \in \Omega\}$, then X_Ω (from the properties of τ_ν) is a compact subset of X and since X is strongly K-analytic (via F) there is $\sigma \in \Sigma$ such that $X_\Omega \subseteq F(\sigma)$. It follows immediately that

$$\Omega \subseteq \Phi(\sigma). \tag{2}$$

It is clear that it remains to show that Φ is an upper semicontinuous map. So let $\sigma \in \Sigma$ and V be an open subset of $(\mathcal{K}(X), \tau_\nu)$ with

$$\Phi(\sigma) \subseteq V. \tag{3}$$

Since $\Phi(\sigma)$ is compact in $(\mathcal{K}(X), \tau_\nu)$ we may assume that V is equal to a finite union of basic open sets. Assume for simplicity that V is the union of two basic open sets, that is, $V = V_1 \cup V_2$ where $V_k = \beta_k(V_{1,k} \dots, V_{n_k,k})$, $k = 1, 2$ and $V_{1,1}, \dots, V_{n_1,1}$ and $V_{1,2}, \dots, V_{n_2,2}$ are open non empty subsets of X. It then follows from (3) and the definition of basic open sets of Vietoris topology that for every $K \in \Phi(\sigma)$ we have that

$$\text{either } \Big(K \cap V_{i,1} \neq \emptyset \;\text{ for }\; i = 1, 2, \dots, n_1 \;\text{ and }\; K \subseteq \bigcup_{i=1}^{n_1} V_{i,1}\Big) \tag{4}$$

$$\text{or } \Big(K \cap V_{i,2} \neq \emptyset \;\text{ for }\; i = 1, 2, \dots, n_2 \;\text{ and }\; K \subseteq \bigcup_{i=1}^{n_2} V_{i,2}\Big). \tag{5}$$

We also have that for every $t \in F(\sigma)$

$$\Big(\text{either } \; t \in \bigcup_{i=1}^{n_1} V_{i,1} \;\text{ or }\; t \in \bigcup_{i=1}^{n_2} V_{i,2}\Big). \tag{6}$$

It follows from (6) that if $K \in \Phi(\sigma)$ then

$$K \subseteq \Big(\bigcap_{i=1}^{n_1} V_{i,1}\Big) \cup \Big(\bigcap_{i=1}^{n_2} V_{i,2}\Big). \tag{7}$$

Assume without loss of generality that (4) holds. Set

$$V_0 = \Big\{K \in \mathcal{K}(X) : K \subseteq \Big[\Big(\bigcap_{i=1}^{n_1} V_{i,1}\Big) \cup \Big(\bigcap_{i=1}^{n_2} V_{i,2}\Big)\Big] \cap \Big(\bigcup_{i=1}^{n_1} V_{i,1}\Big)\Big\}.$$

Then V_0 is a basic open set in $(\mathcal{K}(X), \tau_\nu)$ so that $\Phi(\sigma) \subseteq V_0 \subseteq V_1 \cup V_2$. Indeed, if $K \in \Phi(\sigma)$ then $K \subseteq \bigcap_{i=1}^{n_1} V_{i,1}$ and $K \subseteq \left(\bigcap_{i=1}^{n_1} V_{i,1}\right) \cup \left(\bigcap_{i=1}^{n_2} V_{i,2}\right)$ from (4) and (7) respectively, hence $K \in V_0$. Let now $K \in V_0$; if $K \cap \left(\bigcap_{i=1}^{n_1} V_{i,1}\right) \neq \emptyset$ then since $K \subseteq \bigcup_{i=1}^{n_1} V_{i,1}$ we find that $K \in V_1$; if $K \cap \left(\bigcap_{i=1}^{n_1} V_{i,1}\right) = \emptyset$ then obviously $K \subseteq \bigcap_{i=1}^{n_2} V_{i,2} \subseteq \bigcup_{i=1}^{n_2} V_{i,2}$ hence $K \in V_2$.

Since F is upper semicontinuous there is $n_0 \in \mathbb{N}$ so that

$$F(V_{\sigma\,|\,n_0}) \subseteq \Big[\Big(\bigcap_{i=1}^{n_1} V_{i,1}\Big) \cup \Big(\bigcap_{i=1}^{n_2} V_{i,2}\Big)\Big] \cap \Big(\bigcup_{i=1}^{n_1} V_{i,1}\Big),$$

(where $V_{\sigma\,|\,n_0} = \{\tau \in \Sigma : \tau\,|\,n_0 = \sigma\,|\,n_0\}$). It follows easily that $\Phi(V_{\sigma\,|\,n_0}) \subseteq V_0$, hence Φ is upper semicontinuous.

(ii) $\Rightarrow$ (iii). It is obvious.

(iii) $\Rightarrow$ (ii). Let $\Phi : \Sigma \to \mathcal{K}(\mathcal{K}(X))$ be an upper semicontinuous map which makes $(\mathcal{K}(X), \tau_\nu)$ a K-analytic topological space. Assume without loss of generality that Φ has the further property

$$\text{if } \ \sigma, \tau \in \Sigma,\ \sigma \leq \tau \ \text{ then } \ \Phi(\sigma) \subseteq \Phi(\tau).$$

Set $X_\sigma = \bigcup\{K : K \in \Phi(\sigma)\}$, for $\sigma \in \Sigma$, then it is clear that we have the following:

(a) X_σ is compact in X for all $\sigma \in \Sigma$,

(b) if $\sigma, \tau \in \Sigma$, $\sigma \leq \tau$ then $X_\sigma \subseteq X_\tau$ and

(c) if $K \in \mathcal{K}(X)$ then there is $\sigma \in \Sigma$ with $K \subseteq X_\sigma$ (indeed, $K \in \Phi(\sigma)$ for some $\sigma \in \Sigma$, thus $K \subseteq X_\sigma$).

We shall show that the map $F : \Sigma \to \mathcal{K}(X) : F(\sigma) = X_\sigma$, $\sigma \in \Sigma$ is upper semicontinuous. Let $\sigma \in \Sigma$ and V be an open subset of X with $X_\sigma \subseteq V$, then the set $\beta(V) = \{K \in \mathcal{K}(X) : K \subseteq V\}$ is open in $(\mathcal{K}(X), \tau_\nu)$ and it is easily seen that $\Phi(\sigma) \subseteq \beta(V)$. Let now $n_0 \in \mathbb{N}$ be such that $\Phi(V_{\sigma\,|\,n_0}) \subseteq \beta(V)$; then it is easy to show that $F(V_{\sigma\,|\,n_0}) \subseteq V$.

