# Intersections of minimal prime ideals in the rings of continuous functions

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*Abstract.* A space $X$ is called $\mu$-compact by M. Mandelker if the intersection of all free maximal ideals of $C(X)$ coincides with the ring $C_K(X)$ of all functions in $C(X)$ with compact support. In this paper we introduce $\phi$-compact and $\phi'$-compact spaces and we show that a space is $\mu$-compact if and only if it is both $\phi$-compact and $\phi'$-compact. We also establish that every space $X$ admits a $\phi$-compactification and a $\phi'$-compactification. Examples and counterexamples are given.

*Keywords:* minimal prime ideal, $P$-space, $F$-space, $\mu$-compact space, $\phi$-compact space, $\phi'$-compact space, round subset, almost round subset, nearly round subset

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## 1. Introduction

By a space we always mean a completely regular Hausdorff space. It is well-known that if $X$ is realcompact, then the intersection of all free maximal ideals of $C(X)$ coincides with the ring $C_K(X)$ of all functions in $C(X)$ with compact support ([1, 8.19]). A space with the latter property is called $\mu$-compact by M. Mandelker in 1971 ([5]). A subset $A$ of $\beta X$ is called round by M. Mandelker in 1969 if for any zero set $Z$ of $X$, $\mathrm{cl}_{\beta X} Z$ is a neighbourhood of $A$ whenever $\mathrm{cl}_{\beta X} Z \supseteq A$ ([4, 4]). In 1973, D.G. Johnson and M. Mandelker have shown that for any space $X$, there is a smallest $\mu$-compact space $\mu X$ lying between $X$ and $\beta X$ ([3, 4.1]). They have also proved that $\mu X$ is the smallest subspace of $\beta X$ containing $X$ for which $\beta X - \mu X$ is round ([3, 4.3]). We define $\phi$-compact spaces in terms of intersections of minimal prime ideals of $C(X)$. The class of all $\phi$-compact spaces extends the class of all $\mu$-compact spaces. We prove that for any space $X$, there is a smallest $\phi$-compact space $\phi X$ lying between $X$ and $\beta X$. Mandelker's definition of round subsets of $\beta X$ characterizes $P$-spaces. In fact, $X$ is a $P$-space if and only if every subset of $\beta X$ is round ([4, 5.6]). The question is what type of subsets of $\beta X$ characterize $F$-spaces? We define almost round subsets of $\beta X$. It turns out that a space $X$ is an $F$-space if and only if every subset of $\beta X$ is almost round. We also establish that $\phi X$ is the smallest subspace of $\beta X$ containing $X$ for which $\beta X - \phi X$ is almost round. Our motivation to define $\phi'$-compact spaces is the theorem in which we show that a space is $\mu$-compact if and only if it is both $\phi$-compact and $\phi'$-compact. We prove that for any space $X$, there is a smallest $\phi'$-compact space $\phi' X$ lying between $X$ and $\beta X$. We define nearly round subsets of $\beta X$ and similar results as for round and almost round subsets are established. Finally we show that an $F$-space $X$ is a $P$-space if and only if every subset of $\beta X$ is nearly round.

## 2. Maximal, prime and minimal prime ideals

As usual, $\beta X$ is the Stone-Čech compactification of $X$. There is a one-one correspondence between the points of $\beta X$ and the maximal ideals of $C(X)$, described in the following theorem ([1, 7.3]).

**Theorem 2.1** ([Gelfand-Kolmogoroff]). *The maximal ideals of $C(X)$ are given by $M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}(p \in \beta X)$, here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of $f$.*

Also the set $O^p = \{f \in C(X) : \mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p\}$ is an ideal of $C(X)$, for each $p \in \beta X$.

An ideal $I$ of $C(X)$ is called a $z$-ideal if $Z(f) = Z(g)$ and $f \in I$ implies $g \in I$.

It is clear that for each $p \in \beta X$, $M^p$ and $O^p$ are $z$-ideals of $C(X)$.

We now write down the following important theorem given in [1, 7.15].

**Theorem 2.2.** *Every prime ideal $P$ of $C(X)$ contains $O^p$ for a unique $p$ and $M^p$ is the unique maximal ideal that contains $P$.*

It is well-known that $X$ is an $F$-space if and only if $O^p$ is prime for each $p \in \beta X$ ([1, 14.25]), and $X$ is a $P$-space if and only if $O^p = M^p$ for each $p \in \beta X$ ([1, 14.29]). Clearly every $P$-space is an $F$-space, the converse is not true. The space $\beta\mathbb{R}\backslash\mathbb{R}$ is a compact $F$-space ([1, 14.27]). It fails to be a $P$-space since every compact $P$-space is finite ([1, 4k, 2]).

