# Fourier inversion of distributions on projective spaces

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*Abstract.* We show that the Fourier-Laplace series of a distribution on the real, complex or quarternionic projective space is uniformly Cesàro-summable to zero on a neighbourhood of a point if and only if this point does not belong to the support of the distribution.

*Keywords:* distribution, projective space, Fourier-Laplace series, Cesàro summability

*Classification:* Primary 46F12; Secondary 42C10

## 1. Introduction

In [5] Kahane and Salem characterized the closed sets of uniqueness in the unit circle $\mathbb{S}^1$ by using the support of distributions. In particular they proved that, given a distribution $T$ on $\mathbb{S}^1$ whose Fourier transform $\mathcal{F}T$ vanishes at infinity and $E$ a closed set in $\mathbb{S}^1$, the support of $T$ is in $E$ if and only if for all $x \in \mathbb{S}^1 \setminus E$

$$\lim_{N \to +\infty} \sum_{k=-N}^{N} \mathcal{F}T(k) \exp(2\pi i x k) = 0.$$

Later Walter showed that the Fourier series

$$\sum_{k=-\infty}^{\infty} \mathcal{F}T(k) \exp(2\pi i x k)$$

of any distribution $T$ on $\mathbb{S}^1$ is Cesàro-summable to zero for all $x$ out of the support of $T$ ([7]). However, this is not sufficient to characterize the support of $T$, since, as Walter himself remarked, the Fourier series of the first derivative of the Dirac measure at a point $s \in \mathbb{S}^1$, $\delta'_s$, is summable in Cesàro means of order 2 to zero everywhere.

In fact, a point $x$ is out of the support of $T$ if and only if the Fourier series of $T$ is *uniformly* Cesàro-summable to zero on a neighbourhood of $x$. In [2] we established this result for the general case of a distribution $T$ on $\mathbb{S}^{n-1}$ ($n \geq 2$) and its Fourier-Laplace series (see Section 2 below). Here we will show in Section 4 that from the result on the sphere we can obtain the similar result about the Fourier-Laplace expansion of distributions on real, complex and quaternionic projective spaces. In Section 3 we introduce the necessary tools on projective spaces.

## 2. Fourier inversion on the sphere

We write $\sum_{m=0}^{+\infty} b_m = B \ (C,k)$ to say that the series of complex numbers $\sum_{m\geq 0} b_m$ is summable in Cesàro means of order $k$ to $B \in \mathbb{C}$ (see [3]).

The restriction to $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$, of the non-radial part of the Laplace operator $\Delta$ on $\mathbb{R}^n$ is the *Laplace-Beltrami operator* on $\mathbb{S}^{n-1}$, $\Delta_{\mathbb{S}}$. It is self-adjoint with respect to the scalar product of $L^2(\mathbb{S}^{n-1}, d\sigma_{n-1})$ and commutes with rotations (we choose $d\sigma_{n-1}$ normalized).

A *spherical harmonic of degree* $l$ *on* $\mathbb{S}^{n-1}$ ($l \in \mathbb{N}_0$) is the restriction to $\mathbb{S}^{n-1}$ of a polynomial on $\mathbb{R}^n$ which is harmonic and homogeneous of degree $l$. We write $\mathcal{S}H_l(\mathbb{S}^{n-1})$ the set of spherical harmonics of degree $l$. Every non zero element of $\mathcal{S}H_l(\mathbb{S}^{n-1})$ is an eigenfunction of $\Delta_{\mathbb{S}}$ with eigenvalue $-l(n+l-2)$. Let $(E_1^l, \ldots, E_{d_l}^l)$ be an orthonormal basis of $\mathcal{S}H_l(\mathbb{S}^{n-1})$. The function $Z_l(\zeta,\eta) := \sum_{j=1}^{d_l} E_j^l(\zeta)\overline{E_j^l(\eta)}$ is called *zonal of degree* $l$. For all $\zeta$, $\eta \in \mathbb{S}^{n-1}$, $Z_l(\zeta,\eta) = Z_l(\eta,\zeta) \in \mathbb{R}$ and

$$Z_l(\rho\zeta,\eta) = Z_l(\zeta,\rho^{-1}\eta) \tag{1}$$

if $\rho \in O(n)$ ([6, Lemma 2.8, p. 143]).