The proof of the theorem is complete. □

The following result gives some elementary stability properties of strongly K-analytic topological spaces.

Proposition 1.13. (i) *If X is any strongly K-analytic topological space and Y is a closed subset of X then Y is also strongly K-analytic.*

(ii) *If (X_n) is any sequence of strongly K-analytic topological spaces then their cartesian product $X = \prod_{n=1}^{\infty} X_n$ is strongly K-analytic.*

Proof: (i) Let $F : \Sigma \to \mathcal{K}(X)$ be an upper semicontinuous map such that for every compact subset K of X there exists $\sigma \in \Sigma$ with $K \subseteq F(\sigma)$. Set $U = X \backslash Y$ and $V = \{\sigma \in \Sigma : F(\sigma) \subseteq U\}$, then clearly V is an open subset of Σ. We define a map $\Phi : \Sigma\backslash V \to \mathcal{K}(Y)$ by the rule $\Phi(\sigma) = F(\sigma) \cap Y$, $\sigma \in \Sigma\backslash V$; since the set $\Sigma\backslash V$ is closed in Σ it is a Polish space, hence there exists a continuous onto map $f : \Sigma \to \Sigma\backslash Y$. Now it is easy to verify that the map $\sigma \in \Sigma \to \Phi(f(\sigma)) \in \mathcal{K}(Y)$ makes Y a strongly K-analytic space.

(ii) Let $F_i : \Sigma \to \mathcal{K}(X_i)$, $i \in \mathbb{N}$ be an upper semicontinuous map that makes X_i a strongly K-analytic space. Set for every $(\sigma_1, \dots, \sigma_n, \dots) \in \Sigma^{\mathbb{N}}$

$$F(\sigma_1, \dots, \sigma_n, \dots) = \prod_{i=1}^{\infty} F_i(\sigma_i).$$

The space $\Sigma^{\mathbb{N}}$ is (obviously) homeomorphic to the space Σ and the map $F : \Sigma^{\mathbb{N}} \to \mathcal{K}(X)$ defined above compact valued. The map F is also upper semicontinuous; for a proof of this fact we refer the reader to [J-R, Theorem 2.5.4, p. 23]. Let K be a compact subset of X; it is clear that $K \subseteq \prod_{i=1}^{\infty} K_i$, where $K_i = \pi_i(K)$, $i \in \mathbb{N}$ and $\pi_i : X \to X_i$ is the projection at coordinate i. Since K is compact K_i is compact for all $i \in \mathbb{N}$; hence there is for every $i \in \mathbb{N}$ (using the strong K-analyticity of X_i) $\sigma_i \in \Sigma$ such that $K_i \subseteq F_i(\sigma_i)$. It then clearly follows that $K \subseteq \prod_{i=1}^{\infty} K_i \subseteq F(\sigma_1, \dots, \sigma_n, \dots)$, therefore F makes X a strongly K-analytic space. □

Now we give some easy consequences of the above results concerning strong $\mathcal{K}$-analyticity.

Corollary 1.14. *Let X be a separable Banach space which is not SWKA (for instance $X = c_0$). Then the space $\mathcal{K}(X)$ (resp. $\mathcal{K}(B_X)$) of weakly compact subset of X (resp. of B_X) with Vietoris topology (induced by its weak topology) is not $\mathcal{K}$-analytic, in particular it is not analytic.*

Proof: It is an immediate consequence of Theorem 1.12. □

Note. We notice that every separable Banach space with the weak topology is an analytic space. Indeed, $(X, \|\cdot\|)$ is a Polish space and the identity map $\mathrm{id} : (X, \|\cdot\|) \to (X, w)$ is obviously continuous.

The following result says that a countable metric space is not necessarily strongly K-analytic. Since clearly a countable metric space is analytic, we get in particular that the class of strongly K-analytic (metric) spaces is not stable under continuous images.

Proposition 1.15. *The space of rational numbers $\mathbb{Q}$ (with the usual topology from $\mathbb{R}$) is not strongly K-analytic.*

Proof: Since (as it is well known) $\mathbb{Q}$ is not a Polish space, the result follows immediately from Remark 1.11.1(2). □

Remark 1.16. (1) It follows immediately from Proposition 1.13 that the class of SWKA Banach spaces is stable under closed subspaces and finite products. On the other hand it is not stable under continuous linear maps (c_0 is a quotient of ℓ^1).

(2) If a Banach space X has a Čech-complete unit ball in its weak topology (that is, B_X is a G_δ subset of $(B_{X^{**}}, \omega^*)$) then X is SWKA. This is so, because then X is isomorphic to $Y \oplus Z$, where Y is a Polish and Z a reflexive Banach space (see [E-W] and [H-H-Z, Theorem 315]).

(3) It follows immediately from Theorem 1.12 that a Banach space X is SWKA iff the space $K(X)$ with Vietoris topology (induced by the weak topology of X) is K-analytic.

2

In this section we are concerned with separable SWKA Banach spaces which do not contain ℓ^1. We shall show that such Banach spaces have separable duals and thus they are Polish according to Proposition 1.9. We shall need for this purpose a nice generalization of Rosenthal's ℓ^1-theorem due to Stern ([S, Theorem 2.1], see also [To, Theorem 12]); we also need a deep result of Stegall, that Stegall used in his study of dual Banach spaces with Radon-Nikodym property (RNP) (see [St, Theorem 2.3]). It should be recalled that in the special case when the separable space X (which does not contain ℓ^1) is a subspace of a SWCG space then X is reflexive (see Remark 1.10.1).

The following result from [M, Remark 1.4.1] will be used in the sequel; we prove it for completeness (the proof given in [M] is different).

Lemma 2.1. *Let Γ be a non empty set and $(\Gamma_\sigma)_{\sigma\in\Sigma}$ a family of subsets of Γ such that*

(i) $\Gamma = \bigcup_{\sigma\in\Sigma} \Gamma_\sigma$,
(ii) *if $\sigma, \tau \in \Sigma$ and $\sigma \leq \tau$ then $\Gamma_\sigma \subseteq \Gamma_\tau$ and*
(iii) *each Γ_σ is a finite set.*

Then Γ is at most countable.