Every $z$-ideal in $C(X)$ is an intersection of prime ideals ([1, 2.8]). Since $O^p$ is a $z$-ideal we have the following theorem.

**Theorem 2.3.** *The ideal $O^p$ is the intersection of all minimal prime ideals containing it.*

Let $\mathcal{P}_{\min}(X)$ denote the class of all minimal prime ideals of $C(X)$. We define the relation '$\sim$' on $\mathcal{P}_{\min}(X)$ by $P \sim Q$ if and only if $P, Q$ are contained in a same maximal ideal. Obviously '$\sim$' is an equivalence relation on $\mathcal{P}_{\min}(X)$. All the minimal prime ideals of $C(X)$ contained in $M^p$ (i.e. containing $O^p$) for some $p \in \beta X$ form an equivalence class which will be denoted by $E_p$. We state the following important characterization of minimal prime ideals of $C(X)$ which is an immediate consequence of [2, Lemma 1.1].

**Theorem 2.4.** *Let $P$ be a prime ideal of $C(X)$. Then $P$ is minimal if and only if for any $f \in P$, there exists $g \in C(X) - P$ such that $fg = 0$.*

**Notations 2.5.** Let $X \subseteq Y \subseteq \beta X$ and $p \in \beta X$. The ideal $\{f \in C(X) : \mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p\}$ of $C(X)$ will be denoted by $O^p_X$ and the ideal $\{f \in C(Y) : \mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p\}$ of $C(Y)$ will be denoted by $O^p_Y$.

We note that every minimal prime ideal in $C(X)$ is a $z$-ideal ([1, 14.7]). Now we prove the following theorem.

**Theorem 2.6.** *Let $X \subseteq Y \subseteq \beta X$ and $p \in \beta X$. If $P_Y$ is a minimal prime ideal of $C(Y)$ with $P_Y \supseteq O^p_Y$ and if $f \in P_Y$ then there exists a minimal prime ideal $P_X$ of $C(X)$ with $P_X \supseteq O^p_X$ such that $f|_X \in P_X$. Also if $P_X$ is a minimal prime ideal of $C(X)$ with $P_X \supseteq O^p_X$ and if $f \in P_X$ with $f^Y \in C(Y)$ then there exists a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P^Y$, here $f^Y$ is the continuous extension of $f$ over $Y$.*

Proof: Let $f \in P_Y$ where $P_Y$ is a minimal prime ideal of $C(Y)$ with $P_Y \supseteq O^p_Y$. Then there exists $g \in C(Y)$ such that $fg = 0$ and $g \notin P_Y$ (Theorem 2.4). Clearly, $g \notin O^p_Y$. Let $g' = g|_X$. Then $Z(g') \subseteq Z(g)$ and hence $g' \notin O^p_X$. Let $f' = f|_X$. Clearly, $f'g' = 0$. Now $g' \notin O^p_X$ implies that there exists a minimal prime ideal $P_X$ of $C(X)$ with $P_X \supseteq O^p_X$ such that $g' \notin P_X$. Thus $f' = f|_X \in P_X$.

Conversely let, $f \in P_X$ with $f^Y \in C(Y)$ where $P_X$ is a minimal prime ideal of $C(X)$ such that $P_X \supseteq O^p_X$. Now there exists $g \in C(X)$ with $fg = 0$ such that $g \notin P_X$ (Theorem 2.4). Let $h = g \wedge 1$. Since $g \notin P_X$ and $P_X$ is a $z$-ideal, $h \notin P_X$. Clearly $fh = 0$. Let $h^Y$ be the continuous extension of $h$ over $Y$. Then, $f^Y h^Y = 0$. We claim that there exists a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $h^Y \notin P_Y$. If not, then $h^Y \in O^p_Y$ and so there is a neighbourhood $V$ of $p$ in $\beta X$ $(= \beta Y)$ such that $Z(h^Y) \supseteq V \cap Y$ ([1, 7.12(a)]). Thus, $Z(h) = X \cap Z(h^Y) \supseteq V \cap Y \cap X = V \cap X$ and so, $h \in O^p_X$ ([1, 7.12(a)]). Hence $g \in O^p_X$ since $O^p_X$ is a $z$-ideal. This shows that $g \in P_X$, a contradiction. So, $h^Y \notin P_Y$ for some minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O^p_Y$ and thus $f^Y \in P_Y$. □

## 3. $\phi$-compact spaces and almost round subsets

Recall the equivalence relation introduced in Section 2. Let us now give the following definition.

