Continuous selections on spaces of continuous functions

Ángel Tamariz-Mascarúa

Abstract. For a space Z, we denote by $\mathcal{F}(Z)$, $\mathcal{K}(Z)$ and $\mathcal{F}_2(Z)$ the hyperspaces of non-empty closed, compact, and subsets of cardinality ≤ 2 of Z, respectively, with their Vietoris topology. For spaces X and E, $C_p(X,E)$ is the space of continuous functions from X to E with its pointwise convergence topology.

We analyze in this article when $\mathcal{F}(Z)$, $\mathcal{K}(Z)$ and $\mathcal{F}_2(Z)$ have continuous selections for a space Z of the form $C_p(X,E)$, where X is zero-dimensional and E is a strongly zero-dimensional metrizable space. We prove that $C_p(X,E)$ is weakly orderable if and only if X is separable. Moreover, we obtain that the separability of X, the existence of a continuous selection for $\mathcal{K}(C_p(X,E))$, the existence of a continuous selection for $\mathcal{F}_2(C_p(X,E))$ and the weak orderability of $C_p(X,E)$ are equivalent when X is $\mathbb{N}$-compact.

Also, we decide in which cases $C_p(X,2)$ and $\beta C_p(X,2)$ are linearly orderable, and when $\beta C_p(X,2)$ is a dyadic space.

Keywords: continuous selections, Vietoris topology, linearly orderable space, weakly orderable space, space of continuous functions, dyadic spaces

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0. Definitions and notations

In order to simplify our statements and proofs, *all spaces will be assumed Tychonoff and with more than one point, unless we explicitly mention the contrary. Moreover, each cardinal number κ, when considered as a space, will be the discrete space of cardinality κ; and, of course, it will be greater or equal to* 2.

For a topological space Z, we denote by $\mathcal{A}(Z)$ the collection of non-empty subsets of Z, and $\mathcal{F}(Z)$ is the collection of all non-empty closed subsets of Z. For a subset $\mathcal{G}$ of $\mathcal{A}(Z)$, we consider $\mathcal{G}$ with the Vietoris topology τ_V; that is, the topology generated by sets of the form

$$\langle V_1,\dots,V_n\rangle_{\mathcal{G}} = \{H \in \mathcal{G} : H \subseteq \bigcup_{i\in\{1,\dots,n\}} V_i \quad \text{and} \quad H\cap V_i \neq \emptyset \ \forall\, i\},$$

where $n \in \mathbb{N}$ and each V_i is an open subset of Z. A *continuous selection on* Z for $\mathcal{G}$ is a continuous function $\phi : (\mathcal{G},\tau_V) \to Z$ such that $\phi(F) \in F$ for each $F \in \mathcal{G}$. The symbol $\mathcal{F}_2(Z)$ will denote the collection of non-empty subsets of Z with $\leq$ two elements, and $\mathcal{K}(Z)$ is the family of non-empty compact subsets of Z. For $\mathcal{G} \subseteq \mathcal{A}(Z)$, we will denote by $\mathcal{S}el(\mathcal{G})$ the set of continuous selections defined for $\mathcal{G}$.

Recall that a space $(Z, \mathcal{T})$ is a *LOTS* or a *linearly orderable topological space* if there is a linear order $\leq$ in Z such that the topology generated by $\leq$, $\mathcal{T}_{\leq}$, coincides with $\mathcal{T}$. A topological space is a *generalized ordered space* (a *GO-space*) (also called *suborderable space*) if it can be embedded in a linearly orderable space as a subspace. Finally, a space $(Z, \mathcal{T})$ is *weakly orderable* if there is a linear order $\leq$ in Z such that $\mathcal{T}_{\leq} \subset \mathcal{T}$. Of course, every LOTS is a GO-space, and each GO-space is weakly orderable. (The Sorgenfrey line is an example of a GO-space which is not a LOTS, and the space $\{(0,0)\} \cup \{(x, \sin(1/x)) : x > 0\} \subseteq \mathbb{R}^2$ is a weakly orderable space which is not a GO-space.)

For two spaces X and E, $C_p(X,E)$ will denote the space of continuous functions defined on X and with values in E, equipped with the pointwise convergence topology which is the topology inherited from the Tychonoff product E^X. When E is the real line, we will write $C_p(X)$ instead of $C_p(X,E)$.

In this article *iff* means *if and only if* and *clopen* means *closed and open*. For a finite set $\{x_1, \dots, x_n\}$ in X and a finite set $\{A_1, \dots, A_n\}$ of open subsets of E, we will denote the canonical open set $\{f \in C_p(X,E) : f(x_i) \in A_i \ \forall\ i \in \{1, \dots, n\}\}$ of $C_p(X,E)$ as $[x_1, \dots, x_n; A_1, \dots, A_n]$ or $[x_1, \dots, x_n; A]$ if every A_i is equal to A.

The concepts, terminology and notations used and not defined in this article can be found in [E].

1. Introduction and basic facts

In this article we are going to study continuous selections defined in some classes of closed subsets of spaces of continuous functions with their topology of pointwise-convergence.

Engelking, Hewitt and Michael proved the following facts about continuous selections which are currently well known; the first of them was also proved, independently, by M. Čoban:

1.1 Theorem ([EHM], [Č]). *There is always a continuous selection for $\mathcal{F}(Z)$ if Z is a strongly zero-dimensional completely metrizable space.* (*In particular, there is a continuous selection for $\mathcal{F}(\mathbb{P})$.*)

1.2 Theorem ([EHM]). *The real line $\mathbb{R}$ does not have a continuous selection for $\mathcal{F}(\mathbb{R})$.*

1.3 Theorem ([EHM]). *The space of rational numbers $\mathbb{Q}$ does not have a continuous selection for $\mathcal{F}(\mathbb{Q})$.*

Afterwards, van Mill, Pelant and Pol [vMPP] completed the picture with the following theorem:

1.4 Theorem. *For a metrizable space M, if there is a continuous selection for $\mathcal{F}(M)$, then M has to be completely metrizable.*

So, if we consider spaces of continuous functions with their topology of pointwise convergence, we can deduce the following results:

1.5 Corollary. *For a countable space X and a metrizable space E, we have:*

(1) *if the space $C_p(X,E)$ has a continuous selection for $\mathcal{F}(C_p(X,E))$, then X is discrete and E is completely metrizable;*

(2) *if X is discrete and E is a strongly zero-dimensional completely metrizable space, then $C_p(X,E)$ has a continuous selection for $\mathcal{F}(C_p(X,E))$.*

Proof: 1. Since E has at least two distinct points, $C_p(X,2)$ is naturally embedded as a closed subset of $C_p(X,E)$; so, $\mathcal{S}el\mathcal{F}(C_p(X,2)) \neq \emptyset$ provided $\mathcal{S}el\mathcal{F}(C_p(X,E)) \neq \emptyset$. Thus, it suffices to show that X is discrete if $C_p(X,2)$ has a continuous selection for $\mathcal{F}(C_p(X,2))$. To this end, we will proceed by contradiction: assume that $\mathcal{S}el\mathcal{F}(C_p(X,2)) \neq \emptyset$ and that X is not discrete. Let x_0 be a non-isolated point of X. Define a map $g : X \to \{0,1\}$ by $g(x) = 0$ if $x = x_0$ and $g(x) = 1$ if $x \neq x_0$. Clearly, g is not continuous. Now, consider the translation $\Phi : 2^X \to 2^X$ defined by

$$\Phi(f) = g + f \pmod 2.$$

The map $\Phi : 2^X \to 2^X$ is a homeomorphism. Hence, $D = \Phi[C_p(X,2)]$ is a dense G_δ-subset of 2^X because so is $C_p(X,2)$ being completely metrizable by hypothesis (see Theorem 1.4). But $\Phi(f) \notin C_p(X,2)$ for every $f \in C_p(X,2)$ because g is not continuous. Thus, $D \cap C_p(X,2) = \emptyset$ which is impossible because both sets D and $C_p(X,2)$ are dense G_δ-sets in the Cantor set 2^X (X is countable). The contradiction so obtained implies that X must be discrete.