We write $\mathcal{D}(\mathbb{S}^{n-1})$ for the set of test functions and $\mathcal{D}'(\mathbb{S}^{n-1})$ for the set of *distributions* on $\mathbb{S}^{n-1}$. The support of $T \in \mathcal{D}'(\mathbb{S}^{n-1})$ is denoted by $\operatorname{supp} T$. The *Fourier-Laplace series* of a distribution $T$ on $\mathbb{S}^{n-1}$ is $\sum_{l=0}^{+\infty} \Pi_l(T)$, where $\Pi_l(T)(\zeta) := T[\eta \mapsto Z_l(\zeta,\eta)]$ for $\zeta \in \mathbb{S}^{n-1}$; this series converges to $T$ in the sense of distributions. In [2, Theorem 1 and Remark 2] we obtained:

**Proposition 1.** *Let $T \in \mathcal{D}'(\mathbb{S}^{n-1})$ be of order $m \in \mathbb{N}_0$.*

(i) *If there exist $k \geq 0$ and $U$ an open subset of $\mathbb{S}^{n-1}$ on which*

$$\sum_{l=0}^{+\infty} \Pi_l(T)(\zeta) = 0 \quad (C,k) \tag{2}$$

*holds uniformly (in $\zeta$), then $T$ is zero on $U$.*

(ii) *Conversely, if $k > n-2+2m$, then* (2) *holds uniformly on every closed subset of $\mathbb{S}^{n-1} \setminus \operatorname{supp} T$.*

(iii) *Moreover, if $\operatorname{supp} T$ has at least two points, then* (2) *holds uniformly on every closed subset of $\mathbb{S}^{n-1} \setminus \operatorname{supp} T$ as soon as $k > n/2 - 1 + m$.*

## 3. Projective spaces

Here we will write $\mathbb{K}$ for either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$ (the algebra of quaternions) and let $d := \dim_{\mathbb{R}} \mathbb{K}$. We also define $U(\mathbb{K}) := \{k \in \mathbb{K} : \|k\| = 1\}$ and note $dk$ the normalized Haar measure of $U(\mathbb{K})$.

For $x, y \in \mathbb{K}^{n+1} \setminus \{0\}$, write $x \sim y$ if there exists $k \in \mathbb{K}^*$ such that $x = ky$, and let $[x]$ be the equivalence class of $x$. The *projective space of dimension* $n$ *on* $\mathbb{K}$ is $P^n(\mathbb{K}) := \mathbb{K}^{n+1} \setminus \{0\}/\sim$; it is a compact symmetric space of rank one (see [4]). Identifying $\mathbb{K}^{n+1}$ with $\mathbb{R}^{dn+d}$, we see that $P^n(\mathbb{K}) = \mathbb{S}^{dn+d-1}/\sim$. The connected component of the identity in the group of isometries of $P^n(\mathbb{K})$ is a group we write $S\mathbb{K}(n+1)$; in fact $S\mathbb{K}(n+1) = SO(n+1)$, $SU(n+1)$ or $Sp(n+1)$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively (see [1]). Moreover, the action of $S\mathbb{K}(n+1)$ on $P^n(\mathbb{K})$ is the one induced by the action on $\mathbb{S}^{dn+d-1}$ of $S\mathbb{K}(n+1)$ as a subgroup of $SO(dn+d)$. We also have, from the action of $U(\mathbb{K})$ on $\mathbb{K}^{n+1}$,