**Definition 3.1.** Let $A \subseteq \beta X$. A family $\mathcal{F}$ of minimal prime ideals of $C(X)$ is said to be adequate for $A$ if $\mathcal{F} \cap E_p \neq \phi$ $\forall p \in A$. A space $X$ is defined to be $\phi$-compact if $\bigcap \mathcal{F} \subseteq C_K(X)$ for every family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$.

**Examples 3.2.** (a) Every $F$-space is $\phi$-compact. In fact, if $X$ is an $F$-space then $E_p = \{O^p\}$ $\forall p \in \beta X$. So if $\mathcal{F}$ is a family of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ then $O^p \in \mathcal{F}\ \forall p \in \beta X - X$. Clearly, $\bigcap \mathcal{F} \subseteq \bigcap_{p\in\beta X - X} O^p = C_K(X)$ and thus $X$ is $\phi$-compact.

(b) Every $\mu$-compact space is $\phi$-compact (hence every realcompact space is $\phi$-compact). In fact, if $\mathcal{F}$ is any family of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ then $\bigcap \mathcal{F} \subseteq \bigcap_{p\in\beta X - X} M^p$. Now if $X$ is $\mu$-compact then $\bigcap_{p\in\beta X-X} M^p = C_K(X)$ and thus $\bigcap \mathcal{F} \subseteq C_K(X)$. So $X$ becomes $\phi$-compact.

(c) The Tychonoff plank $T$ is not $\phi$-compact. We know that there is only one free maximal ideal, say $M^t$ in $C(T)$. Also $O^t$ is not prime ([1, 8$J$, 6]). Thus if $P$ is any minimal prime ideal of $C(T)$ with $P \subseteq M^t$ then $O^t \subsetneqq P$ and hence $T$ cannot be $\phi$-compact.

Our next theorem shows that every space $X$ admits a $\phi$-compactification.

**Theorem 3.3.** *For every space $X$, there is a smallest $\phi$-compact space $\phi X$ lying between $X$ and $\beta X$. So $X$ is $\phi$-compact if and only if $X = \phi X$.*

Proof: Let $\Phi$ denote the set of all $\phi$-compact spaces lying between $X$ and $\beta X$. Clearly $\Phi \neq \emptyset$ since $\beta X \in \Phi$. Let $\phi X = \bigcap \Phi$. To complete the theorem we shall show that $\phi X$ is $\phi$-compact. Consider any family $\mathcal{F}$ of minimal prime ideals of $C(\phi X)$, adequate for $\beta(\phi X) - \phi X$ $(= \beta X - \phi X)$ and suppose $f \in \bigcap \mathcal{F}$. Let $Y \in \Phi$ and $p \in \beta X - Y$. Then $p \in \beta X - \phi X$. Since $\mathcal{F}$ is adequate for $\beta X - \phi X$, there is a minimal prime ideal $P_{\phi X}$ of $C(\phi X)$ in $\mathcal{F}$ with $P_{\phi X} \supseteq O^p_{\phi X}$. So $f \in P_{\phi X}$. Clearly $f \in C^*(\phi X)$ and let $f^Y$ be the continuous extension of $f$ over $Y$. By Theorem 2.6, there is a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P_Y$. Thus $\mathcal{F}' = \{P_Y : P_Y$ is a minimal prime ideal of $C(Y)$ with $f^Y \in P_Y\}$ is adequate for $\beta Y - Y$ and $f^Y \in \bigcap \mathcal{F}'$. Since $Y$ is $\phi$-compact, $f^Y \in C_K(Y)$. So, $\mathrm{cl}_Y(Y - Z(f^Y))$ is compact and hence so is $\bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Clearly, $\mathrm{cl}_{\phi X}(\phi X - Z(f)) \subseteq \bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Let $p \in \bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Then $p \in Y\ \forall Y \in \Phi$ and so $p \in \phi X$. Take any neighbourhood $U$ of $p$ in $\phi X$. Then there is a neighbourhood $V$ of $p$ in Y (where $Y \in \Phi$) such that $V \cap \phi X = U$. Also, $V \cap (Y - Z(f^Y)) \neq \emptyset$. Thus, $V \cap (Y - Z(f^Y))$ is a non-void open set in $Y$. Since $\phi X$ is dense in $Y$, $\phi X \cap V \cap (Y - Z(f^Y)) \neq \emptyset$ i.e. $U \cap (\phi X - Z(f)) \neq \emptyset$. So $p \in \mathrm{cl}_{\phi X}(\phi X - Z(f))$. Thus, $\mathrm{cl}_{\phi X}(\phi X - Z(f)) = \bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Hence $f \in C_K(\phi X)$ and $\phi X$ becomes $\phi$-compact. □

We now define almost round subsets as follows.