Proof: We set $\Gamma_\emptyset = \Gamma$ and $\Gamma_s = \bigcup_{s<\sigma} \Gamma_\sigma$ for $s \in S$. Assume that Γ is uncountable. Since $\Gamma = \Gamma_\emptyset = \bigcup_{n=1}^\infty \Gamma_n$ there is $n_1 \in \mathbb{N}$ such that $|\Gamma_{n_1}| \geq \omega^+$; since $\Gamma_{n_1} = \bigcup_{n=1}^\infty \Gamma_{n_1 n}$ there is $n_2 \in \mathbb{N}$ with $|\Gamma_{n_1 n_2}| \geq \omega^+$. We proceed by induction and find a sequence $n_1, n_2, \dots, n_k, \dots$ of natural numbers such that $\Gamma_{n_1} \supseteq \Gamma_{n_1 n_2} \supseteq \dots \supseteq \Gamma_{n_1 n_2 \dots n_k} \supseteq \dots$, and $|\Gamma_{n_1 n_2 \dots n_k}| \geq \omega^+$ for $k \in \mathbb{N}$. Set $\sigma = (n_1, n_2, \dots, n_k, \dots) \in \Sigma$, then the sequence of sets $(\Gamma_{\sigma\,|\,k})_{k\in\mathbb{N}}$ is decreasing and consists of uncountable sets. Pick a sequence (γ_k) of distinct points of Γ and $(\sigma_k) \subseteq \Sigma$ such that $\sigma\,|\,k < \sigma_k$ and $\gamma_k \in \Gamma_{\sigma_k}$ for $k \in \mathbb{N}$. It is clear that the sequence (σ_k) is convergent in the space Σ and $\sigma_k \to \sigma$, thus the set $\{\sigma_k : k \in \mathbb{N}\} \cup \{\sigma\}$ is compact in Σ. Let $\tau \in \Sigma$ with $\sigma_k \leq \tau$ for all $k \in \mathbb{N}$, therefore $\Gamma_{\sigma_k} \subseteq \Gamma_\tau$ for all $k \in \mathbb{N}$. It clearly follows that $\{\gamma_k : k \in \mathbb{N}\} \subseteq \Gamma_\tau$; but this is a contradiction because the set $\{\gamma_k : k \in \mathbb{N}\}$ is infinite and the set Γ_τ is finite. $\square$

Throughout this paper T stands for the *dyadic tree*; i.e., $T = \bigcup_{n=0}^{\infty} \{0,1\}^n$ ordered (as the tree $S = \bigcup_{n=0}^{\infty} \mathbb{N}^n$) by the relation "$s$ is an initial segment of t", denoted by $s \leq t$. By a *subtree* T' of T we mean any subset of T having a unique minimal element and such that any its element has exactly two immediate successors; thus in particular T' is isomorphic to T. By the term chain (resp. antichain) of T we mean a set of pairwise comparable (resp. incomparable) elements of T; a branch of T is any maximal chain of T. Note that if T' is any subtree of T then the chains and antichains of T' are chains and antichains of T.

Theorem 2.2 (Stern). *Let X be a Banach space and $(x_s)_{s\in T}$ a bounded family of elements of X; then there exists a subtree T' of T such that one of the following alternatives holds:*

(i) *for any branch δ' of T', the sequence $(x_{\delta' \mid n})$ is weak-Cauchy;*
(ii) *for any branch δ' of T' the sequence $(x_{\delta' \mid n})$ is equivalent to the usual ℓ^1-basis.*

Denote by Δ the Cantor set, i.e., $\Delta = \{0,1\}^{\mathbb{N}}$. A family $(h_s)_{s\in T} \subseteq C(\Delta)$ is said to be a *Haar system* on Δ if the following hold:

(i) for every $s \in T$, $h_s = \mathcal{X}_{A_s}$ where A_s is a non empty open and closed subset of Δ;
(ii) $A_\emptyset = \Delta$ and for every $s \in T$, $A_s = A_{s0} \cup A_{s1}$ and $A_{s0} \cap A_{s1} = \emptyset$;
(iii) for every $\delta \in \Delta$ the intersection $\bigcap_{n=1}^{\infty} A_{\delta \mid n}$ is one point set (see [St, p. 215]).

Theorem 2.3 (Stegall). *Let X be a separable Banach space with non separable dual. Then for any $\varepsilon > 0$ there exist a subset Δ_ε of the unit sphere of X^* which is weak* homeomorphic to the Cantor set Δ, a Haar system $(h_s)_{s\in T}$ on Δ and a family $(e_s)_{s\in T} \subseteq X$ with $\|e_s\| \leq 1+\varepsilon$ for all $s \in T$ such that if $\Phi : X \to C(\Delta_\varepsilon)$ is the canonical evaluation operator (i.e., $\Phi(x)(x^*) = x^*(x)$, for $x \in X$ and $x^* \in \Delta_\varepsilon$), then*

$$\sum_{n=0}^{\infty} \Big(\sum_{s\in\{0,1\}^n} \|\Phi(e_s) - h_s\| \Big) < \varepsilon. \tag{1}$$

Note. In the following lemmas until to the proof of our main result (Theorem 2.6), X stands for a separable Banach space with non separable dual not containing ℓ^1 and $(e_s)_{s\in T}$, $(h_s)_{s\in T}$, Φ, for the families and the operator of Stegall's theorem.