$$U(\mathbb{K}) < O(dn+d). \tag{3}$$

If $g$ is a $U(\mathbb{K})$-invariant function on $\mathbb{S}^{dn+d-1}$, we can define a function $g\downarrow$ on $P^n(\mathbb{K})$ by $g\downarrow([\eta]) := g(\eta)$. Conversely, if $f$ is a function on $P^n(\mathbb{K})$, we get, by putting $f\uparrow(\eta) := f([\eta])$, a $U(\mathbb{K})$-invariant function $f\uparrow$ on $\mathbb{S}^{dn+d-1}$ with $(f\uparrow)\downarrow = f$. Now, given an arbitrary function $g$ on $\mathbb{S}^{dn+d-1}$, we define a $U(\mathbb{K})$-invariant function $g^\sharp$ on $\mathbb{S}^{dn+d-1}$ by $g^\sharp(\eta) := \int_{U(\mathbb{K})} g(k\eta)\,dk$ (when $g$ is $U(\mathbb{K})$-invariant, $g^\sharp = g$). We then put $g^\flat := (g^\sharp)\downarrow$. If $T$ is a distribution on $P^n(\mathbb{K})$, we let, for $\varphi \in \mathcal{D}(\mathbb{S}^{dn+d-1})$, $T\uparrow(\varphi) := T(\varphi^\flat)$. Then $T\uparrow$ is a distribution on $\mathbb{S}^{dn+d-1}$ of the same order as $T$ and $\operatorname{supp} T\uparrow = \{\eta \in \mathbb{S}^{dn+d-1} : [\eta] \in \operatorname{supp} T\}$.

We write $dp_n$ for the unique normalized Radon measure on $P^n(\mathbb{K})$ which is $S\mathbb{K}(n+1)$-invariant. The link between $dp_n$ and $d\sigma_{dn+d-1}$ is:

$$\int_{\mathbb{S}^{dn+d-1}} g(\zeta)\,d\sigma_{dn+d-1}(\zeta) = \int_{P^n(\mathbb{K})} g^\flat(z)\,dp_n(z)$$

for every $g \in \mathcal{D}(\mathbb{S}^{dn+d-1})$. Finally, we can define the Laplace-Beltrami operator $\Delta_P$ on $P^n(\mathbb{K})$ by $\Delta_P(f) := (\Delta_S(f\uparrow))\downarrow$, using (3) and the facts that $f\uparrow$ is $U(\mathbb{K})$-invariant and $\Delta_S$ commutes with all rotations. Then $\Delta_P$ commutes with all elements of $S\mathbb{K}(n+1)$.

## 4. Fourier inversion on $P^n(\mathbb{K})$

Given $T \in \mathcal{D}'(P^n(\mathbb{K}))$, $\Pi_l(T\uparrow) \in \mathcal{S}H_l(\mathbb{S}^{dn+d-1})$ is $U(\mathbb{K})$-invariant:

$$\begin{aligned}
\Pi_l(T\uparrow)(u\zeta) &= T\uparrow(\eta \mapsto Z_l(u\zeta, \eta)) \\
&= T(Z_l(u\zeta, \cdot)^\flat) \\
&= T\Big([\eta] \mapsto \int_{U(\mathbb{K})} Z_l(u\zeta, k\eta)\,dk\Big) \\
&= T\Big([\eta] \mapsto \int_{U(\mathbb{K})} Z_l(\zeta, u^{-1}k\eta)\,dk\Big)
\end{aligned}$$

$$= T([\eta] \mapsto \int_{U(\mathbb{K})} Z_l(\zeta, k\eta)\, dk)$$
$$= T(Z_l(\zeta,\cdot)^\flat) \;=\; \Pi_l(T\uparrow)(\zeta)$$