**Definition 3.4.** A subset $A$ of $\beta X$ is said to be almost round if $\bigcap \mathcal{F} \subseteq \bigcap_{p\in A} O^p$ for every family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $A$.

Obviously $X$ is $\phi$-compact if and only if $\beta X - X$ is almost round. We also note that the union of any collection of almost round subsets of $\beta X$ is almost round.

We now prove the following two lemmas.

**Lemma 3.5.** *Let $X \subseteq Y \subseteq vX$. Then $f \in O^p_X$ if and only if $f^Y \in O^p_Y$ where $f^Y$ is the continuous extension of $f$ over $Y$.*

Proof: The lemma follows from the fact that $\mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X} Z(f^Y)$. □

**Lemma 3.6.** *Let $X \subseteq Y \subseteq vX$. Then $Y$ is $\phi$-compact if and only if $\beta X - Y$ is almost round (with respect to $X$).*

Proof: Let $Y$ be $\phi$-compact and let $\mathcal{F}$ be a family of minimal prime ideals of $C(X)$, adequate for $\beta X - Y$. Suppose $f \in \bigcap \mathcal{F}$ and $f^Y$ is the continuous extension of $f$ over $Y$. If $p \in \beta X - Y$ then there is a minimal prime ideal $P_X \in \mathcal{F}$ with $P_X \supseteq O^p_X$, $\mathcal{F}$ being adequate for $\beta X - Y$. So by Theorem 2.6, there is a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P^Y$. Thus $\mathcal{F}' = \{P_Y : P_Y$ is a minimal prime ideal of $C(Y)$ with $f^Y \in P_Y\}$ is adequate for $\beta X - Y$ and $f^Y \in \bigcap \mathcal{F}'$. Since $Y$ is $\phi$-compact, $f^Y \in C_K(Y)$. Thus $f^Y \in O^p_Y$ $\forall p \in \beta X - Y$. So by Lemma 3.5, $f \in O^p_X$ $\forall p \in \beta X - Y$. Consequently, $\bigcap \mathcal{F} \subseteq \bigcap_{p \in \beta X - Y} O^p_X$ and so $\beta X - Y$ is almost round.

Conversely let $\beta X - Y$ be almost round. Suppose $\mathcal{F}'$ is any family of minimal prime ideals of $C(Y)$, adequate for $\beta Y - Y$ $(= \beta X - Y)$ and suppose $f \in \bigcap \mathcal{F}'$. Let $f_1 = f|_X$ and $p \in \beta X - Y$. Since $\mathcal{F}'$ is adequate for $\beta X - Y$, there is a minimal prime ideal $P_Y \in \mathcal{F}'$ such that $P_Y \supseteq O^p_Y$. Also $f \in P_Y$. By Theorem 2.6, there is a minimal prime ideal $P_X$ of $C(X)$ with $P_X \supseteq O^p_X$ such that $f_1 \in P_X$. Thus $\mathcal{F} = \{P_X : P_X$ is a minimal prime ideal of $C(X)$ with $f_1 \in P_X\}$ becomes adequate for $\beta X - Y$ and $f_1 \in \bigcap \mathcal{F}$. Since $\beta X - Y$ is almost round, $f_1 \in O^p_X$ $\forall p \in \beta X - Y$ and so by Lemma 3.5, $f \in O^p_Y$ $\forall p \in \beta X - Y$. So $\bigcap \mathcal{F}' \subseteq \bigcap_{p \in \beta X - Y} O^p_Y = C_K(Y)$ and hence $Y$ is $\phi$-compact. □

**Corollary 3.7.** *For any space $X$, $\beta X - \phi X$ is almost round.*

We now use Lemma 3.6 to prove the following theorem.