Lemma 2.4. *Let (t_n) be a chain and (s_n) an antichain of T. Then the sequence $(e_{s_n} - e_{t_n})$ of X has no weakly convergent subsequence.*

Proof: Assume without loss of generality that the given sequence is itself weakly convergent in X, say $w - \lim(e_{s_n} - e_{t_n}) = x$, hence $w - \lim \Phi(e_{s_n} - e_{t_n}) = \Phi(x)$. Then we have:

$$\begin{aligned} \|\Phi(e_{s_n} - e_{t_n}) - (h_{s_n} - h_{t_n})\| &= \|(\Phi(e_{s_n}) - h_{s_n}) - (\Phi(e_{t_n}) - h_{t_n})\| \\ &\leq \|\Phi(e_{s_n}) - h_{s_n}\| + \|\Phi(e_{t_n}) - h_{t_n}\| \to 0, \\ &\quad \text{for} \quad n \to \infty \end{aligned}$$

(the fact that this limit is zero is a consequence of inequality (1) of Stegall's theorem). It then follows that $w - \lim(h_{s_n} - h_{t_n}) = \Phi(x)$ which is a contradiction because clearly the sequence (h_{s_n}) is weakly null and the sequence (h_{t_n}) is weakly Cauchy but not weakly convergent. □

Lemma 2.5. *Let T' be a subtree of T such that for every branch δ of T' the sequence $(e_{\delta\,|\,n})$ is weakly Cauchy. Then the Banach space X is not SWKA.*

Proof: Assume for the purpose of contradiction that X is SWKA and let $(W_\sigma)_{\sigma\in\Sigma}$ be a family of weakly compact sets in X satisfying assertion (ii) of Proposition 1.7. It is clear that the set of branches Δ' of T' can be identified with the (Cantor) set $\{0,1\}^{\mathbb{N}}$; we associate with every branch δ of T' the weakly null sequence

$$\varphi(\delta) = \{e_{\delta\,|\,n-1} - e_{\delta\,|\,n} : n \in \mathbb{N}\}.$$

It is obvious that for every $\delta \in \Delta'$ there exists $\sigma_\delta \in \Sigma$ such that

$$\varphi(\delta) \subseteq W_{\sigma_\delta}.$$

Set for $\sigma \in \Sigma$, $X_\sigma = \{\delta \in \Delta' : \varphi(\delta) \subseteq W_{\sigma_\delta}\}$; it follows then from the properties of the family $(W_\sigma)_{\sigma\in\Sigma}$ that

(i) $\Delta' = \bigcup_{\sigma\in\Sigma} X_\sigma$,
(ii) if $\sigma, \tau \in \Sigma$ and $\sigma \leq \tau$ then $X_\sigma \subseteq X_\tau$.

Since the set Δ' is uncountable, Lemma 2.1 implies that there is $\sigma_0 \in \Sigma$ so that the set X_{σ_0} is infinite. Let (δ_k) be a sequence of distinct points of X_{σ_0} which is convergent in the Cantor set Δ', say $\delta_k \to \delta$. Set $d(\delta_k, \delta) = \frac{1}{m_k}, k \in \mathbb{N}$ (where d is the usual metric in the Cantor set $\Delta' = \{0,1\}^{\mathbb{N}}$) and assume without loss of generality that $2 \leq m_1 < m_2 < \ldots < m_k < \ldots$, for $k \in \mathbb{N}$. It is clear that

$$\delta\,|_{m_k-1} = \delta_k\,|_{m_k-1} \quad \text{and} \quad \delta_k(m_k) \neq \delta(m_k), \quad \text{for} \quad k \in \mathbb{N}.$$

So we get that the sequence $\varepsilon_k = \delta_k \,|\, m_k$, $k \in \mathbb{N}$ consists of pairwise incomparable elements of T' (and thus of T) and also that, $e_{\varepsilon_k} - e_{\delta\,|\,m_k-1} = e_{\delta_k\,|\,m_k} - e_{\delta_k\,|\,m_k-1}$, for every $k \in \mathbb{N}$. But this is a contradiction because on one hand the sequence $e_{\delta_k \,|\, m_k} - e_{\delta_k \,|\, m_k - 1}$, $k \in \mathbb{N}$, is contained in the weakly compact set W_{σ_0} and on the other hand the sequence $e_{\varepsilon_k} - e_{\delta \,|\, m_k - 1}$, $k \in \mathbb{N}$, cannot have a weakly convergent subsequence by Lemma 2.4. □

Theorem 2.6. *Let X be a separable Banach space not containing ℓ^1. If X is SWKA then X has separable dual and thus is a Polish Banach space.*

PROOF: Assume that X has non separable dual and let $(e_s)_{s\in T}$ be the family given in Stegall's theorem. We apply Stern's theorem for $(e_s)_{s\in T}$; since X does not contain ℓ^1 the first alternative of this result must hold, but this contradicts our assumption according to Lemma 2.5. So the space X has separable dual and thus is Polish by Proposition 1.9. □

Corollary 2.7. *Every SWKA Banach space X not containing ℓ^1 is Asplund and (hence) WCG.*

PROOF: Every separable subspace of X is Polish by Theorem 2.6 and thus it has separable dual, so X is Asplund. As X is Asplund and WKA it is WCG (see Corollary 4.4, Chapter VI of [D-G-Z]). □

By using Stern's theorem and the techniques used before we can also prove the following

Theorem 2.8. *Let X be a Banach space and $(e_s)_{s\in T}$ be a bounded family of X. Assume that*

(i) *for no chain (t_n) of T the sequence (e_{t_n}) is weakly convergent and*
(ii) *for every antichain (s_n) of T there is a subsequence (s'_n) of (s_n) so that the sequence $(e_{s'_n})$ is weakly convergent.*

Then X is not SWKA.

PROOF: We first remark that as it follows from (i) and (ii), for every pair (t_n), (s_n) where (t_n) is a chain and (s_n) an antichain of T the sequence $(e_{s_n} - e_{t_n})$ of X has no weakly convergent subsequence.

Let T' be a subtree of T given by Stern's result so that one of the following alternatives holds.

(1) For every chain (t_n) of T', (e_{t_n}) is a weak Cauchy sequence in X.
(2) For every chain (t_n) of T', (e_{t_n}) is equivalent to the usual ℓ^1-basis.

Assume without loss of generality that $T' = T$.

If alternative (1) holds, then we proceed in exactly the same way as in the proof of Lemma 2.5 and (by using the above mentioned remark) show that X is not SWKA.

Now assume that (2) holds for T. We notice that a dyadic tree contains uncountable many antichains. We give a proof of this (simple) argument which has been taken from [S-W, Example 2.6, p. 391]. Let $\theta(0) = 1$ and $\theta(1) = 0$; we define for $\delta = (\delta_n) \in \{0,1\}^{\mathbb{N}}$ a sequence $\varepsilon_1(\delta) = (\delta_1)$, $\varepsilon_2(\delta) = (\theta(\delta_1), \delta_2), \dots, \varepsilon_n(\delta) = (\theta(\delta_1), \dots, \theta(\delta_{n-1}), \delta_n), \dots$ in T. Then $(\varepsilon_n(\delta))$ is (obviously) an antichain of T and the map $\delta \in \Delta \to (\varepsilon_n(\delta)) \in T^{\mathbb{N}}$ is $1-1$.

