Removing sets from connected product spaces while preserving connectedness

Melvin Henriksen, Amir Nikou

Abstract. As per the title, the nature of sets that can be removed from a product of more than one connected, arcwise connected, or point arcwise connected spaces while preserving the appropriate kind of connectedness is studied. This can depend on the cardinality of the set being removed or sometimes just on the cardinality of what is removed from one or two factor spaces. Sometimes it can depend on topological properties of the set being removed or its trace on various factor spaces. Some of the results are complicated to prove while being easy to state. Sometimes proofs for different kinds of connectedness are similar, but different enough to require separate proofs. Many examples are given to show that part of the hypotheses of theorems cannot be dropped, and some examples describe results about spaces whose connectedness can be established directly but not with the help of our results. A large number of examples are given for such purposes.

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1. Introduction and preliminary material

The real line $\mathbb{R}$ is connected, but $\mathbb{R} \setminus \mathbb{Q}$ is not connected, where $\mathbb{Q}$ denotes the subspace of rational numbers. Many will guess that $\mathbb{R} \times \mathbb{R} \setminus \mathbb{Q} \times \mathbb{Q}$ will also be disconnected, but that this is not the case is left as an exercise in Section 27 of [W70]. It is not difficult to generalize this [Willard's] observation to show that $\mathbb{R}^n \setminus \mathbb{Q}^n$ is connected for every integer $n \geq 2$. Below, we generalize this result in many ways. Our results will consist largely of solutions of special cases of the following problem suggested by O.A.S. Karamzadeh.

($\mathcal{P}$) *Suppose* $\{X_i\}_{i\in I}$ *is a family of more than one infinite connected spaces,* $X = \prod_{i\in I} X_i$ *is their topological product, and* S *is a nonempty proper subspace of* X*. When will* $X \setminus S$ *remain connected*?

While many of our results are valid under less restrictive assumptions, the reader may assume that all spaces we consider are Tychonoff spaces (i.e., are subspaces of compact Hausdorff spaces).

Many papers written about connected spaces are concerned with connectifications; i.e., studying when a topological space can be embedded densely in one that is connected. This paper is concerned with an opposite problem; namely what can be removed from a product of more than one connected spaces without losing connectedness? To help us in this endeavor, we adopt enough separation axioms to get pathwise connected spaces to be arcwise connected, and for all infinite connected subspaces to have cardinality at least 2^ω. By restricting ourselves to product spaces, we avoid having to deal with indecomposable or irreducibly connected spaces and such a product has no cut points. We also pay a penalty in that we were unable to find any other paper on the subject of the title. In addition to considering spaces that are connected and arcwise connected, we examine connected spaces such that any point p in it is an end point of an arc connecting it to another point q. A connected space with this latter property is said to be *point arcwise connected*.

The sets we remove for the purpose of the title are sometimes restricted to being $< 2^\omega$ in cardinality, or to being discrete, or being the complement of a dense set. Often, restrictions need to be imposed only on one or two factor spaces, and under very weak hypotheses, such a product space has no countable cut set.

We begin by recalling some definitions most of which are standard and may be found in familiar texts.

1.1 Definitions. A topological space X is said to be:

(a) *connected* if it cannot be written as the union of two disjoint nonempty closed subsets;

(b) *locally connected* at *a point* $p \in X$ if each open neighborhood of p contains a connected open neighborhood of p. X is called *locally connected* if it is locally connected at each of its points;

(c) *pathwise* (*resp. arcwise*) connected if for each $x \neq y$ in X, there is a continuous map (resp. homeomorphism) $f : [0.1] \to X$ such that $f(0) = x$ and $f(1) = y$. When this happens, we say that x and y are connected by a path (resp. arc) beginning at x and ending at y, and call x and y the *end points* of the path or arc. $x \cup y$ will denote the set of arcs with end points x and y and two arcs in $x \cup y$ whose only points in common are x and y are said to be *disjoint arcs connecting them*;

(d) *point arcwise connected* if it is connected and for each point $p \in X$, there is some point $p' \in X$ with $p' \neq p$ such that there is an arc connecting p to p' lying in X.

Recall that a point p of a connected space X such that $X \setminus \{p\}$ is not connected is called a *cut point* of X.

Clearly every arcwise connected space is point arcwise connected. The next example will help to enable us to distinguish between these various kinds of connectedness.