**Theorem 3.8.** *For any space $X$, $\phi X$ is the smallest subspace of $\beta X$ containing $X$ for which $\beta X - \phi X$ is almost round.*

Proof: Let $X \subseteq Y \subseteq \beta X$ such that $\beta X - Y$ is almost round. Then $(\beta X - \phi X) \cup (\beta X - Y) = \beta X - (\phi X \cap Y)$ is almost round. Clearly $X \subseteq \phi X \cap Y \subseteq vX$ and so Lemma 3.6 implies that $\phi X \cap Y$ is $\phi$-compact. Since $\phi X$ is the smallest $\phi$-compact space between $X$ and $\beta X$, $\phi X \subseteq \phi X \cap Y$. So $\phi X \subseteq Y$ and the theorem follows. □

Almost round subsets characterize $F$-spaces in the following way.

**Theorem 3.9.** *$X$ is an $F$-space if and only if every subset of $\beta X$ is almost round.*

Proof: The necessity follows from the fact that for an $F$-space $X$, $E_p = \{O^p\}$ $\forall p \in \beta X$.

To prove the sufficiency let $p \in \beta X$. Since $\{p\}$ is almost round, $O^p = P$ for any minimal prime ideal $P$ with $P \supseteq O^p$. Thus $O^p$ is prime and so $X$ is an $F$-space.

Let $X$ be a $\phi$-compact space. If $\tau : X \to Y$ is a homeomorphism then $\tau$ has an extension to a homeomorphism $\tau_1 : \beta X \to \beta Y$ such that $\tau|_{\beta X - X} : \beta X - X \to \beta Y - Y$ is also a homeomorphism. Also the map $\psi : C(Y) \to C(X)$ defined by $f \to f_\circ \tau$ is an isomorphism. If $\mathcal{F} = \{P_Y^\alpha : \alpha \in \wedge\}$ is a family of minimal prime ideals of $C(Y)$, adequate for $\beta Y - Y$ then clearly $\mathcal{F}_X = \{\psi(P_Y^\alpha) : \alpha \in \wedge\}$ becomes a family of minimal prime ideals of $C(X)$, adequate for $\beta X - X$. It is now easy to see that $Y$ is $\phi$-compact. Hence we have the following theorem. $\square$

**Theorem 3.10.** *$\phi$-compactness is a topological property.*

**Example 3.11.** Let $Y = \beta N - \{p\}$ where $p \in \beta N - N$. Then $Y$ is an $F$-space and hence $\phi$-compact. The lone free maximal ideal of $C(Y)$ is $M_Y^p = \{f \in C(Y) : p \in \mathrm{cl}_{\beta Y} Z(f)\}$. Clearly $p \in \mathrm{cl}_{\beta N}(Y - N)$. Define $f : N \to R$ by $f(n) = \frac{1}{n}$ and suppose $h = f^\beta|_Y$. Then $h \in C(Y)$ and $Z(h) = Y - N$. Thus $h \in M_Y^p$. Now $\mathrm{cl}_Y(Y - Z(h)) = \mathrm{cl}_Y N = Y$ which is not compact and so $h \notin C_K(Y)$. Hence $Y$ is not $\mu$-compact.

## 4. $\phi'$-compact spaces and nearly round subsets

Recall the definition of a family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ (Definition 3.1). Let us now give the following definition.

**Definition 4.1.** A space $X$ is said to be $\phi'$-compact if for any $f \in \bigcap_{p \in \beta X - X} M^p$, there is a family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ such that $f \in \bigcap \mathcal{F}$.

**Example 4.2.** Every $\mu$-compact space is $\phi'$-compact (hence every realcompact space is $\phi'$-compact). In fact, if $X$ is $\mu$-compact and if $f \in \bigcap_{p \in \beta X - X} M^p$ then $f \in C_K(X)$ and so $f$ is in every free minimal prime ideal of $C(X)$. So if $\mathcal{F}$ is the collection of all free minimal prime ideals in $C(X)$ then $f \in \bigcap \mathcal{F}$. Clearly $\mathcal{F}$ is adequate for $\beta X - X$.

The following theorem relates $\mu$-compact spaces, $\phi$-compact spaces and $\phi'$-compact spaces.

**Theorem 4.3.** *A space is $\mu$-compact if and only if it is both $\phi$-compact and $\phi'$-compact.*

Proof: Necessity follows from 3.2(b) and 4.2.

For sufficiency we assume that $X$ is both $\phi$-compact and $\phi'$-compact. Let $f \in \bigcap_{p \in \beta X - X} M^p$. Since $X$ is $\phi'$-compact, there is a family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ such that $f \in \bigcap \mathcal{F}$. Now $\phi$-compactness of $X$ implies $\bigcap \mathcal{F} \subseteq C_K(X)$. Thus $f \in C_K(X)$ and so $X$ is $\mu$-compact. $\square$

**Example 4.4.** Recall the space $Y = \beta N - \{p\}$ where $p \in \beta N - N$ given in 3.11. The space is $\phi$-compact but not $\mu$-compact. Hence the space is also not $\phi'$-compact by the previous theorem.