Now we assume for the purpose of contradiction that X is SWKA and proceed in a similar way as in the proof of Lemma 2.5. So let $(W_\sigma)_{\sigma\in\Sigma}$ be a family of weakly compact subsets of X that satisfies assertion (ii) of Proposition 1.7. It follows clearly from our hypothesis that for every antichain (s_n) of T the set $\{e_{s_n} : n \in \mathbb{N}\}$ is weakly relatively compact in X, hence it is contained in some W_σ. So it follows in particular that for every $\delta \in \{0,1\}^{\mathbb{N}}$ there is $\sigma_\delta \in \Sigma$ such that $\{e_{\varepsilon_n(\delta)} : n \in \mathbb{N}\} \subseteq W_{\sigma_\delta}$. Set for $\sigma \in \Sigma$, $X_\sigma = \{\delta \in \{0,1\}^{\mathbb{N}} : \{e_{\varepsilon_n(\delta)} : n \in \mathbb{N}\} \subseteq W_\sigma\}$; it is then clear that

(a) $\{0,1\}^{\mathbb{N}} = \bigcup_{\sigma\in\Sigma} X_\sigma$ and
(b) if $\sigma, \tau \in \Sigma$ and $\sigma \le \tau$ then $X_\sigma \subseteq X_\tau$.

So it follows from Lemma 2.1 that there is $\sigma_0 \in \Sigma$ with X_{σ_0} an infinite set. Let (δ_n) be a non trivial sequence in X_{σ_0} which is convergent in the Cantor space, say $\delta_n \to \delta$. It is now easy to choose inductively an infinite chain $\{\varepsilon_k : k \in \mathbb{N}\} \subseteq \bigcup_{n=1}^{\infty} \{\varepsilon_m(\delta_n) : m \in \mathbb{N}\}$, thus $\{e_{\varepsilon_k} : k \in \mathbb{N}\} \subseteq W_{\sigma_0}$. But this is a contradiction because the set W_{σ_0} is weakly compact and the sequence $\{e_{\varepsilon_k} : k \in \mathbb{N}\}$ equivalent to the usual ℓ^1-basis. $\square$

As an application of Theorem 2.8 we show that a weakly sequentially complete, (even separable) space need not be SWKA; the example X_0 which we are going to consider is due to Batt and Hiermeyer. It is proved in [S-W] that X_0 is not SWCG. We show here that X_0 is not even SWKA, in particular X_0 is not a subspace of a SWCG space.

Example 2.9. Let α be the set of finite chains of the dyadic tree T; it is clear that if $A \subseteq B \in \alpha$ then $A \in \alpha$.

Let

$$X_0 = \{f : T \to \mathbb{R}, \|f\| < +\infty\},$$

where

$$\|f\| = \sup\Big\{\Big[\sum_{i\in I}\Big(\sum_{s\in A_i} |f(s)|\Big)^2\Big]^{1/2} : I \text{ finite}, \ A_i \cap A_j = \emptyset, \ i \ne j \text{ and } A_i \in \alpha \text{ for } i \in I\Big\}$$

(see [A-M, Definition 3.13]).

It is easy to verify that $(X_0, \|\cdot\|)$ is a Banach space, having the set $\{e_s : s \in T\}$ as an unconditional boundedly complete (normalized) basis. Therefore X_0 is a weakly sequentially complete separable dual space (it has the Radon-Nikodym property) and it does not contain c_0 (see [L-T, Theorem 1.c.10]).

Let $A \subseteq T$. Then it easily follows from the definition of the space X_0 that:

(i) if A is a chain of T then the family $\{e_s : s \in A\}$ is equivalent to the usual ℓ^1-basis;

(ii) if A is an antichain of T then $\{e_s : s \in A\}$ is equivalent to the usual ℓ^2-basis, hence in particular it is a weakly relatively compact subset of X_0.

So we get immediately from (i), (ii) and Theorem 2.8 that X_0 cannot be a SWKA space.

3

Our aim in this last section is to give an example X of a subspace of $L^1[0,1]$ (more exactly of the space $L^1[0,1]^{\mathbb{N}}$) which is not SWCG. It should be noticed that since X is a subspace of a SWCG space, it is itself a SWKA space (see Proposition 1.5). This example answers in the negative a question posed in [S-W, Question (A), p. 397]: "Must a WCG subspace of an SWCG space again be SWCG? Specifically, must every subspace of a separable SWCG space again be SWCG? This is of particular interest for the space $L^1[0,1]$."

Our construction of the space X is similar and it is based on the classical construction of Rosenthal of a subspace Y of $L^1[0,1]^{\mathfrak{c}}$ which is not WCG. This example answered (in the negative) the "heredity" problem in the class of WCG spaces (see [R1]). It is clear that our example has the similar property for the class of separable SWCG spaces.

The following result is crucial for our purposes.

Proposition 3.1. *Let X be a Banach space with a normalized unconditional basis Γ. If X is SWCG then there exists a countable family $(\Gamma_n)_n$ of subsets of Γ such that:*

(i) Γ_n is a weakly relatively compact subset of Γ for all $n \in \mathbb{N}$ (hence each $\Gamma_n \cup \{0\}$ is weakly compact);

(ii) if A is any weakly relatively compact subset of Γ then there is $n \in \mathbb{N}$ with $A \subseteq \Gamma_n$.