1.2 Example. Consider the subspaces $A = \{(x, \sin(1/x)) : 0 < x \leq 1\}$ and $B = \{(0, y) : -1 \leq y \leq 1\}$ of $\mathbb{R} \times \mathbb{R}$. Then $\mathrm{Ts} = A \cup B$ is called the *topologists* *sine curve*. It is connected but is not locally connected since no point of B has a connected neighborhood of radius less than 1. It fails to be either arcwise or pathwise connected because no point of A can lie in an arc or path containing a point of B. It is, however, point arcwise connected because any point of A lies in arc containing any other point of A, and the same can be said about points of B.

The next proposition is shown in 31.6 of [W70].

1.3 Proposition. *A Hausdorff space is pathwise connected if and only if it is arcwise connected.*

Because we are concerned only with Tychonoff spaces, we will use the terms path connected and arcwise connected spaces interchangeably.

We will try to find solutions to problem ($\mathcal{P}$) also in case the spaces X_i have some of the connectivity properties listed among the definitions given in 1.1.

A continuous mapping $f : X \to Y$ is said to be a *perfect map* if it is closed and $f^{-1}(y)$ is compact whenever $y \in Y$. Note that every continuous mapping from a compact space is perfect. It is shown in 4.4.15 of [E89] that the image under a perfect mapping of a metrizable space is metrizable. It follows that compact connected spaces and continuous mappings form a category and many research workers who study connected spaces confine their main efforts to this category. See, for example [N92].

Proofs of the parts of the next proposition and remark can be found in [E89], [L74], [M75], [N92], or [W70], and some are exercises.

1.4 Proposition.

(a) *A continuous image of an* [*arcwise*] *connected space is* [*arcwise*] *connected.*

(b) *The union of a family of* [*arcwise*] *connected spaces with a point in common is* [*arcwise*] *connected.*

(c) *A topological* (*or Cartesian*) *product of connected spaces is connected and a metrizable product of arcwise connected spaces is arcwise connected.* (See also [L74] and 2.3 below.)

(d) *If for each pair* $\{a, b\}$ *of points in a space* X, *there is a connected subspace of* X *containing both* a *and* b, *then* X *is connected.*

(e) *If for each pair* $\{a, b\}$ *of distinct points of a space* X, *there is an arc containing* a *and* b, *then* X *is arcwise connected.*

(f) *If* $\{a, b, c\}$ *are distinct points of a space* X, *and there are arcs in* $a \cup b$ *and* $b \cup c$, *then there is an arc in* $a \cup c$.

We close this section with the observation that the class $\mathcal{PC}$ of spaces that are products of more than one infinite connected (Tychonoff) spaces is contained properly in the class of infinite connected spaces. For example, no member of $\mathcal{PC}$ has a cut point or fails to be decomposable. (X is *decomposable* if has an infinite proper closed connected subspace. See [N92].)

2. Some solutions of Problem ($\mathcal{P}$) for connected products

2.1 Notational conventions that will be used often below. Suppose $\{X_i\}_{i\in I}$ is a family of more than one infinite spaces, and $X = \prod_{i\in I} X_i$ is their (topological) product.

Let $\pi_j : X \to X_j$ denote the projection map that sends $\boldsymbol{x} = \{x_i\}_{i\in I} \in X$ to its jth coordinate x_j.

If $S \subset X$ is nonempty, let $S_j = \pi_j[S]$ denote the set of all $\pi_j(s)$ as s ranges over S, and let P_S denote the topological product of the spaces S_i.

Let $D[S] = D = \{j \in I : S_j$ is a proper subspace of $X_j\}$.

The right hand side of $x = \{x_i\}_{i\in I}$ or other descriptions points in a product space will often be omitted when clear from context.

2.2 Remark. In [L74], B. Lehman defines an arc to be a compact Hausdorff continuum with at most two non cut points (called its endpoints), and calls a space arcwise connected if any two of its points are the endpoints of an arc. We will call such spaces *L-arcwise connected*. Using this definition, Lehman shows that an arbitrary product of L-arcwise connected spaces is L-arcwise connected, and notes that these two concepts coincide in metrizable spaces. In this paper this result is improved by showing that an arbitrary product of arcwise (resp. point arcwise) connected spaces is arcwise (resp. point arcwise) connected, even if the product is not metrizable.

We are grateful to W.W. Comfort for this result and its proof. We are unable to find a reference to this result in the mathematical literature except for countable products of metrizable spaces; perhaps because most of the interesting research on connectedness is done by mathematicians interested exclusively in (separable) metrizable spaces.