**Notations 4.5.** Let $X \subseteq Y \subseteq \beta X$ and $p \in \beta X$. The maximal ideal $\{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ of $C(X)$ will be denoted by $M_X^p$ and the maximal ideal $\{f \in C(Y) : p \in \mathrm{cl}_{\beta Y} Z(f)\}$ of $C(Y)$ will be denoted by $M_Y^p$.

In our next theorem we shall show that every space $X$ admits a $\phi'$-compactification.

**Theorem 4.6.** *For any space $X$, there is a smallest $\phi'$-compact space $\phi' X$ lying between $X$ and $\beta X$. Thus $X$ is $\phi'$-compact if and only if $X = \phi' X$.*

PROOF: Let $\Phi'$ be the family of all $\phi'$-compact spaces lying between $X$ and $\beta X$. Then $\Phi' \neq \emptyset$ since $\beta X \in \Phi'$. Let $\phi' X = \bigcap \Phi'$. To prove the theorem we shall show that $\phi' X$ is $\phi'$-compact. So let $f \in \bigcap_{p \in \beta X - \phi' X} M_{\phi' X}^p$ and let $p \in \beta X - \phi' X$. Then there is $Y \in \Phi'$ such that $p \in \beta X - Y$. Now $f \in C^*(\phi' X)$ and let $f^Y$ be the continuous extension of $f$ over $Y$. Let $q \in \beta X - Y$. Clearly $q \in \beta X - \phi' X$. So $f \in M_{\phi' X}^q$. Hence $q \in \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} Z(f^Y)$. Thus $f^Y \in M_Y^q$. So $f^Y \in \bigcap_{q \in \beta X - Y} M_Y^q$. Since $Y$ is $\phi'$-compact and $p \in \beta X - Y$, there is a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O_Y^p$ such that $f^Y \in P_Y$. So by Theorem 2.6, there is a minimal prime ideal $P_{\phi' X}$ of $C(\phi' X)$ with $P_{\phi' X} \supseteq O_{\phi' X}^p$ such that $f \in P_{\phi' X}$. So $\mathcal{F} = \{P_{\phi' X} : P_{\phi' X}$ is a minimal prime ideal of $C(\phi' X)$ with $f \in P_{\phi' X}\}$ is adequate for $\beta X - \phi' X$ and $f \in \bigcap \mathcal{F}$. Thus $\phi' X$ is $\phi'$-compact. $\square$

We now define nearly round subsets as follows.

**Definition 4.7.** A subset $A$ of $\beta X$ is said to be nearly round if $f \in \bigcap_{p \in A} M^p$ implies $f \in \bigcap \mathcal{F}$ for some family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $A$.

Obviously $X$ is $\phi'$-compact if and only if $\beta X - X$ is nearly round. We note that the union of any collection of nearly round subsets of $\beta X$ is nearly round. We also note that a subset of $\beta X$ is round if and only if it is both almost round and nearly round.

We now prove the following lemma.

**Lemma 4.8.** *Let $X \subseteq Y \subseteq vX$. Then $Y$ is $\phi'$-compact if and only if $\beta X - Y$ is nearly round (with respect to $X$).*

PROOF: Let $Y$ be $\phi'$-compact and let $f \in \bigcap_{p \in \beta X - Y} M_X^p$. Let $f^Y$ be the continuous extension of $f$ over $Y$. Then $\mathrm{cl}_{\beta X} Z(f^Y) = \mathrm{cl}_{\beta X} Z(f)$ and thus $f^Y \in \bigcap_{p \in \beta X - Y} M_Y^p$. Suppose $p \in \beta X - Y$. Now $\phi'$-compactness of $Y$ implies that there is a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O_Y^p$ such that

$f^Y \in P_Y$. So by Theorem 2.6, there is a minimal prime ideal $P_X$ of $C(X)$ with $P_X \supseteq O_X^p$ such that $f \in P_X$. Thus $\mathcal{F} = \{P_X : P_X$ is a minimal prime ideal of $C(X)$ with $f \in P_X\}$ is adequate for $\beta X - Y$ and $f \in \bigcap \mathcal{F}$. Consequently $\beta X - Y$ is nearly round.