Proof: Let K be a weakly compact convex symmetric subset of X that strongly generates X (see Definition 1.1). Assume without loss of generality that $K \subseteq B_X$. We consider the operator $T : X^* \to C(K)$ defined by the rule $T(x^*) = x^*|_K$, $x^* \in X^*$. It is easy to see that T is a bounded linear one-to-one operator ($\|T\| \leq 1$) with the further property that its restriction on B_{X^*} is a weak* to pointwise continuous map. Therefore the compact space (B_{X^*}, w^*) is affinely homeomorphic to the compact space $(T(B_{X^*}), \tau_p)$, where τ_p is the topology of pointwise convergence on $C(K)$. Since the set $T(B_{X^*})$ is bounded in $C(K)$ it follows from Grothendieck's theorem (see [R1, p. 102]) that the topologies of pointwise convergence τ_p and the weak topology coincide on $T(B_{X^*})$; it follows in particular that

T is weakly compact (cf. the proof of Corollary 3.4 of [R1]). As it is proved in [S-W, Theorem 2.1], a Banach space X is SWCG if and only if the space (B_{X^*}, τ), where τ is the Mackey topology of X^* is metrizable. It follows from the method of the proof of implication (c) $\Rightarrow$ (a) of Theorem 2.1 that the norm-metric of $C(K)$ metrizes the Mackey topology of B_{X^*}, that is, $(B_{X^*}, \tau) \approx (T(B_{X^*}), \|\cdot\|_1)$, where $\|\cdot\|_1$ denotes the supremum norm of $C(K)$.

We set for every $n \in \mathbb{N}$, $\Gamma_n = \{\gamma \in \Gamma : \|T(\gamma^*)\|_1 \geq \frac{1}{n}\}$, where γ^* denotes the biorthogonal functional of $\gamma \in \Gamma$.

Now, by using the fact that T is a weakly compact operator and following the method of the proof of implication (4) $\Rightarrow$ (1) of a result of Johnson (see Remark 3.2 below) we can prove that every set Γ_n, $n \in \mathbb{N}$, is weakly relatively compact in X and also that $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. We shall show that this is the desired family.

Claim. Let $A \subseteq \Gamma$. Then A is a weakly relatively compact subset of X if and only if there is $\delta > 0$ with $\|T(\gamma^*)\|_1 \geq \delta$ for all $\gamma \in A$.

PROOF OF THE CLAIM: Assume that A is a weakly relatively compact subset of X and also assume for the purpose of contradiction that there is a sequence $\gamma_1, \dots, \gamma_n, \dots$ of distinct points of A so that $\lim_{n\to\infty} \|T(\gamma_n^*)\|_1 = 0$, equivalently $\tau - \lim_{n\to\infty} \gamma_n^* = 0$. Since the Mackey topology of X^* coincides with the topology of uniform convergence on weakly compact subsets of X we conclude that,

$$a_n = \sup\{|\gamma_n^*(x)| : x \in A\} \to 0, \quad \text{for} \quad n \to \infty,$$

because A is a weakly relatively compact subset of X. On the other hand, since $\gamma_n \in A$ for $n \in \mathbb{N}$, we get that $\gamma_n^*(\gamma_n) = 1$ for all $n \in \mathbb{N}$, hence $a_n \geq 1$ for all $n \in \mathbb{N}$, which is a contradiction.

For the "if" assertion of the Claim consider a positive δ so that $\|T(\gamma^*)\|_1 \geq \delta$, for every $\gamma \in A$. Pick $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} \leq \delta$, then it is obvious that $A \subseteq \Gamma_{n_0}$ and thus A is also a weakly relatively compact set.

It is clear that this Claim finishes the proof of the proposition. $\square$

Remark 3.2. The result of Johnson mentioned in the proof of Proposition 3.1 states that: For a Banach space X with an unconditional basis Γ, X is WCG if and only if there exists a sequence (Γ_n) of subsets of Γ such that, $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ and each Γ_n is a weakly relatively compact subset of X (see [R1, Proposition 1.3]). This result was used for the proof of some properties of Rosenthal's example.

Let $(X, \mathcal{M}, \mu)$ be a probability measure space. Following Rosenthal [R1], we define for every $f \in L^1(\mu)$ the modulus of absolute continuity $\omega(f, \delta)$ of f by the rule

$$\omega(f, \delta) = \sup\{\int_E |f|\, d\mu : E \in \mathcal{M}, \mu(E) \leq \delta\},\ \delta \in (0, 1].$$

So a function $\omega(f,\cdot) : (0,1] \to \mathbb{R}$ is defined which has the following properties:

(i) $\omega(f,\cdot)$ is increasing on $(0,1]$ and
(ii) $\lim_{\delta\to 0} \omega(f,\delta) = 0$ (see [R1, p. 89]).

Using the function $\omega(f,\cdot)$ the classical characterization of relatively weakly compact subsets of $L^1(\mu)$ may be reformulated as

Lemma 3.3. *Let S be a non empty bounded subset of $L^1(\mu)$. Then S is relatively weakly compact if and only if* $\lim_{\delta\to 0}[\sup_{f\in S} \omega(f,\delta)] = 0$ (*see Lemma* 1.4 *of* [R1]).

Now we let,

$$\mathcal{R} = \Big\{ r : [0,1] \to \mathbb{R},\ r \in L^1[0,1],\ \int_0^1 r\, dx = 0 \ \text{ and } \ \int_0^1 |r|\, dx = 1 \Big\}$$

(see [R1, p. 86]).

The following result is well known (see [R1, p. 90]). We give a (different) proof of this for completeness.

Lemma 3.4. *The set $\mathcal{R}$ is not a σ-relatively weakly compact subset of* $L^1[0,1]$ (*that is, $\mathcal{R}$ is not a countable union of relatively weakly compact sets in* $L^1[0,1]$).

Proof: We define a map $\Lambda : L^1[0,1] \to \mathbb{R}$ by the rule $\Lambda(f) = \int_0^1 f\, dx$, $f \in L^1[0,1]$. It is clear that Λ is a non trivial continuous linear functional on $L^1[0,1]$, therefore the set $Z = \{f \in L^1[0,1] :\ \Lambda(f) = 0\}$ is a closed hyperplane of $L^1[0,1]$. The set $\mathcal{R}$ is obviously the unit sphere S_Z of Z. Since Z is not reflexive (Z is isomorphic to $L^1[0,1]$) the conclusion follows immediately from the following simple result: A Banach space X is reflexive if and only if its unit sphere S_X is a σ-relatively weakly compact set. $\square$

Remark 3.5. It is clear that the set $\mathcal{R}$ endowed with the norm-metric is a complete separable metric space.

Let μ denote the product Lebesgue measure on the compact space $\Omega = [0,1]^{\mathcal{R}}$. We associate with each function $r \in \mathcal{R}$ the μ-integrable function $f_r : [0,1]^{\mathcal{R}} \to \mathbb{R}$ defined by $f_r = r \circ \pi_r$, where π_r is the projection to the rth coordinate (see [R1, p. 86]).