2. If X is discrete and E is a strongly zero-dimensional completely metrizable space, then $C_p(X,E) = E^\omega$ is also a strongly zero-dimensional completely metrizable space. So, we have only to apply Theorem 1.1. □

Nevertheless, if we consider the hyperspace of compact subsets of $C_p(X,\kappa)$ the situation changes. Observe that when X is a subset of a topological space Z, then, of course, $\mathcal{K}(X) \subseteq \mathcal{K}(Z)$ and $\mathcal{F}_2(X) \subseteq \mathcal{F}_2(Z)$, and if X is closed in Z, $\mathcal{F}(X) \subseteq \mathcal{F}(Z)$. Moreover, the Vietoris topology τ_V in $\mathcal{F}(X)$, $\mathcal{K}(X)$, $\mathcal{F}_2(X)$ coincides with its topology inherited from the Vietoris topology in $\mathcal{F}(Z)$, $\mathcal{K}(Z)$, $\mathcal{F}_2(Z)$, respectively.

1.6 Corollary. *For every countable space X and every strongly zero-dimensional completely metrizable space E, $C_p(X,E)$ has a continuous selection for $\mathcal{K}(C_p(X,E))$.*

Proof: The space E^X has a continuous selection ϕ for $(\mathcal{K}(E^X), \tau_V)$ (Theorem 1.1). So, ϕ restricted to $\mathcal{K}(C_p(X,E))$ is a continuous selection when we consider the topology in $\mathcal{K}(C_p(X,E))$ inherited by the Vietoris topology of $\mathcal{K}(E^X)$. But this is the Vietoris topology in $\mathcal{K}(C_p(X,E))$. □

Moreover, every space E is homeomorphic to a closed subset of $C_p(X,E)$. So, if E does not have any continuous selection for $\mathcal{F}(E)$, then $C_p(X,E)$ does not have any continuous selection for $\mathcal{F}(C_p(X,E))$. This is the case, for example, when $E = \mathbb{R}$ (or $E = \mathbb{Q}$). Next, we put this comment in a more explicit form.

1.7 Proposition. *For every space X, $\mathcal{S}el(\mathcal{F}(C_p(X))) = \emptyset$.*

Some other well known facts about continuous selections are the following results. Observe that the existence of continuous selections is related to the existence of some kind of linear order relations.

1.8 Theorem ([M]). *In a connected space $(Z, \mathcal{T})$, there is a continuous selection for $\mathcal{F}(Z)$ if and only if there is a lineal order $\leq$ for Z such that the topology generated by $\leq$, $\mathcal{T}_{\leq}$, is contained in $\mathcal{T}$ and every $\mathcal{T}$-closed subset of Z has a $\leq$-first element.*

1.9 Theorem ([vMW]). *A compact space Z possesses a continuous selection for $\mathcal{F}(Z)$ if and only if there exists a continuous selection for $\mathcal{F}_2(Z)$, if and only if Z is linearly orderable, if and only if Z is weakly orderable.*

1.10 Theorem ([GS]). *A pseudocompact space Z possesses a continuous selection for $\mathcal{F}_2(Z)$ if and only if Z is weakly orderable, if and only if βZ is linearly orderable.*

1.11 Remarks. (1) If a space Z is weakly orderable, then there is a continuous selection for $\mathcal{F}_2(Z)$. In fact, let $\leq$ be a linear order relation on Z which testifies the weak orderability of Z. The function $\phi : \mathcal{F}_2(Z) \to Z$ defined as $\phi(\{a, b\}) = \min\{a, b\}$ is continuous, where the minimum is taken with respect to $\leq$.

(2) For a space $(Z, \mathcal{T})$, if we have a continuous selection $\phi : \mathcal{F}_2(Z) \to Z$, we obtain a relation $<_\phi$ in Z defined by $a <_\phi b$ iff $\phi(\{a, b\}) = a$. This relation is not necessarily transitive. Nevertheless, for each $a \in Z$ the sets $(a, \rightarrow) = \{x \in Z : a <_\phi x\}$ and $(\leftarrow, a) = \{x \in Z : x <_\phi a\}$ are open (see [M, Lemma 7.2.1]).

(3) If $\kappa > \aleph_0$ and $\tau \geq 2$, then there is no continuous selection in τ^κ for $\mathcal{F}_2(\tau^\kappa)$. Indeed, every linearly ordered space is hereditarily normal and 2^{ω_1} contains a subspace homeomorphic to $\mathbb{N}^{\omega_1}$. But this space is not normal; so, 2^κ does not have a selection for $\mathcal{F}_2(2^\kappa)$, because if it had, since it is compact, it would be linearly orderable.

All these results and remarks presented up to here lead us to ask about the existence of continuous selections for $\mathcal{G} \subset \mathcal{F}(Z)$ where Z is a space of the form $C_p(X, E)$. So, this article is devoted indeed to analyzing continuous selections for $\mathcal{G} \subset \mathcal{F}(Z)$ and some other related linearly order type properties in the class of spaces of continuous functions $C_p(X, E)$. In Section 2 we study the case when E is either $\mathbb{R}$ or the unit interval $[0, 1]$. In Section 3 we prove that *continuous selections for the hyperspace of closed subsets of $C_p(X, E)$ exist if and only if X is countable and discrete, when X is zero-dimensional and E is strongly zero-dimensional and completely metrizable.* In Section 4 we obtain our main results (Theorems 4.5 and 4.10): (1) *For a zero-dimensional space X and a strongly zero-dimensional and metrizable space E, $C_p(X, E)$ is weakly orderable iff X is separable*; and (2) *if, in addition, X is $\mathbb{N}$-compact, X is separable iff there is a continuous selection for $\mathcal{K}(C_p(X, E))$, iff there is a continuous selection for*

$\mathcal{F}_2(C_p(X,E))$, *iff* $C_p(X,E)$ *is weakly orderable*. In Section 5 we obtain a result which is similar to the proposition in (2) whenever E is compact, metrizable and zero-dimensional, by deciding in what situations $\beta C_p(X,E)$ coincides with E^X which will lead us to determine when $\beta C_p(X,E)$ is a dyadic space. The last section is devoted to prove some results about linear order type properties on $C_p(X,E)$.

2. Selections for $\mathcal{F}_2(C_p(X))$ and $\mathcal{F}_2(C_p(X,I))$

We have already mentioned in Section 1 that $\mathcal{S}el(\mathcal{F}(C_p(X))) = \emptyset$ for all X because $\mathcal{S}el(\mathbb{R}) = \emptyset$ and $\mathbb{R}$ is contained as a closed subset of $C_p(X)$. Now we are going to decide when $C_p(X)$ has a continuous selection for $\mathcal{K}(C_p(X))$ and for $\mathcal{F}_2(C_p(X))$.

Assume that x_0, x_1 are two different elements in X. Let Y be the subspace $\{f \in C_p(X) : f(x_0) = 0\}$ of $C_p(X)$. The map $\psi : C_p(X) \to Y \times \mathbb{R}$ given by: $\psi(g) = (g - g(x_0), g(x_0))$ is a homeomorphism. Moreover, if $h \in Y$ is such that $h(x_1) \neq 0$, then $\phi : \mathbb{R} \to Y$ defined by $\phi(t) = t \cdot h$ is an embedding. So, for a space X with more than one point, $C_p(X)$ contains a copy of $\mathbb{R}^2$ which does not have a continuous selection for its $\mathcal{F}_2(\mathbb{R}^2)$. So, we obtain:

2.1 Proposition. *Let $\mathcal{G}$ be a subcollection of $\mathcal{K}(C_p(X))$ containing $\mathcal{F}_2(C_p(X))$. The following statements are equivalent:*

(1) *the space $C_p(X)$ has a continuous selection for $\mathcal{G}$;*
(2) *the space $C_p(X)$ is weakly orderable;*
(3) *the space $C_p(X)$ is a GO-space;*
(4) *the space $C_p(X)$ is a LOTS;*
(5) *X contains only one point.*

Since $C_p(X,I)$ is a subset of $C_p(X)$, and $C_p(X,(0,1)) \cong C_p(X)$ is a subspace of $C_p(X,I)$, we have a similar result of Proposition 2.1 for $C_p(X,I)$.