(where $u \in U(\mathbb{K})$, $\zeta \in \mathbb{S}^{dn+d-1}$), using (1) and (3) for the fourth equality. Hence we can define a function $\Xi_l(T)$ on $P^n(\mathbb{K})$ by $\Xi_l(T) := (\Pi_l(T\uparrow))\downarrow$. Since $\Pi_l(T\uparrow)$ is either 0 or an eigenfunction of $\Delta_S$, $\Xi_l(T)$ is either 0 or an eigenfunction of $\Delta_P$. Moreover, if $l \neq m$, $\Xi_l(T)$ and $\Xi_m(T)$ are orthogonal in $L^2(P^n(\mathbb{K}), dp_n)$. This justifies the name *Fourier-Laplace series of* $T$ we give to $\sum_{l=0}^{+\infty} \Xi_l(T)$; this series converges to $T$ in the sense of distributions:

$$\begin{aligned}
\lim_{N\to+\infty} \sum_{l=0}^{N} \Xi_l(T)(\varphi) &= \lim_{N\to+\infty} \sum_{l=0}^{N} \int_{P^n(\mathbb{K})} \Xi_l(T)(z)\varphi(z)\, dp_n(z) \\
&= \lim_{N\to+\infty} \sum_{l=0}^{N} \int_{\mathbb{S}^{dn+d-1}} \Pi_l(T\uparrow)(\zeta)\varphi\uparrow(\zeta)\, d\sigma_{dn+d-1}(\zeta) \\
&= \lim_{N\to+\infty} \sum_{l=0}^{N} \Pi_l(T\uparrow)(\varphi\uparrow) \\
&= T\uparrow(\varphi\uparrow) \;=\; T((\varphi\uparrow)^\flat) \;=\; T(\varphi)
\end{aligned}$$

if $\varphi \in \mathcal{D}(P^n(\mathbb{K}))$. From the preceding section and Proposition 1 we deduce:

**Proposition 2.** *Let $T \in \mathcal{D}'(P^n(\mathbb{K}))$ be of order $m \in \mathbb{N}_0$.*

(i) *If there exist $k \geq 0$ and $U$ an open subset of $P^n(\mathbb{K})$ on which*

$$\sum_{l=0}^{+\infty} \Xi_l(T)(z) = 0 \quad (C,k) \tag{4}$$

*holds uniformly (in $z$), then $T$ is zero on $U$.*

(ii) *Conversely, if $k > (dn+d)/2 - 1 + m$, then* (4) *holds uniformly on every closed subset of $P^n(\mathbb{K}) \setminus \operatorname{supp} T$.*

**Remarks. 1.** Naturally (4) can hold for some $k \leq (dn+d)/2 - 1 + m$. For example, take $\mathbb{K} = \mathbb{R}$, $n \geq 2$, pick a point $z_0$ in $P^n(\mathbb{R})$ and consider the ball $B$ with centre $z_0$ whose radius is the diameter of $P^n(\mathbb{R})$; its boundary $\partial B$ can be identified with $P^{n-1}(\mathbb{R})$. For all $\varphi \in \mathcal{D}(P^n(\mathbb{R}))$ we let

$$\mu_{n-1}(\varphi) := \int_{\partial B} \varphi(z)\, dp_{n-1}(z);$$

this defines a measure $\mu_{n-1}$ on $P^n(\mathbb{R})$. Then the distribution $\Delta_P^q \mu_{n-1}$ ($q \in \mathbb{N}_0$) has order $2q$ and support $\partial B$; its Fourier-Laplace series is $(C,k)$-summable to 0 at all points outside $\partial B \cup \{z_0\}$ if and only if $k > 2q$ and at $z_0$ if and only if $k > (n-1)/2 + 2q$; this follows from [2, Proposition 1].

**2.** If (4) holds uniformly on a subset $A$ of $P^n(\mathbb{K})$, it holds uniformly on the closure of $A$. Hence, when the interior of $\operatorname{supp} T$ is empty, (4) does not hold uniformly on $P^n(\mathbb{K}) \setminus \operatorname{supp} T$; this is the case in the preceding example.

**3.** Since $P^n(\mathbb{K})$ is a sphere when $n = 1$, Proposition 2(ii) gives a partial refinement of Proposition 1(ii):

**Corollary.** *Let $d = 1, 2$ or $4$ and $T \in \mathcal{D}'(\mathbb{S}^d)$ be of order $m \in \mathbb{N}_0$. If $k > d-1+m$, then* (2) *holds uniformly on every closed subset of $\mathbb{S}^d \setminus \operatorname{supp} T$.*

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