2.3 Theorem (Comfort)**.** *A topological product of infinite arcwise* (*resp. point arcwise*) *connected spaces is arcwise connected.*

Proof: Suppose X is the product of a family $\{X_i\}_{i\in I}$ of infinite arcwise connected spaces. If $p(0) \neq p(1)$ are distinct points of X, we will produce an arc in X whose endpoints are $p(0)$ and $p(1)$. To produce the desired mapping $f : [0,1] \to X$, we begin by defining for each $i \in I$, a mapping $f_i : [0,1] \to X_i$ such that $f_i(0) = (p(0))_i$ and $f_i(1) = (p(1))_i$. If $(p(0))_i = (p(1))_i$, let $f_i([0,1]) = (p(0))_i = (p(1))_i$, and let f_i be a homeomorphism into X_i otherwise.

Now define $f : [0,1] \to X$ by letting $(f(r))_i = f_i(r)$ for each $r \in [0,1]$ and $i \in I$.

It is easy to verify that f is an injection that meets the specifications described above and hence is continuous by 2.3.6 of [E89]. Since $p(0)$ and $p(1)$ are arbitrary, it follows that X is pathwise connected. Thus X is arcwise connected by 1.3 above. (This latter reference can be avoided by using the fact that each f_i is either constant or a homeomorphism.)

The proof in the point arcwise connected case is similar. □

The next lemma will be used below.

2.4 Lemma. *Suppose X is the product of a family $\{X_i\}_{i\in I}$ of more than one infinite topological spaces and $x \in X$. If A_r is a nonempty subspace of X_r such that $x_r \in A_r$ for some $r \in I$, and $A = \{y \in X : y_r \in A_r$ and $y_i = x_i$ if $i \neq r\}$, then:*

(a) *A_r and A are homeomorphic.*
(b) *If there is a $z \in A$ such that $z_r \neq x_r$ and A_r is an arc lying in X_r with $\{x_r, z_r\}$ as endpoints, then A is an arc lying in X with $\{x, z\}$ as endpoints.*
(c) *The set B fixing the rth coordinate of a point of X and $\prod_{i\neq r} X_i$ are homeomorphic. Hence*
 (i) *if each X_i is connected, then so is B, and*
 (ii) *if each X_i is arcwise connected, then so is B.*
(d) *Suppose $\mathrm{Card}(I) > 2$, $d \neq d' \in I$, let $J = I \setminus \{d, d'\}$, and let H denote a subset of $X_d \times X_{d'}$ that contains $(x_d, x_{d'})$. If $M = \{z \in X : (z_d, z_{d'}) \in H$ and $z_j = x_j$ for each $j \in J\}$, then M and H are homeomorphic.*

Proof: (a) It is easy to see that the map $H : A_r \to X$ such that $H(t)_r = t$ for each $t \in A_r$ and $H(t)_i = x_i$ if $i \neq r$ is the desired homeomorphism.

(b) The proof is routine once it is observed that if $h_1 : [0, 1] \to A_r$ is a homeomorphism and $h_2 : A_r \to A$ are homeomorphisms, then so is $h_2 \circ h_1$.

(c) The proof of (i) is an exercise, and (ii) holds by Theorem 2.3.

(d) Use will be made of part (a). Let $J = I \setminus \{d, d'\}$ and $I^{\#} = J \cup \{t\}$, where $t = (d, d') \in I \times I$. Then let $\mathbf{X}' = \prod_{j\in J}(X_j)$, let $Y_t = X_d \times X_{d'}$, and observe that X and $\mathbf{X}' \times Y_t$ are homeomorphic. If in the hypothesis of this lemma, we replace I by $I^{\#}$, r by t, and A by M, the desired conclusion follows. □

The next theorem is an essential tool for many results that follow.

2.5 Theorem. *Using the notation of 2.1, if X is the product of a family $\{X_i\}_{i\in I}$ of more than one infinite connected spaces, $S \subset X$ is nonempty, and $\mathrm{Card}(D[S] = D) > 1$, then:*

(a) *$F = X \setminus P_S$ is connected;*
(b) *if each of the spaces X_i is arcwise connected, then F is arcwise connected;*
(c) *if X_r is point arcwise connected for some $r \in I$, then so is F;*
(d) *if X_r is point arcwise connected for some $r \in I$, then so is X.*

Proof: (a) We will show that any two points of F are contained in one of its connected subsets. Suppose x and y are distinct points in F. Then there are r and s in D such that $x_r \notin S_r$ and $y_s \notin S_s$. If $U_r = \{\boldsymbol{z} \in F : z_r = x_r\}$ and $V_s = \{\boldsymbol{z} \in F : z_s = y_s\}$, then by Lemma 2.4(c), these sets are connected since each X_i is. We consider three cases.