Another concept applicable to topological spaces and related with the concept of orderability is that of butterflying local base (see for example [W]):

A point x in a space X has a *butterflying local base* if there are two collections $\mathcal{U}_1$ and $\mathcal{U}_2$, of open sets, subject to:

(1) $\mathcal{U}(x) = \{U_0 \cup U_1 \cup \{x\} : U_i \in \mathcal{U}_i,\ i \in \{1,2\}\}$ is a local base at x,
(2) $(\mathcal{U}_i, \supset)$ is linearly ordered for each $i \in \{1,2\}$, and
(3) for each pair $(U_0, U_1) \in \mathcal{U}_1 \times \mathcal{U}_2$, $U_0 \cap U_1 = \emptyset$.

A space X is a *butterfly* space when each of its points has a butterflying local base. Obviously every GO-space is a butterfly space. In [W], proofs of the following assertions can be found: every GO-space contains a dense orderable subspace, and for every topological group G, G has a dense orderable subspace if and only if it contains a dense butterfly subspace.

Some other definitions: For a space X, the collection $\mathcal{R}(X)$ of all regular-open subsets of X is a complete Boolean algebra, and thus its Stone space $\mathcal{S}(\mathcal{R}(X))$ is a compact extremely disconnected space. The subspace of $\mathcal{S}(\mathcal{R}(X))$ consisting of ultrafilters in $\mathcal{R}(X)$ converging in X is denoted by $\mathcal{E}(X)$ and is called the *absolute* of X. We say that a space X is *co-absolute* with a space Y when $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are homeomorphic. A space X is *non-Archimedean* if it has a base in which every two elements are either disjoint or related by inclusion. A space X is *proto-metrizable* if it is paracompact and has an ortho-base.

2.2 Theorem ([W, Theorem 4.2]). *Suppose $X = \prod_{\alpha\in\kappa} X_\alpha$ is an infinite product of infinite spaces. If X has a dense orderable subspace, then $|\kappa| = \omega$ and X has a dense metrizable subspace.*

By applying this and other results contained in [W] to our continuous function spaces, we obtain:

2.3 Theorem. *For a topological space X, the following statements are equivalent:*

(1) *$C_p(X)$ has a dense orderable subspace;*
(2) *$C_p(X)$ has a dense butterfly subspace;*
(3) *$C_p(X)$ has a dense butterfly subspace and $\beta C_p(X)$ is co-absolute with a LOTS;*
(4) *$C_p(X)$ has countable character and $\beta C_p(X)$ is co-absolute with a LOTS;*
(5) *$C_p(X)$ has a dense non-Archimedean orderable subspace;*
(6) *$C_p(X)$ contains a dense proto-metrizable subspace;*
(7) *$C_p(X)$ has a dense metrizable subspace;*
(8) *$C_p(X)$ is metrizable;*
(9) *the space X is countable.*

Proof: A topological group has a dense orderable subspace iff it has a dense butterfly subspace ([W, Theorem 2.3]); thus, (1) $\Leftrightarrow$ (2) holds. The equality (1) $\Leftrightarrow$ (3) is Proposition 1.3 in [W]. (4) $\Rightarrow$ (5) is Proposition 2.2 in [W]. If $C_p(X)$ is first countable, then X must be countable; so, (4) $\Rightarrow$ (9) holds. (9) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) and (5) $\Rightarrow$ (1) are trivial (observe that (7) $\Leftrightarrow$ (8) is true even for topological groups). (5) $\Leftrightarrow$ (6) is Theorem 2.1 in [W]. And the implication (1) $\Rightarrow$ (9) is a consequence of Theorem 2.2. Finally, (9) plus (3) implies (4). We have finished the proof. $\square$

In a similar form, we can obtain an analogous result to Theorem 2.3 for $C_p(X, [0,1])$, excluding the statement in (2).

3. Continuous selections for $\mathcal{F}(C_p(X,E))$ when X is zero-dimensional

For a topological space X, $\mathcal{G} \subseteq \mathcal{P}(X)$ is a *cellular collection* of subsets of X if every $G \in \mathcal{G}$ is not empty and every two different elements in $\mathcal{G}$ have an empty intersection. The number $c(X)$ is the supremum of the cardinalities of every cellular family of open subsets of X. The one-point compactification of a discrete space of cardinality τ will be denoted by A_τ.

3.1 Proposition. *Let E be a topological space. If a zero-dimensional space X has a cellular family of open subsets with cardinality $\geq \tau \geq \aleph_0$, then $C_p(X,E)$ has a subspace homeomorphic to A_τ.*

Proof: Let a, b be two different elements in E and take two disjoint open subsets A, B of E such that $a \in A$ and $b \in B$. Let $\mathcal{G} = \{U_\lambda : \lambda < \tau\}$ be a cellular family of open subsets of X. Since X is zero-dimensional and each $U \in \mathcal{G}$ is not empty, then we can assume, without loss of generality, that each U_λ is clopen. For each $\lambda < \tau$, let f_λ be the function defined by

$$f_\lambda(x) = \begin{cases} a & \text{if } x \notin U_\lambda \\ b & \text{if } x \in U_\lambda. \end{cases}$$

Since U_λ is clopen, f_λ is continuous. Now, observe that the set $D = \{f_\lambda : \lambda < \tau\}$ is relatively discrete. In fact, if $x_\lambda \in U_\lambda$, then $[x_\lambda; B] \cap D = \{f_\lambda\}$. Now, take the constant function equal to a which we will denote by c_a. Consider an arbitrary open set $W = [x_1, \ldots, x_k; V]$ containing c_a. It happens that W contains all but a finite collection of elements f_λ. Therefore, $D \cup \{c_a\}$ is homeomorphic to A_τ. □

3.2 Remark. Observe that, for every $\tau > \omega$, A_τ is not a GO-space because the tightness of A_τ, $t(A_\tau)$, is equal to $\aleph_0$, its character $\chi(A_\tau)$ is equal to τ, and these two cardinal functions coincide in GO-spaces.

3.3 Corollary. *If X is a zero-dimensional space with $c(X) \geq \omega_1$ and E is a topological space, then the set $\mathcal{S}el(\mathcal{F}_2(C_p(X,E)))$ is empty.*

Proof: Because of Proposition 3.1, $C_p(X,E)$ contains a subspace Y homeomorphic to A_{ω_1}. If $\mathcal{S}el(\mathcal{F}_2(C_p(X,E))) \neq \emptyset$, then $\mathcal{S}el(\mathcal{F}_2(Y)) \neq \emptyset$. But, since Y is compact, Y is orderable (Theorem 1.9), which is not possible (Remark 3.2). □

We have already assigned the symbol A_τ to the one-point compactification of the discrete space of cardinality τ. We write L_τ in order to designate the one-point Lindelöfication of the discrete space of cardinality τ. The Σ-product $\Sigma_0 2^\tau = \{f \in 2^\tau : |\{\lambda < \tau : f(\lambda) = 1\}| \leq \aleph_0\}$ is homeomorphic to $C_p(L_\tau, 2)$ and the σ-product $\sigma_0 2^\tau = \{f \in 2^\tau : |\{\lambda < \tau : f(\lambda) = 1\}| < \aleph_0\}$ is homeomorphic to $C_p(A_\tau, 2)$. So, since $c(A_\tau)$ and $c(L_\tau)$ are equal to τ, the sets $\mathcal{S}el(\mathcal{F}_2(\Sigma_0 2^\tau))$ and $\mathcal{S}el(\mathcal{F}_2(\sigma_0 2^\tau))$ are empty if (and only if) τ is uncountable (Corollary 3.3).

Recall that a topological space X is a *P-space* if every G_δ-set in X is open.

3.4 Lemma. *A zero-dimensional space X contains a cellular family $\{B_n : n < \omega\}$ of clopen sets such that $\bigcup_{n<\omega} B_n$ is not closed if and only if X is not a P-space.*

Proof: Assume that X contains a cellular family $\{B_n : n < \omega\}$ of clopen sets such that $\bigcup_{n<\omega} B_n$ is not closed. Take $A_n = X \setminus B_n$. We have that each A_n is clopen and $\bigcap_{n<\omega} A_n = X \setminus \bigcup_{n<\omega} B_n$ is not open because $\bigcup_{n<\omega} B_n$ is not closed.