It is easy to verify that for every $r \in \mathcal{R}$ we have

$$(1) \qquad \text{(i)} \quad \int_\Omega f_r\, d\mu = 0 \text{ and } \qquad \text{(ii)} \quad \int_\Omega |f_r|\, d\mu = 1.$$

The following two lemmas are due to Rosenthal (see [R1, pp. 89–90] and the references given there).

Lemma 3.6. *The family $\widetilde{\mathcal{R}} = \{f_r : r \in \mathcal{R}\}$ is an unconditional basis for its closed linear span Y in the space $L^1(\mu)$ (with unconditional constant ≤ 2).*

Lemma 3.7. (i) *If $r \in \mathcal{R}$ then $\omega(r,\delta) = \omega(f_r,\delta)$ for all $\delta \in (0,1]$, thus*

(ii) *if $S \subseteq \mathcal{R}$ then S is a relatively weakly compact subset of $L^1[0,1]$ if and only if $\widetilde{S} = \{f_r : r \in S\}$ is a relatively weakly compact subset of $L^1(\mu)$.*

We notice that it is easy to prove assertion (i). Assertion (ii) is a consequence of (i) and Lemma 3.3.

Remark 3.8. It follows from Lemmas 3.4 and 3.7 that the set $\widetilde{\mathcal{R}}$ is not a σ-relatively weakly compact subset of $L^1(\mu)$.

Let now Δ be a norm-dense subset of the complete separable metric space $\mathcal{R}$ endowed with the norm metric (see Remark 3.5). Set $\widetilde{\Delta} = \{f_r : r \in \Delta\}$ and denote by Y_Δ the closed linear span of $\widetilde{\Delta}$ in $L^1(\mu)$; also denote by Y the space $Y_{\mathcal{R}}$. Then the following result is proved.

Theorem 3.9. (i) (Rosenthal [R1]). *The space Y is not WCG.*

(ii) *The space Y_Δ is not SWCG.*

Proof: We are interested in assertion (ii) of course but for completeness we also give the easy proof (due to Rosenthal) of assertion (i). The set $\widetilde{\mathcal{R}}$ is from Lemma 3.6 an unconditional basis of Y and, by Remark 3.8, it is not a σ-relatively weakly compact subset of $L^1(\mu)$. Now the result of Johnson (see Remark 3.2) proves immediately (i).

In order to prove (ii) assume for the purpose of contradiction that the space Y_Δ is SWCG. Since the family $\widetilde{\Delta}$ is an unconditional basis of Y_Δ there exists by Proposition 3.1 a sequence (Δ_n) of subsets of Δ such that:

(a) each $\widetilde{\Delta}_n$ is a relatively weakly compact subset of Y_Δ (hence, Δ_n is a relatively weakly compact subset of $L^1[0,1]$);

(b) if $K \subseteq \Delta$ and $\widetilde{K}$ is a relatively weakly compact subset of Y_Δ then there is $n_0 \in \mathbb{N}$ so that $\widetilde{K} \subseteq \widetilde{\Delta}_{n_0}$ (hence, $K \subseteq \Delta_{n_0}$).

Denote by E_n the weak closure of Δ_n in $L^1[0,1]$ for $n \in \mathbb{N}$, hence each E_n is weakly compact. We shall show that $\mathcal{R} = \bigcup_{n=1}^{\infty}(\mathcal{R} \cap E_n)$; since each of the sets $\mathcal{R} \cap E_n$ is a relatively weakly compact subset of $L^1[0,1]$, this equality clearly contradicts Lemma 3.4. So let $r \in \mathcal{R}$; since Δ is norm-dense in $\mathcal{R}$ there is a sequence $(r_n) \subseteq \Delta$ with $\lim_{n\to\infty} \|r_n - r\|_1 = 0$. Therefore the set $\{r_n : n \in \mathbb{N}\} \cup \{r\}$ is norm and thus weakly compact subset of $L^1[0,1]$. It follows then from Lemma 3.7(ii) that the set $\{f_{r_n} : n \in \mathbb{N}\}$ is a relatively weakly compact subset of $L^1(\mu)$ and thus of Y_Δ. So there exists (from (b)) $n_0 \in \mathbb{N}$ with $\{f_{r_n} : n \in \mathbb{N}\} \subseteq \widetilde{\Delta}_{n_0}$ which implies that $\{r_n : n \in \mathbb{N}\} \subseteq \Delta_{n_0}$ and hence $w - \lim r_n = r \in E_{n_0}$. The obvious inclusion $\bigcup_{n=1}^{\infty}(\mathcal{R} \cap E_n) \subseteq \mathcal{R}$, finishes the proof of assertion (ii). $\square$

Corollary 3.10. *Let Δ be a countable norm dense subset of $\mathcal{R}$. Then the space Y_Δ is a non SWCG subspace of the space $L^1[0,1]^{\mathbb{N}}$ ($\cong L^1[0,1]$).*

Proof: It is an obvious consequence of the construction of Y_Δ and of Theorem 3.9(ii). □

Note. The space Y_Δ (where Δ is a norm-dense subset of $\mathcal{R}$) is weakly sequentially complete as a subspace of $L^1[0,1]^\Delta$ and it has an unconditional basis. Therefore its basis is boundedly complete and Y_Δ is isomorphic to a dual Banach space (see [L-T, Theorem 1.c.10]).

H. Rosenthal has proved, assuming Martin's axiom plus the negation of continuum hypothesis (MA $+\rceil$CH), the following result: If μ is a probability measure on some measurable space and Y is a closed linear subspace of $L^1(\mu)$ of density character $\dim Y < \mathfrak{c}$, then Y is WCG (see Theorem 2.7 of [R1]). The first named author has proved several years ago (it was about 1993) a generalization of this result, which still remains unpublished. We decided to include this result in this note because it improves considerably the result of Rosenthal and also because its proof is short and sweet.

We shall need a definition and also the main result of [M] (see also [M-N] and [D-G-Z]).