Conversely let $\beta X - Y$ be nearly round and let $f \in \bigcap_{p \in \beta X - Y} M_Y^p$. Let $f|_X = g$. Then $\operatorname{cl}_{\beta X} Z(f) = \operatorname{cl}_{\beta X} Z(g)$ and so $g \in \bigcap_{p \in \beta X - Y} M_X^p$. Let $q \in \beta X - Y$. Since $\beta X - Y$ is nearly round, there is a minimal prime ideal $P_X$ of $C(X)$ with $P_X \supseteq O_X^q$ such that $g \in P_X$. Hence by Theorem 2.6, there is a minimal prime ideal $P_Y$ of $C(Y)$ with $P_Y \supseteq O_Y^q$ such that $f \in P_Y$. Thus $\mathcal{F}' = \{P_Y : P_Y$ is a minimal prime ideal of $C(Y)$ with $f \in P_Y\}$ is adequate for $\beta X - Y$ and $f \in \bigcap \mathcal{F}'$. Thus $Y$ is $\phi'$-compact. $\square$

**Corollary 4.9.** *For any space $X$, $\beta X - \phi' X$ is nearly round.*

We now use Lemma 4.8 to prove the following theorem.

**Theorem 4.10.** *For any space $X$, $\phi' X$ is the smallest subspace of $\beta X$ containing $X$ for which $\beta X - \phi' X$ is nearly round.*

Proof: Let $X \subseteq Y \subseteq \beta X$ such that $\beta X - Y$ is nearly round. Then $(\beta X - \phi' X) \cup (\beta X - Y) = \beta X - (\phi' X \cap Y)$ is nearly round. Clearly $X \subseteq \phi' X \cap Y \subseteq vX$ and so by Lemma 4.8, $\phi' X \cap Y$ is $\phi'$-compact. Since $\phi' X$ is the smallest $\phi'$-compact space between $X$ and $\beta X$, $\phi' X \subseteq \phi' X \cap Y$. So $\phi' X \subseteq Y$ and the proof is complete. $\square$

The following theorem gives a necessary and sufficient condition for an $F$-space to be a $P$-space.

**Theorem 4.11.** *An $F$-space $X$ is a $P$-space if and only if every subset of $\beta X$ is nearly round.*

Proof: Let $X$ be a $P$-space and $A \subseteq \beta X$. Suppose $f \in \bigcap_{p \in A} M^p$. Then $f \in \bigcap_{p \in A} O^p$. Thus $\mathcal{F} = \{O^p : p \in A\}$ is a family of minimal prime ideals of $C(X)$, adequate for $A$ with $f \in \bigcap \mathcal{F}$. So $A$ is nearly round.

Conversely let $X$ be an $F$-space and every subset of $\beta X$ be nearly round. Let $p \in \beta X$ and suppose $f \in M^p$. Since $\{p\}$ is nearly round there is a minimal prime ideal $P$ of $C(X)$ with $P \supseteq O^p$ such that $f \in P$. Also since $X$ is an $F$-space, $P = O^p$ and thus $f \in O^p$. So $O^p = M^p$ and hence $X$ is a $P$-space.

Let $X$ be a $\phi'$-compact space. If $\tau : X \to Y$ is a homeomorphism then $\tau$ has an extension to a homeomorphism $\tau_1 : \beta X \to \beta Y$ such that $\tau_1|_{\beta X - X} : \beta X - X \to \beta Y - Y$ is also a homeomorphism. Also the map $\psi : C(Y) \to C(X)$ defined by $f \to f_\circ \tau$ is an isomorphism. If $f$ is in the intersection of all free maximal ideals of $C(Y)$ then $\psi(f)$ is in the intersection of all free maximal ideals of $C(X)$. Now $\phi'$-compactness of $X$ implies that there is a family $\mathcal{F}_X = \{P_X^\alpha : \alpha \in \wedge\}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ with $\psi(f) \in \bigcap \mathcal{F}_X$. Then $\mathcal{F}_Y = \{\psi^{\leftarrow}(P_X^\alpha) : \alpha \in \wedge\}$ becomes a family of minimal prime ideals of $C(Y)$

adequate for $\beta Y - Y$ and $f \in \bigcap \mathcal{F}_Y$. Thus $Y$ is also $\phi'$-compact. So we have the following theorem. $\square$

**Theorem 4.12.** *$\phi'$-compactness is a topological property.*

**Notation 4.13.** Let $\omega_1$ denote the space of all countable ordinals. Let $T^* = (\omega_1 + 1) \times (\omega_0 + 1)$ and $T = T^* - \{(\omega_1, \omega_0)\}$ be the Tychonoff plank.