Now, suppose that X is not a P-space. So, there is a sequence $\{A_n : n < \omega\}$ of clopen sets such that $\bigcap_{n<\omega} A_n$ is not open. Without loss of generality, we can assume that $A_0 = X$ and $A_{n+1} \subseteq A_n$ with $A_{n+1} \neq A_n$ for each n. Let $B_n = A_n \setminus A_{n+1}$ for each $n < \omega$. Thus, $\{B_n : n < \omega\}$ is a cellular family of clopen sets and $\bigcup_{n<\omega} B_n = X \setminus \bigcap_{n<\omega} A_n$ is not closed. □

3.5 Lemma. *For a zero-dimensional space X, there exists a countable non discrete space Z and a (surjective) quotient $q : X \to Z$ if and only if X is not a P-space.*

Proof: If X is not a P-space, there exists a cellular family $\{B_n : n < \omega\}$ of clopen sets such that $B = \bigcup_{n<\omega} B_n$ is not closed (Lemma 3.4). Let Z be the quotient of X obtained from the partition $\{X \setminus B\} \cup \{B_n : n < \omega\}$. Let p represent $X \setminus B$ and x_n represent B_n in Z. So p is the only point in Z which is not isolated. Then, Z is countable and not discrete.

Now, assume that there exists a countable non discrete space Z and a surjective quotient $q : X \to Z$. Since Z is not discrete, there is a point $p \in Z$ which is not isolated. Since Z is countable and zero-dimensional, there is a sequence $\{B_n : n < \omega\}$ of clopen sets in Z such that $\{p\} = \bigcap_{n<\omega} B_n$. We can take $(B_n)_{n<\omega}$ in such a way that $B_0 = Z$ and $B_{n+1} \subseteq B_n$ with $B_{n+1} \neq B_n$ for all n. Let us denote by A_n the set $B_n \setminus B_{n+1}$ and by C_n the set $q^{-1}[A_n]$. Since q is a continuous function, each C_n is clopen. Moreover, $C_n \cap C_m = \emptyset$ if $n \neq m$. Furthermore, $\bigcup_{n<\omega} C_n$ is not closed. Indeed, $q^{-1}(p)$ is not open because p is not isolated, and $X = q^{-1}(p) \cup \bigcup_{n<\omega} C_n$. Now, we only have to apply Lemma 3.4. □

3.6 Theorem. *Let X be a zero-dimensional space which is not a P-space, and let E be a completely metrizable space. Then $C_p(X, E)$ does not have a continuous selection for $\mathcal{F}(C_p(X, E))$.*

Proof: Because of Lemma 3.5, there is a countable non discrete space Z and a surjective quotient $q : X \to Z$. It is possible to prove that the function $q^{\#} : C_p(Z, E) \to C_p(X, E)$ defined by $q^{\#}(f) = f \circ q$ is an embedding onto a closed subset of $C_p(X, E)$. Therefore, $C_p(X, E)$ does not have a continuous selection for $\mathcal{F}(C_p(X, E))$ because $C_p(Z, E)$ does not have a continuous selection for $\mathcal{F}(C_p(Z, E))$ (see Corollary 1.5). □

It is proved in [Arh2, Corollary I.3.3] that $C_p(X)$ is Čech-complete iff X is countable and discrete. The proof uses the facts: (1) every Čech-complete space is a space of point countable type, (2) $\mathbb{R}$ is a topological group, and (3) if a dense subset of a Tychonoff product Z^τ has a proper non-empty compact subspace K

with $\chi(K,Y) \leq \aleph_0$, then $\tau \leq \aleph_0$. So, in a similar way, the following can be proved:

3.7 Lemma. *For a zero-dimensional space X, the space $C_p(X,2)$ is Čech complete if and only if X is discrete and countable.*

3.8 Theorem. *Let X be a zero-dimensional space and $\kappa \geq 2$. Then, the following assertions are equivalent:*

(1) *$C_p(X,\kappa)$ has a continuous selection for $\mathcal{F}(C_p(X,\kappa))$;*
(2) *$C_p(X,\kappa)$ is completely metrizable;*
(3) *X is countable and discrete.*

Proof: Of course (3) $\Rightarrow$ (2) holds, and (2) $\Rightarrow$ (3) is a consequence of Lemma 3.7 because $C_p(X,2)$ is a closed subset of $C_p(X,\kappa)$. Moreover, Theorem 1.1 guarantees that implication (3) $\Rightarrow$ (1) is also true.

(1) $\Rightarrow$ (3): If $C_p(X,\kappa)$ has a continuous selection for $\mathcal{F}(C_p(X,\kappa))$, then X must be a P-space and $c(X) \leq \omega$ (Theorem 3.6 and Corollary 3.3). This means, because X is zero-dimensional, that X has countable character. But X is a P-space, so it is discrete. Moreover, κ^{ω_1} does not have a continuous selection (see Remarks 1.11). Thus, $|X| \leq \aleph_0$. □

3.9 Corollary. *Let X be a zero-dimensional space and let E be a strongly zero-dimensional metrizable space. Then, the following assertions are equivalent:*

(1) *$C_p(X,E)$ has a continuous selection for $\mathcal{F}(C_p(X,E))$;*
(2) *$C_p(X,E)$ is completely metrizable;*
(3) *X is countable and discrete, and E is completely metrizable.*

Proof: Using similar arguments to those given in the proof of Theorem 3.8, we have that implications (2) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (1) hold.

Now, if $C_p(X,E)$ has a continuous selection for $\mathcal{F}(C_p(X,E))$, then $C_p(X,2)$ must admit a continuous selection for $\mathcal{F}(C_p(X,2))$. But this means that X is countable and discrete (Theorem 3.8). Hence, the metric space E^ω has a continuous selection for $\mathcal{F}(E^\omega)$. By Theorem 1.4, E^ω is completely metrizable; so is E. □

3.10 Remark. Observe that the equivalence (2) $\Leftrightarrow$ (3) in Corollary 3.9 holds for an arbitrary metric space E without requiring any dimensional hypothesis referring to E. It can be proved, using Lemma 3.7, that *for a zero-dimensional space X and a Tychonoff space E, $C_p(X,E)$ is Čech-complete iff X is countable and discrete and E is Čech-complete*.

4. Continuous selections on $\mathcal{K}(C_p(X,\kappa))$

4.1 Theorem. *Let X be a separable space and κ a cardinal number. Then, $C_p(X,\kappa)$ is weakly orderable.*

Proof: Let D be a countable dense subset of X. The function $\pi : C_p(X,\kappa) \to \kappa^D$ defined by $f \mapsto f|_D$ is continuous, while π is injective because $D \subseteq X$ is dense. The Baire space $B(\kappa) = \kappa^D$ is orderable (see, for instance, [H]), say by the order relation $\leq$. Hence, the topology $\mathcal{T}_{\preceq}$ in $C(X,\kappa)$ defined by the order relation $\preceq$ determined by $\leq$ and π ($f \prec g$ iff $\pi(f) < \pi(g)$), is contained in the pointwise convergence topology in $C(X,\kappa)$. Therefore, $C_p(X,\kappa)$ is weakly orderable. $\square$

4.2 Corollary. *If X is separable and E is a strongly zero-dimensional metrizable space, then $C_p(X,E)$ is weakly orderable.*

Proof: The space E is a subset of the Baire space $B(\kappa) = \kappa^\omega$ where κ is equal to the weight of E. So, $C_p(X,E)$ is a subspace of $C_p(X,\kappa^\omega)$. Since weak orderability is a hereditary property, $C_p(X,E)$ is weakly orderable if $C_p(X,\kappa^\omega)$ is weakly orderable. But $C_p(X,\kappa^\omega)$ is homeomorphic to $C_p(\bigoplus_{n\in\mathbb{N}} X_n, \kappa)$ where $\bigoplus_{n\in\mathbb{N}} X_n$ is the free topological sum of spaces X_n and each X_n is homeomorphic to X for every $n \in \mathbb{N}$. Since X is separable, so is $\bigoplus_{n\in\mathbb{N}} X_n$. Therefore, Theorem 4.1 implies that $C_p(\bigoplus_{n\in\mathbb{N}} X_n, \kappa)$ is weakly orderable, and the proof is finished. $\square$