Definition 3.11 ([M, Definition 1.1]). For a topological space X we denote by $c_1(X)$ the closed linear subspace of $\ell^\infty(X)$ which consists of all bounded functions $f : X \to \mathbb{R}$ such that for every $\varepsilon > 0$ the set $\sigma_\varepsilon(f) = \{t \in X : |f(t)| \geq \varepsilon\}$ is a closed and discrete subset of X.

Theorem 3.12 ([M, Theorem 4.1]). *A Banach space Y is WKA if and only if there exists a bounded linear one-to-one operator $T : Y^* \to c_1(\Sigma \times \{0,1\}^a)$, where $\dim Y \leq 2^a$, which is weak* to pointwise continuous.*

Note. It is obvious that the space $\Sigma \times \{0,1\}^\omega$ is homeomorphic to the Baire space Σ, hence if $\dim Y \leq 2^\omega = \mathfrak{c}$, then the operator T of Theorem 3.12 takes values in the space $c_1(\Sigma)$.

Now we state the above mentioned generalization of Rosenthal's result.

Theorem 3.13 (MA $+\rceil$CH). *Let Y be a WKA Banach space with $\dim Y < \mathfrak{c}$. Then Y is WCG.*

Proof: Since Y is a WKA Banach space with $\dim Y < \mathfrak{c}$ there exists according to Theorem 3.12 a bounded linear one-to-one operator $T : Y^* \to c_1(\Sigma)$ which is weak* to pointwise continuous. Therefore the space $\Omega = T(B_{Y^*})$ endowed with the topology of pointwise convergence is homeomorphic to the compact space (B_{Y^*}, w^*). Set $\Sigma' = \bigcup\{\operatorname{supp} f : f \in \Omega\}$ (where for $f \in c_1(\Sigma)$, $\operatorname{supp} f = \{\sigma \in \Sigma : f(\sigma) \neq 0\}$); then it is easy to see that the cardinality $|\Sigma'|$ of Σ' is equal to the topological weight of Ω and thus of (B_{Y^*}, w^*); hence

$$|\Sigma'| = \dim Y < \mathfrak{c}.$$

Now we recall a consequence of Martin's axiom: Every subset Σ' of the Baire space Σ of cardinality smaller than $\mathfrak{c}$ is a σ-relatively compact subset of Σ (see [J-R, pp. 118–119]). So let (K_n) be a sequence of compact subsets of Σ with $\Sigma' \subseteq \bigcup_{n=1}^{\infty} K_n$. We set $\Gamma = \bigcup_{n=1}^{\infty} K_n$ and define a map $\Phi : Y^* \to c_0(\Gamma)$ in the following way:

$$\Phi(y^*) = \left(\frac{T(y^*)\,|_{K_n}}{n} \right)_{n \in \mathbb{N}}.$$

It is easy to verify that Φ is a well defined bounded linear one-to-one operator which is weak* to pointwise continuous. Since the range of Φ is the Banach space $c_0(\Gamma)$ we conclude immediately from the classical theorem of Amir-Lindenstrauss that Y is a WCG Banach space (see [N], [M-N] and [D-G-Z]). □

The following open questions concern a (non separable) SWKA Banach space X.

(a) Is X a subspace of a WCG Banach space, or at least a $K_{\sigma\delta}$ subset of (X^{**}, w^*)? What happens if we assume furthermore that X is a dual Banach space?

(b) Assume that X does not contain ℓ^1. Does then X have a Čech-complete unit ball? (see Remark 1.16(2) and Corollary 2.7).

References

[A-M] Argyros S., Mercourakis S., *On weakly Lindelöf Banach spaces*, Rocky Mountain J. Math. **23** (1993), 395–446.

[Ch] Christensen J.P.R., *Topology and Borel Structure*, North-Holland, Amsterdam, 1974.

[D-G-Z] Deville R., Godefroy G., Zizler V., *Smoothness and renorming in Banach spaces*, Longman, Harlow, 1993.

[E] Engelking R., *General Topology*, PWN, Warszawa, 1977.

[E-W] Edgar G.A., Wheller R.F., *Topological properties of Banach spaces*, Pacific J. Math. **115** (1984), no. 2, 317–350.

[H-H-Z] Habala P., Hájek P., Zizler V., *Introduction to Banach Spaces I and II*, Matfyzpress, Praha, 1996.

[J-R] Jayne J.E., Rogers C.A., *K-analytic sets*, in Analytic Sets, Academic Press, London, 1980.

[K] Kuratowski K., *Topology*, Vol. I (1966), Vol. II (1968), Academic Press, New York-London.

[L-T] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces I*, Springer, Berlin-New York, 1977.

[M] Mercourakis S., *On weakly countably determined Banach spaces*, Trans. Amer. Math. Soc. **300** (1987), 307–327.

[M-N] Mercourakis S., Negrepontis S., *Banach Spaces and Topology II*, Recent Progress in General Topology, M. Hušek and J. Van Mill (eds.), North-Holland, Amsterdam, 1992, pp. 493–536.

[N] Negrepontis S., *Banach Spaces and Topology*, Handbook of Set-Theoretic Topology, K. Kunen and J. Vaughan (eds.), North-Holland, Amsterdam, 1984, pp. 1045–1142.

[R0] Rosenthal H.P., *A characterization of Banach spaces containing ℓ^1*, Proc. Nat. Acad. Sci. (USA) **71** (1974), 2411–2413.

[R1] Rosenthal H.P., *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **28** (1974), 83–111.

[R2] Rosenthal H.P., *Weak*-Polish Banach spaces*, J. Funct. Anal. **76** (1988), 267–316.

[St] Stegall C., *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213–223.

[S] Stern J., *A Ramsey theorem for trees, with an application to Banach spaces*, Israel J. Math. **29** (1978), 179–188.

[S-W] Schlüchtermann G., Wheeler R.F., *On Strongly WCG Banach spaces*, Math. Z. **199** (1988), 387–398.

[T] Talagrand M., *Espaces de Banach faiblement $\mathcal{K}$-analytiques*, Ann. of Math. **110** (1979), 407–438.

[To] Todorčević S., *Compact subsets of the first Baire class*, J. Amer. Math. Soc. **12** (1999), no. 4, 1179–1212.

S. Mercourakis, E. Stamati:
University of Athens, Department of Mathematics, Panepistemiopolis,
15784 Athens, Greece

E-mail: smercour@math.uoa.gr

(*Received* July 12, 2005, *revised* March 7, 2006)