Let us denote for computational convenience, $(\alpha, \omega_1) \times \{n\}$ ($(\alpha, \omega_1] \times \{n\}$) by $(\alpha, \{n\})$ ($(\alpha, \{n\}]$, respectively), where $\alpha \lneqq \omega_1$ and $n \in (\omega_0 + 1)$.

**Lemma 4.14.** *For each $f \in M^t - O^t$, there exists $g \notin O^t$ such that $fg = 0$ where $t = \{(\omega_1, \omega_0)\}$.*

PROOF: Since $f \in M^t$, i.e. $t \in \mathrm{cl}_{\beta T} Z(f)$, every neighbourhood of $t$ must meet $Z(f)$. Also $f \notin O^t$ and so $\mathrm{cl}_{\beta T} Z(f)$ is not a neighbourhood of $t$. Now any neighbourhood of $t$ is of the form $(\alpha, \omega_1] \times N'$, where $N' \subseteq \omega_0 + 1$, $\alpha \lneqq \omega_1$ and $(\omega_0 + 1) - N'$ is at most a finite set. Thus there exist infinite subsets $N_1$, $N_2$ of $\omega_0$ with $N_1 \cup N_2 = \omega_0$ and $\alpha \lneqq \omega_1$, such that, for each $n \in N_1$, $f((\alpha, \{n\}]) = 0$ and for each $n \in N_2$, $f((\alpha, \{n\}]) \neq 0$. The choice of single $\alpha$ is possible here because of the non-cofinality character of any denumerable subset of $\omega_1$. Also $f((\alpha, \{\omega_0\})) = 0$. Choose $g : T \to \mathbb{R}$ by defining $g((\alpha, \{n\}]) = \frac{1}{n}$, for each $n \in N_1$, $g((\alpha, \{n\}]) = 0$ for each $n \in N_2$ and assign 0 on rest of the region. Clearly, $g$ is continuous in $[0, \alpha] \times (\omega_0 + 1)$. Choose $(\gamma, n) \in (\alpha, \{n\}]$, $n \in \omega_0$. Then $(\alpha, \{n\}]$ is an open neighbourhood of $(\gamma, n)$ and $g((\alpha, \{n\}])$ is either $= 0$ or $\frac{1}{n}$. Thus $f$ is continuous at $(\gamma, n)$. If now $(\gamma, \omega_0) \in (\alpha, \{\omega_0\})$, then $g((\gamma, \omega_0)) = 0$. Choose any $\epsilon \gneqq 0$. Then there exists $n \in \omega_0$ such that $\frac{1}{n} \lneqq \epsilon$. Take $M = (\omega_0 + 1) - \{r \in \omega_0 : r \leqq n\}$. Then $(\alpha, \omega_1] \times M - \{t\}$ is an open neighbourhood of $(\gamma, \omega_0)$ and $g(((\alpha, \omega_1] \times M) - \{t\})$ is contained in $(-\epsilon, \epsilon)$. Hence $g$ is continuous at $(\gamma, \omega_0)$. Thus $g$ is continuous on $T$. Also since $T - Z(g)$ contains $(\alpha, \omega_1] \times N_1$, $g \notin O^t$. Clearly, $fg = 0$. $\square$

Using the above lemma, we now show that the Tychonoff plank $T$ is $\phi'$-compact but not $\mu$-compact.

**Example 4.15.** Since $T$ is not $\phi$-compact (Example 3.2(c)), it is neither $\mu$-compact. We now show that $T$ is $\phi'$-compact. So let $f \in \bigcap_{p \in \beta T - T} M^p$ i.e. $f \in M^t$. We have to produce a family $\mathcal{F}$ of minimal prime ideals of $C(T)$, adequate for $\beta T - T = \{t\}$ such that $f \in \bigcap \mathcal{F}$. If $f \in O^t$, then it becomes obvious, if not then $fg = 0$ for some $g \notin O^t$ by Lemma 4.14. Since $O^t$ is the intersection of all minimal prime ideals containing it, there is a minimal prime ideal, say $P$ containing $O^t$ such that $g \notin P$. So $f \in P$ since $P$ is prime. Let $\mathcal{F} = \{P\}$. Clearly $\mathcal{F}$ is adequate for $\beta T - T$ and $f \in \bigcap \mathcal{F}$.

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