Now, we are going to prove, in Theorem 4.5 below, that the converse of Corollary 4.2 is also true when X is zero-dimensional. It is well known that for every space X, $d(X) = iw(C_p(X)) = \psi(C_p(X))$ and $d(C_p(X)) = iw(X)$ (see [Arh2]). (Recall that the i-weight of a space Z, $iw(Z)$, is the minimum of the cardinal numbers κ such that there is a bijective and continuous function from Z onto a space of weight κ). In [C, Propositions 4.12–4.20], something equivalent for zero-dimensional spaces was proved ($iw_2(Z)$ is the minimum of the cardinal numbers κ such that there is a bijective and continuous function from Z onto a zero-dimensional space of weight κ):

4.3 Lemma. *If X and E are zero-dimensional and E is second countable, then*

$$d(X) = iw_2(C_p(X,E)) = \psi(C_p(X,E)).$$

We are going to use these equalities in that which follows.

4.4 Lemma. *Let X be a zero-dimensional space and let E be a zero-dimensional second countable space. Then, the following assertions are equivalent:*

(1) *X is separable;*
(2) *the space $C_p(X,E)$ is weakly orderable;*
(3) *the topology of $C_p(X,E)$ contains a topology $\mathcal{T}$ for $C(X,E)$ which is T_2, second countable and zero-dimensional.*

Proof: We have already proved in Corollary 4.2 that if X has a countable dense subset, then $C_p(X,E)$ is weakly orderable.

Now, assume that $C_p(X,E)$ is weakly orderable. So, there is a linear order $\leq$ in $C(X,E)$ such that the topology generated by $\leq$, $\mathcal{T}_{\leq}$, is contained in the pointwise convergence topology $\mathcal{T}_p$. Since the cellularity of $C_p(X,E)$ is equal to $\aleph_0$,

$c(C_{\leq}(X,E)) \leq \aleph_0$. Since $C_{\leq}(X,E)$ is a LOTS, $\psi(C_{\leq}(X,E)) \leq c(C_{\leq}(X,E)) \leq \aleph_0$. But $\mathcal{T}_{\leq} \subset \mathcal{T}_p$, so $\psi(C_p(X,E)) \leq \aleph_0$. By Lemma 4.3, X is separable.

Finally, (1) $\Leftrightarrow$ (3) is a consequence of Lemma 4.3. $\square$

4.5 Theorem. *Let X be a zero-dimensional space and let E be a strongly zero-dimensional metrizable space. Then, X is separable if and only if $C_p(X,E)$ is weakly orderable.*

PROOF: By Corollary 4.2, if X is separable then $C_p(X,E)$ is weakly orderable. On the other hand, we are assuming that $|E| \geq 2$, so $C_p(X,2)$ can be considered a subspace of $C_p(X,E)$. If this last space is weakly orderable, so is $C_p(X,2)$. By Lemma 4.4, X is separable. $\square$

So, it happens that spaces as $C_p(\Psi(\mathcal{A}), 2^\omega)$, $C_p(\beta\omega, \mathbb{Q})$ and $C_p(2^{\mathfrak{c}}, 2)$ are weakly orderable.

Lemma 7.5.1 in [M] and Corollary 4.2 establish that *if X is separable and E is a strongly zero-dimensional metrizable space, then $\mathcal{S}el(\mathcal{K}(C_p(X,E))) \neq \emptyset$.* If X is a Corson compact space (that is, a compact subspace of a Σ-product of real lines), then $d(X) = c(X)$. Therefore, if X is a zero-dimensional Corson compact space and E is a strongly zero-dimensional metrizable space, then $\mathcal{S}el(\mathcal{K}(C_p(X,E))) \neq \emptyset$ *iff $\mathcal{S}el(\mathcal{F}_2(C_p(X,E))) \neq \emptyset$, iff X is separable* (Corollary 3.3 and Theorem 4.1). In general, for compact zero-dimensional spaces X, this last result is also true, as we are going to see next:

4.6 Definitions.

(1) A space X is an $\mathbb{N}$-*compact space* if it is homeomorphic to a closed subset of a product of copies of $\mathbb{N}$.

(2) For a zero-dimensional space X and a cardinal number τ, a function $f : X \to 2$ is a *strictly τ-continuous function* if for every subset F of X of cardinality $\leq \tau$, there is a $g \in C(X,2)$ such that $g \restriction F = f \restriction F$.

(3) For a zero-dimensional space X, the number $t_2(X)$ will be the minimum cardinal τ such that every strictly τ-continuous function $f : X \to 2$ is continuous.

A. Contreras proved in [C] the following result.

4.7 Lemma. *For an $\mathbb{N}$-compact space X, $t_2(C_p(X,2)) \leq \aleph_0$.*

4.8 Lemma. *Let Z be a zero-dimensional space. If $\mathcal{S}el(\mathcal{F}_2(Z)) \neq \emptyset$, then $\psi(Z) \leq t_2(Z)$.*

PROOF: Let $\phi : \mathcal{F}_2(Z) \to Z$ be a continuous selection, and let $<$ be the relation in Z defined by $a < b$ iff $\phi(\{a,b\}) = a$, for two different elements a, b. As we have said in Remark 1.11, for each $a \in Z$ the sets $(a, \rightarrow) = \{x \in Z : a < x\}$ and $(\leftarrow, a) = \{x \in Z : x < a\}$ are open. Furthermore, $Z = (\leftarrow, a) \cup \{a\} \cup (a, \rightarrow)$,

$(\leftarrow, a) \cap (a, \rightarrow) = \emptyset$ and a is the only possible point belonging to the boundary of each set $(a, \rightarrow)$ and $(\leftarrow, a)$.

Denote by τ the cardinal number $t_2(Z)$. If $a \in \mathrm{cl}(a, \rightarrow)$, then there is $F \subset (a, \rightarrow)$ of cardinality $\leq \tau$ such that $a \in \mathrm{cl}\, F$. Indeed, assume that this is not true. Define $f : Z \to 2$ as follows: $f(x) = 0$ if $x \in (\leftarrow, a) \cup \{a\}$ and $f(x) = 1$ if $x \in (a, \rightarrow)$. Let H be a subset of Z of cardinality $\leq \tau$. Since $a \notin \mathrm{cl}(H \cap (a, \rightarrow))$, there is a clopen subset V with $a \in V$ and $V \cap (H \cap (a, \rightarrow)) = \emptyset$. The function $g : Z \to 2$ defined by $g(x) = 0$ if $x \in (\leftarrow, a) \cup V$ and $g(x) = 1$ otherwise, is a continuous function and $g \restriction H = f \restriction H$. But f is not continuous, contrary to our hypothesis $t_2(Z) = \tau$.

Similarly, if $a \in \mathrm{cl}(\leftarrow, a)$, there is $G \subset (\leftarrow, a)$ of cardinality $\leq \tau$ such that $a \in \mathrm{cl}\, G$.

Observe that if $a \notin \mathrm{cl}(a, \rightarrow)$, then $(\leftarrow, a) \cup \{a\}$ is open, and if $a \notin \mathrm{cl}(\leftarrow, a)$, then the set $\{a\} \cup (a, \rightarrow)$ is open.