Case 1) If $r \neq s$, then $U_r \cup V_s$ is the union of two connected sets with a point in common and hence is connected by Proposition 1.4(b).

Case 2) If $r = s$ and $x_r = y_r$ then the connected set U_r contains both of these points of F.

Case 3) If $r = s$ and $x_r \neq y_r$, then since the cardinality of D is at least 2, there is a $\boldsymbol{z} \in X$ such that $z_h \in X_h \setminus S_h$ for some $h \in D$ with $h \neq r$. For such an h, let $W_h = \{w \in F : w_h = z_h\}$. Now $U_r \cap W_h \neq \emptyset$ because it contains any $\boldsymbol{v} = \{v_i\}_{i \in I}$ such that $v_r = x_r$ and $v_h = z_h$. Hence $E_1 = U_r \cup W_h$ is a connected subset of F by Proposition 1.4(b), and $E_1 \cap V_s$ is nonempty because it contains all $u \in X$ such that $u_h = z_h$ and $u_s = y_s$. So $E_1 \cup V_s$ is a connected subset of F that contains both $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence F is connected and (a) holds.

(b) Since each X_i is arcwise connected, X is arcwise connected by 2.4. If x and y are distinct points in F, then by assumption, there are $r, s \in D$ such that $x_r \notin X_r$ and $y_s \notin X_s$. Then, by Lemma 2.4(c), $U_r = \{\boldsymbol{z} \in F : z_r = x_r\}$ and $V_s = \{\boldsymbol{z} \in F : z_s = y_s\}$ are also arcwise connected. We will show that there is always an arc in $x \cup y$ lying in F. As in the proof of (a), we consider three cases.

Case 1) If $r \neq s$, then $U_r \cup V_s$ is the union of two arcwise connected sets with a point in common and hence is connected by Proposition 1.4(b).

Case 2) If $r = s$ and $x_r = y_r$ then the arcwise connected set U_r contains both of these points of F.

Case 3) If $r = s$ and $x_r \neq y_r$, then since the cardinality of D is at least 2, there is a $z \in X$ such that $z_h \in X_h \setminus S_h$ for some $h \in D$ with $h \neq r$. For such an h, let $W_h = \{w \in F : w_h = z_h\}$. Now $U_r \cap W_h \neq \emptyset$ because it contains any $\boldsymbol{v} = \{v_i\}_{i \in I}$ such that $v_r = x_r$ and $v_h = z_h$. Now $E_1 = U_r \cup W_h$ is an arcwise connected subset of F such that $E_1 \cap V_s$ is nonempty because it contains all $u = \{u_i\}_{i \in I}$ such that $u_h = z_h$ and $u_s = y_s$. So $E_1 \cup V_s$ is an arcwise connected subset of F that contains both $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence F is arcwise connected.

(c) By (a), F is connected. Suppose $\boldsymbol{x} \notin P_S$. Since $\mathrm{Card}(D) > 1$, there is a $d \neq r$ in D such that $x_d \notin S_d$. Because X_r is point arcwise connected, there is a $z_r \neq x_r$ and arc in $x_r \cup z_r$ lying in X_r. Then by Lemma 2.4(b), we can find the desired $\boldsymbol{z} \in F$ to complete the proof.

(d) By adjusting the proof of (c) slightly, this proof becomes an exercise. $\square$

2.6 Example and remarks. The assumption that $\mathrm{Card}(D) > 1$ cannot be omitted from the hypothesis of 2.5(a) or (b) because $\mathbb{R} \times \mathbb{R} \setminus \mathbb{R} \times \mathbb{Q}$ fails to be [arcwise] connected.

Some possible weaker results seem worth considering. Suppose X is the product of a family $\{X_i\}_{i \in I}$ of more than one infinite connected spaces, $S \subset X$ is nonempty, $\mathrm{Card}(D) = 1$, and $D = \{d\}$.

If there is some $r \in I \setminus \{d\}$ such that X_r is point arcwise connected and $X \setminus P_S$ is connected, then $X \setminus P_S$ is point arcwise connected. For if $x \notin P_S$, then since X_r is point arcwise connected there is an $x_d \in X_d \setminus S_d$ and a $z_r \neq x_r$ and some arc A_r in $x_r \cup z_r$ lying in X_r. Then by using Lemma 2.4(b), we can find a z in $(X \setminus P_S)$ and use A_r to find an arc $A \subset X \setminus P_S$.