Therefore, if $a \in \mathrm{cl}(\leftarrow, a) \cap \mathrm{cl}(a, \rightarrow)$, $\{a\} = \bigcap_{x \in F} (\leftarrow, x) \cap \bigcap_{x \in G} (x, \rightarrow)$. If $a \in \mathrm{cl}(\leftarrow, a)$ and $a \notin \mathrm{cl}(a, \rightarrow)$, then $\{a\} = \bigcap_{x \in G} (x, \rightarrow) \cap [(\leftarrow, a) \cup \{a\}]$. We have a similar situation if $a \notin \mathrm{cl}(\leftarrow, a)$ and $a \in \mathrm{cl}(a, \rightarrow)$. That is, $\{a\}$ is a G_τ-set in any case including, of course, when a is an isolated point. □

4.9 Lemma. *For an $\mathbb{N}$-compact space X, X is separable if there is a continuous selection $\phi : \mathcal{F}_2(C_p(X, 2)) \to C_p(X, 2)$.*

Proof: Since X is an $\mathbb{N}$-compact space, $t_2(C_p(X, 2)) \leq \aleph_0$ (Lemma 4.7). Now we have to apply Lemma 4.8 and obtain that $\psi(C_p(X, 2)) \leq \aleph_0$. But this implies that X is separable (Lemma 4.3). □

4.10 Theorem. *For an $\mathbb{N}$-compact space X and a strongly zero-dimensional metrizable space E, the following assertions are equivalent:*

(1) *X is separable;*
(2) *the space $C_p(X, E)$ is weakly orderable;*
(3) *there is a continuous selection $\phi : \mathcal{K}(C_p(X, E)) \to C_p(X, E)$;*
(4) *there is a continuous selection $\phi : \mathcal{F}_2(C_p(X, E)) \to C_p(X, E)$.*

Proof: The implication (1) ⇒ (2) is consequence of Corollary 4.2. Moreover, Lemma 7.5.1 in [M] guarantees that (2) ⇒ (3) holds. The implication (3) ⇒ (4) is trivial, and if there is a continuous selection $\phi : \mathcal{F}_2(C_p(X, E)) \to C_p(X, E)$, the restriction $\phi \restriction \mathcal{F}_2(C_p(X, 2)) : \mathcal{F}_2(C_p(X, 2)) \to C_p(X, 2)$ is continuous too. By Lemma 4.9, X must be separable. □

Lemma 4.9 produces the following questions:

4.11 Problem.

(1) *For a zero-dimensional realcompact space X, does $\mathcal{S}el(\mathcal{F}_2(C_p(X, 2))) \neq \emptyset$ imply the separability of X?*
(2) *For a zero-dimensional space X, is X separable if $\mathcal{S}el(\mathcal{F}_2(C_p(X, 2))) \neq \emptyset$?*

If Problem 4.11.(2) had a positive answer, then we would have a good class of spaces (the spaces $C_p(X,E)$ with X separable zero-dimensional and E strongly zero-dimensional and metrizable) where the *Question* in [vMW] can be answered in the affirmative. The question posed was: *Let X be a space; is X a weakly orderable space if and only if $\mathcal{S}el(\mathcal{F}_2(X)) \neq \emptyset$?* So, a positive answer for 4.11.(2) would imply: *For a zero-dimensional space X, $\mathcal{S}el(\mathcal{K}(C_p(X,\kappa))) \neq \emptyset \Leftrightarrow \mathcal{S}el(\mathcal{F}_2(C_p(X,\kappa))) \neq \emptyset \Leftrightarrow C_p(X,\kappa)$ is weakly orderable $\Leftrightarrow X$ is separable.*

In relation to this problem, we have that the space $X = C_p(A_{\omega_1}, 2)$ is a space with density equal to ω_1. Furthermore, X has countable cellularity. But $\mathcal{S}el(\mathcal{F}_2(C_p(X,2))) = \emptyset$ because $C_p(C_p(A_{\omega_1},2),2)$ contains a copy of A_{ω_1}. This means that, as we have already seen, countable cellularity in X is a necessary condition to have continuous selections for $\mathcal{F}_2(C_p(X,2))$, but it is not a sufficient condition.

5. $\beta C_p(X,K)$ vs K^X

Because of Problem 4.11 and Theorem 1.10, we want to know when a space $C_p(X,E)$ is pseudocompact and when $\beta C_p(X,E)$ is linearly orderable. We determine this in Corollary 5.5 and in Proposition 6.4 below. So, in this section we are going to deal with pseudocompactness of spaces $C_p(X,E)$, and decide when $\beta C_p(X,E)$ is a dyadic space, when E is compact. All this will allow us to obtain an affirmative answer to Problem 4.11 for some kind of spaces X which are not necessarily $\mathbb{N}$-compact, in terms of topological properties for X.

For a subset D of a topological space X and a cardinal number γ, we will say, as usual, that D is a *G_γ-set* in X if it is the intersection of a family of $\leq \gamma$ open subsets of X. The *γ-closure* of D is the set of all points in X such that each G_γ-set containing one of these points has a non empty intersection with D. We denote this γ-closure of D by the symbol D_γ. D is *G_γ-dense* in X if each G_γ-set in X has a non-empty intersection with D. As usual, if X is the Tychonoff product of a family $\{X_s : s \in S\}$ and $N \subset S$, the function $\pi_N : X \to \prod_{s\in N} X_s$ is the canonical projection.

5.1 Proposition. *Let $\{X_s : s \in S\}$ be a collection of spaces of pseudocharacter $\leq \gamma$. Let $X = \prod_{s\in S} X_s$ be the Tychonoff product, and let D be a subset of X. Then D_γ is the greatest subset of X which contains D and has the following property: for each $N \subset S$ of cardinality $\leq \gamma$, the relation $\pi_N(D) = \pi_N(D_\gamma)$ holds.*

PROOF: Let N be a subset of S of cardinality $\leq \gamma$, and assume that $\pi_N(D)$ is a proper subset of $\pi_N(D_\gamma)$. Take $(y_s)_{s\in S} = y \in D_\gamma$ such that $\pi_N(y) \notin \pi_N(D)$. For each $s \in N$, let $\mathcal{N}(y_s)$ be a pseudobase of neighborhoods of y_s in X_s of cardinality $\leq \gamma$. The collection $\mathcal{V} = \{[s;V] : s \in N, V \in \mathcal{N}(y_s)\}$ also has cardinality $\leq \gamma$ and $y \in \bigcap \mathcal{V} = G$. Thus, G is a G_γ-subset of X, $y \in G$ and $G \cap D = \emptyset$, which contradict the definition of D_γ. Therefore, $\pi_N(D) = \pi_N(D_\gamma)$.

Now, let y be an element of $X \setminus D_\gamma$. Then, there exists a family $\{V_\lambda : \lambda < \gamma\}$ of canonical open subsets of X such that $y \in \bigcap_{\lambda<\gamma} V_\lambda = G$ and $G \cap D = \emptyset$. Each V_λ can be written as $[s_1^\lambda, \dots, s_{k_\lambda}^\lambda; A_1^\lambda, \dots, A_{k_\lambda}^\lambda]$. Consider the set $N = \{s_i^\lambda : \lambda < \gamma, 1 \le i \le k_\lambda\}$. It happens that for each $x \in D$, $\pi_N(x) \ne \pi_N(y)$. $\square$

5.2 Proposition. *Let $\{X_s : s \in S\}$ be a collection of second countable spaces. Let X be the product $\prod_{s\in S} X_s$, and let D be a subset of X. Then D_ω is a realcompact space.*

Proof: The space X is realcompact because it is the product of realcompact spaces. For each countable $N \subset S$, $\pi_N(D)$ is realcompact (it is even Lindelöf). So, $\pi_N^{-1}\pi_N(D)$ is a realcompact subspace of X for each countable $N \subset S$ [E, Corollary 3.11.8, p. 215]. This implies that $R = \bigcap\{\pi_N^{-1}\pi_N(D) : N \subset S, |N| \le \aleph_0\}$ is a realcompact space [E, Corollary 3.11.7, p. 215]. It is enough now to show that $D_\omega = R$.

We know that $\pi_N(D_\omega) = \pi_N(D)$ for every countable $N \subset S$. Thus, $D_\omega \subset R$. Now, for $y \in X \setminus D_\omega$, there is $N_0 \subset S$, which is countable, such that $\pi_{N_0}(D)$ is a proper subset of $\pi_{N_0}(D \cup \{y\})$. This means that $y \notin \pi_{N_0}^{-1}\pi_{N_0}(D)$. Therefore, $D_\omega = R$. $\square$

5.3 Proposition. *Let $\{X_s : s \in S\}$ be a collection of second countable spaces. Let $X = \prod_{s\in S} X_s$ and let D be a dense subset of X. Then:*

(1) *D is C-embedded in D_ω;*
(2) *if for each finite subset F of S, the equality $\pi_F(D) = \prod_{s\in F} X_s$ holds, then D_ω is the greatest subspace of X where D is C-embedded;*
(3) *the space D_ω is the Hewitt realcompactification υD of D.*

Proof: (1) Let $f : D \to \mathbb{R}$ be a continuous function. By the Arhangel'skii factorization theorem (see [Arh1]), there exists a countable $N \subset S$ and a continuous function $\phi_0 : \pi_N(D) \to \mathbb{R}$ such that $f = (\phi_0 \circ \pi_N)|_D$. Because of Proposition 5.1, it makes sense to consider the function $\widetilde{f} = (\phi_0 \circ \pi_N)|_{D_\omega}$. Furthermore, $\widetilde{f}$ is continuous and extends f.

(2) Let Y be a subset of X such that $D \subset Y$ and such that there exists $y \in Y \setminus D_\omega$. Since Y is not equal to D_ω, there is a countable subset $N \subset S$ for which $\pi_N(Y)$ properly contains $\pi_N(D)$. Let z be the element $\pi_N(y)$ of $\pi_N(Y) \setminus \pi_N(D)$. Since $\pi_N(Y)$ is metric and $\pi_N(D)$ is dense in $\pi_N(Y)$, there is a sequence $(z_n)_{n<\omega}$ in $\pi_N(D)$ which converges to z. We can assume that $z_i \ne z_j$ if $i \ne j$.

Let $f : \{z_n : n < \omega\} \to \mathbb{R}$ be defined by $f(z_n) = n$. f is a continuous function defined on a closed subset of a normal space; so there is a continuous extension $\widetilde{f} : \pi_N(D) \to \mathbb{R}$ of f.

Consider the function $g = (\widetilde{f} \circ \pi_N)|_D : D \to \mathbb{R}$. This function is continuous. We are now going to prove that g cannot be continuously extended to Y. In particular, it cannot be continuously extended to $D \cup \{y\}$.

For each n, take $A_n = [s_0^n, \dots, s_{k_n}^n; B_0^n, \dots, B_{k_n}^n] \cap \pi_N(D) \subset \widetilde{f}^{-1}(n - 1/3, n + 1/3) \subset \pi_N(D)$ with $z_n \in A_n$. The set A_n is open in $\pi_N(D)$ for each $n < \omega$ and $A_n \cap A_m = \emptyset$ if $n \neq m$. Of course, we can assume that $s_i^n \in N$ for each $n < \omega$ and for each $i \in \{0, \dots, k_n\}$.

Claim: For each neighborhood U of y in $D \cup \{y\}$, there exists $k < \omega$ such that if $m \geq k$, then $U \cap \pi_N^{-1}(A_m) \neq \emptyset$.

Indeed, let $W = [t_0, \dots, t_l; M_0, \dots, M_l]$ be a canonical open subset of X containing y. The set $\pi_N(W)$ is open in $\prod_{s \in N} X_s$ and contains z. Thus, there exists $k < \omega$ such that if $m \geq k$, then $z_m \in \pi_N(W)$. By hypothesis,

$$z_m \in A_m = \left[s_0^m, \dots, s_{k_m}^m; B_0^m, \dots, B_{k_m}^m\right] \cap \pi_N(D).$$

We define $J = \{s \leq l : t_s \in \{s_0^m, \dots, s_{k_m}^m\}\}$. Take a $b_j \in M_j$ for each $j \in \{0, \dots, l\} \setminus J$. By hypothesis there is $d_m \in D$ such that $d_m(t_i) = z_m(t_i)$ if $i \in J$, $d_m(t_i) = b_i$ if $i \in \{0, \dots, l\} \setminus J$ and $d_m(s_j^m) = z_m(s_j^m)$ for all $j \in \{0, \dots, k_m\}$. Now it happens that $d_m \in D \cap W \cap \pi_N^{-1}(A_m)$.

Therefore, it is not possible to continuously extend g to $D \cup \{y\}$.

(3) This is a consequence of Proposition 5.2 and of the statement in (1) of this proposition. □

Recall that a space K is a *dyadic space* if it is the continuous image of 2^κ for a κ. Engelking and Pelczyński proved in [EP] that if the Stone-Čech compactification βX of a space X is dyadic, then X is pseudocompact.

5.4 Theorem. *Let $\{X_s : s \in S\}$ be a family of non-trivial compact second countable spaces. If D is a dense subset of $X = \prod_{s \in S} X_s$, then the following assertions are equivalent:*

1. *D is pseudocompact;*
2. *D is G_δ-dense in X;*
3. *for every countable subset N of S, the function $\pi_N : D \to \prod_{s \in N} X_s$ is surjective;*
4. *for every continuous function $\phi : D \to \mathbb{R}$ there is a countable subset N of S and a continuous function $\phi_0 : \prod_{s \in N} X_s \to \mathbb{R}$ such that $\phi = \phi_0 \circ (\pi_N)|_D$;*
5. *for every continuous function $\phi : D \to [0,1]$ there is a countable subset N of S and a continuous function $\phi_0 : \prod_{s \in N} X_s \to [0,1]$ such that $\phi = \phi_0 \circ (\pi_N)|_D$;*
6. *$\beta D = X$;*
7. *$\upsilon D = X$.*

Proof: The equivalence (1) ⇔ (2) is well known. The equivalence (1) ⇔ (3) is Lemma 4 in [EE].

(3) ⇒ (4): We have only to apply the Arhangel'skii factorization theorem and the surjectivity of each π_N for all countable subsets N of S.

The implication (4) $\Rightarrow$ (5) is trivial.

The implication (5) $\Rightarrow$ (6) can be proved as follows: Let $\phi : D \to [0,1]$ be continuous. By hypothesis, there is a countable $N \subset S$ and a continuous function $\phi_0 : \prod_{s\in N} X_s \to [0,1]$ such that $\phi = \phi_0 \circ (\pi_N)|_D$. It happens then that the function $\phi_0 \circ \pi_N$ is a continuous extension of ϕ to all X. Therefore, $\beta D = X$.

The implication (6) $\Rightarrow$ (1) is a consequence of the Engelking-Pelczyński' Theorem cited above.

Moreover, (2) $\Rightarrow$ (7) is a consequence of Proposition 5.3, and it is easy to prove (7) $\Rightarrow$ (6). $\square$

Let K be a topological space. A subspace Y of a space X is C_K*-embedded* in X if every continuous function $f : Y \to K$ can be continuously extended to all of X. A topological space X is b_K*-discrete* if every countable subset of X is discrete and C_K-embedded in X.

5.5 Corollary. *Let X be a zero-dimensional space and let K be a zero-dimensional compact metrizable space. Then, the following statements are equivalent:*

(1) *$\beta C_p(X,K)$ is equal to K^X;*
(2) *$\beta C_p(X,K)$ is a compact dyadic space;*
(3) *$C_p(X,K)$ is pseudocompact;*
(4) *X is a b_K-discrete space.*

Proof: (1) $\Rightarrow$ (2): Since K is a compact metrizable space, it is a dyadic space. Then the space $K^X = \beta C_p(X,K)$ is a dyadic space too.

(2) $\Rightarrow$ (3): This is a consequence of Engelking-Pelczyński's result.

Since $C_p(X,K)$ is dense in K^X (see [C, Proposicion 2.23]), the equivalence (3) $\Leftrightarrow$ (4) can be proved using Theorem 5.4 and similar techniques to those used to prove Corollary 3.9 in [CT].

(3) $\Rightarrow$ (1): This is a consequence of Theorem 5.4 because $C_p(X,K)$ is dense in K^X. $\square$

By Theorems 1.10 and 4.5, for zero-dimensional spaces X and K where K is compact and metrizable, if $C_p(X,K)$ is pseudocompact (that is, if X is a b_K-discrete space), then $\mathcal{S}el(\mathcal{F}_2(C_p(X,K))) \neq \emptyset$ if and only if $C_p(X,K)$ is weakly orderable, if and only if X is separable (iff X is countable and discrete).

It was proved in [CT] that, for a zero-dimensional space X, $C_p(X,\mathbb{Z})$ is σ-compact if and only if X is an Eberlein compact space; so, also in this case, $\mathcal{S}el(\mathcal{F}_2(C_p(X,\kappa))) \neq \emptyset$ if and only if $C_p(X,\kappa)$ is weakly orderable, if and only if X is separable, for every $\kappa \geq \omega$.

5.6 Problem. Let X be a zero-dimensional space for which $C_p(X,\mathbb{Z})$ is σ-pseudocompact. Does $\mathcal{S}el(\mathcal{F}_2(C_p(X,\mathbb{Z}))) \neq \emptyset$ imply the separability of X?

6. Ordered type properties on $C_p(X,E)$

The existence of continuous selections for $\mathcal{F}_2(X)$ is related with the existence of linear orders on X. When is a space $C_p(X,2)$ a GO-space or a LOTS? In [VRS] it is proved that a topological group G, which is not totally disconnected, is orderable iff G contains an open subgroup topologically isomorphic to the additive real line; and a locally compact totally disconnected nondiscrete group is orderable if it contains an open subgroup homeomorphic to the Cantor set. Our space $C_p(X,2)$ does not satisfy any of the hypotheses just mentioned unless X is countable and discrete, and in this case it trivially contains an open subgroup homeomorphic to the Cantor set. So, it is of interest to know when such a space is a LOTS. We have:

6.1 Proposition. *Let X be a zero-dimensional space and E be a zero-dimensional separable space with $\chi(E)=w(E)$. Then, the following statements are equivalent:*

(1) *$C_p(X,E)$ is a LOTS;*
(2) *$C_p(X,E)$ is a GO-space;*
(3) *$|X|\leq\aleph_0$ and E is second countable.*

PROOF: (2) $\Rightarrow$ (3): If $C_p(X,E)$ is a GO-space, then the inequalities

$$\chi(C_p(X,E))\leq c(C_p(X,E))\leq w(C_p(X,E))$$

hold (see [MH]). But $c(C_p(X,E))$ is always countable because $C_p(X,E)$ is a dense subspace of a product of copies of the separable space E. Moreover, we always have that $\chi(C_p(X,E))=|X|\cdot\chi(E)$ (see [C, Proposition 4.5, p. 90]). Since $\chi(E)=w(E)$ and $\chi(C_p(X,E))=\aleph_0$, then $|X|\leq\aleph_0$ and E is second countable.

(3) $\Rightarrow$ (1): If $|X|\cdot w(E)\leq\aleph_0$, then $C_p(X,E)$ is a metrizable and zero-dimensional space. Even more, it is second countable. Thus, $C_p(X,E)$ is strongly zero-dimensional. But every strongly zero-dimensional metrizable space is a LOTS [E, 6.3.2.(f), p. 373]. $\square$

A variation of the previous result is:

6.2 Proposition. *Let X be a zero-dimensional space and E be a strongly zero-dimensional metrizable space. Then, the following statements are equivalent:*

(1) *$C_p(X,E)$ is a LOTS;*
(2) *$C_p(X,E)$ is a GO-space;*
(3) *$|X|\leq\aleph_0$.*

PROOF: (2) $\Rightarrow$ (3): As a consequence of Theorem 1.3 in [W], every GO-space contains a dense orderable space; so, if $C_p(X,E)$ is a GO-space, it contains a dense orderable space. This means that E^X contains a dense orderable subspace. Now, Theorem 2.2 implies $|X|\leq\aleph_0$.

(3) $\Rightarrow$ (1): If X is countable, then E^X is a subspace of κ^ω where $\kappa = w(E)$. The Baire space κ^ω has a σ-locally finite base consisting of clopen subsets. So, the same property satisfies every subspace of κ^ω. In particular, $C_p(X,E)$ has a σ-locally finite base consisting of clopen subsets. But this means that $C_p(X,E)$ is a strongly zero-dimensional space [E, Lemma 7.3.6]. Thus $C_p(X,E)$ is a LOTS [E, 6.3.2.(f), p. 373]. □

When we consider, in Propositions 6.1 and 6.2, a topological group E we can say even more:

6.3 Theorem. *For a zero-dimensional space X and a separable zero-dimensional topological group E, the following statements are equivalent:*

(1) $C_p(X,E)$ *is a LOTS;*
(2) $C_p(X,E)$ *is a GO-space;*
(3) $C_p(X,E)$ *contains an orderable dense subspace;*
(4) $C_p(X,E)$ *contains a butterfly dense subspace;*
(5) $C_p(X,E)$ *contains a butterfly dense subspace and* $\beta C_p(X,E)$ *is co-absolute with a LOTS;*
(6) $C_p(X,E)$ *is first countable and* $\beta C_p(X,E)$ *is co-absolute with a LOTS;*
(7) $C_p(X,E)$ *contains an orderable non-Archimedean dense subspace;*
(8) $C_p(X,E)$ *contains a dense proto-metrizable subspace;*
(9) $C_p(X,E)$ *contains a dense metrizable subspace;*
(10) $C_p(X,E)$ *is a metrizable space;*
(11) $|X| \leq \aleph_0$ *and* E *is second countable.*

Proof: The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (11) can be proved in a similar way to Proposition 6.1 using the fact that for every topological group E, $w(E) = \chi(E)\cdot d(E)$. Since every GO-space contains a dense orderable subspace, we obtain (2) $\Rightarrow$ (3). On the other hand, a topological group has a dense orderable subspace iff it has a dense butterfly space ([W, Theorem 2.3]); thus, (3) $\Leftrightarrow$ (4) holds. The equality (3) $\Leftrightarrow$ (5) is Proposition 1.3 in [W]. (6) $\Rightarrow$ (7) is Proposition 2.2 in [W]. If $C_p(X,E)$ is first countable, then X must be countable and E first countable (see [C, Proposition 4.5, p. 90]). But E is separable and a topological group; so, E must have countable weight and (6) $\Rightarrow$ (11) holds. And (5) $\Rightarrow$ (4), (7) $\Rightarrow$ (3) and (11) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (8) are trivial.

Assume now that the statement (3) holds. So, by Theorem 4.2 in [W], $|X| = \aleph_0$ and E^X contains a dense metrizable subspace. Since E^X is a topological group, it must be metrizable. But our hypothesis says that E is separable; so E is second countable.

So we have that (1), (2), (3), (4), (5) and (11) are equivalent. Then (5) plus (11) produces (6). Finally, (7) $\Leftrightarrow$ (8) is Theorem 2.1 in [W]. We have finished the proof. □

Corollary 4.2 and Proposition 6.3 give us examples of spaces weakly orderable which are not GO-spaces. Indeed, for every non-countable zero-dimensional separable space X, $C_p(X,2)$ is an example of this.

In [VRS] it was proved that for a space X, if βX is orderable then X is countably compact. In [CT] it was proved that, for a zero-dimensional space X, $C_p(X,2)$ is countably compact if X is a P-space. It is also proved in [VRS] that if a Tychonoff non-compact space Z is either metrizable or paracompact or Lindelöf or separable, then βZ is not orderable. So, if X is countable and non-discrete, then $C_p(X,2)$ is orderable but $\beta C_p(X,2)$ is not orderable.

6.4 Proposition. *For a zero-dimensional space X, $\beta C_p(X,\kappa)$ is orderable if and only if X is countable and discrete and $\kappa < \omega$.*

PROOF: If $\beta C_p(X,\kappa)$ is orderable, then $C_p(X,\kappa)$ is a GO-space. This means that X is countable. So $C_p(X,\kappa)$ is metrizable. Since $\beta C_p(X,\kappa)$ is orderable, $C_p(X,\kappa)$ must be compact [VRS]; that is, X is discrete and κ is finite.

The other implication is trivial. $\square$

By Propositions 6.2 and 6.4, we have:

6.5 Proposition. *For a zero-dimensional space X and a strongly zero-dimensional metrizable space E, $\beta C_p(X,E)$ is orderable if and only if X is countable and discrete and E is compact.*

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Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, Mexico D.F., 04510, México

E-mail: atamariz@servidor.unam.mx

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