If, instead, the only point arcwise connected factor of the product X is X_d,

then we can modify Example 1.2 to show that $X \setminus P_S$ may not be point arcwise connected:
Let $B = \{(x, \sin(1/x) \in \mathbb{R} \times \mathbb{R}, 0 < x \leq 1\}$ and let $T = B \cup \{(0,0)\}$. Clearly $X = T \times [0,1]$ is point arcwise connected. If $S = T \times ([0, 0.5) \cup (0.5, 1])$, then $X \setminus P_S$ is homeomorphic with T which is not point arcwise connected.

The next lemma records some well known facts about connected (Tychonoff) spaces which will be used to prove the major result that follows 2.8.

2.7 Lemma.
(a) *If T is a connected subset of a space Y, then any subspace of Y between T and its closure is connected.*
(b) *If X is a connected Tychonoff space and $p \in X$, then any open neighborhood of p has cardinality at least 2^ω.*

The next theorem provides some solutions of the problem $(\mathcal{P})$ in the special case $\mathrm{Card}(D[S]) > 1$. We continue to use the notation of 2.1.

2.8 Theorem. *If X is the product of a family $\{X_i\}_{i \in I}$ of more than one infinite connected (Tychonoff) spaces, $S \subset X$ is nonempty, and $\mathrm{Card}(D) > 1$, then:*
(a) *If $F = X \setminus P_S$ is dense in $X \setminus S$, then $X \setminus S$ is connected.*
(b) *If $X_d \setminus S_d$ is dense in X_d for some $d \in D$, then X is connected.*
(c) *The following assertions are equivalent.*
  (i) *$X \setminus S$ is connected.*
  (ii) *For every $x \in P_S \setminus S$ there is an $x' \in X \setminus P_S$ and some connected subset $C(x, x')$ of $X \setminus S$ containing both x and x'.*
(d) *If there are $d \neq d'$ in D such that for any $p \in S_d \times S_{d'}$ there is a $p' \in X_d \times X_{d'} \setminus S_d \times S_{d'}$ and a connected subset $C(p, p')$ of $X_d \times X_{d'}$ containing both p and p' that meets $S_d \times S_{d'}$ only in p, then $X \setminus S$ is connected.*
(e) *If there is some $d \in D$ such that for every element $x_d \in S_d$, there is some connected subset $C(x_d, z_d)$ of X_d containing both of x_d and a point $z_d \in X_d \setminus S_d$, such that $C(x_d, z_d) \cap S_d = \{x_d\}$ then $X \setminus S$ is connected.*
(f) *If there is some $d \in D$ such that X_d is locally connected and S_d is a totally disconnected subspace of X_d (in particular if S is a totally disconnected subset of X), then $X \setminus S$ is connected.*

Proof: (a) By 2.5(a) F is connected, so the conclusion follows from 2.7(a).

(b) It will be shown that $F = X \setminus P_S$ is dense in $X \setminus S$; that is every open neighborhood of any point $x \in X \setminus S$ meets F. Let $x \in X \setminus S = F \cup (P_S \setminus S)$. If $x \in F$ then x belongs the closure of F. Now suppose $x \in P_S \setminus S$, and let U be any open neighborhood of x. Since $X_d \setminus S_d$ is dense in X_d, the open set U_d contains an element $z_d \in X_d \setminus S_d$. Let $y_d = z_d$ and for each $i \in I \setminus \{d\}$, let $y_i = x_i$. Clearly $y = \{y_i\}_{i \in I}$ is not in P_S because if y is in P_S, then for each $i \in I$ we would have $y_i \in S_i$. But this is not the case because $y_d \notin S_d$. So $y \in X \setminus P_S = F$. Therefore every open neighborhood of a point $x \in X \setminus S$ contains some element of F; i.e.,

x belongs the closure of F in X. Therefore F is dense in $X \setminus S$ and the desired conclusion follows from part (a).

(c) If (i) holds and $x = \{x_i\}_{i \in I} \in P_S \setminus S$, by assumption there is a $d \in D$ for which there is a $z_d \in X_d \setminus S_d$. If we let $x'_d = z_d$ and let $x'_i = x_i$ when $i \neq d$ is in I, then $x' \in X \setminus P_S$, so by hypothesis there is some connected set $C(x, x')$ containing both of x and x' lying in $X \setminus S$. So (ii) holds.

Suppose (ii) holds. To show (i), it suffices to show that whenever x and y are distinct elements of $X \setminus S$, there is a connected subset $C(x, y)$ of $X \setminus S$ containing both of them. We consider 4 cases.

Case 1) If $x \notin P_S$ and $y \notin P_S$, then since x and y are distinct elements of $F = X \setminus P_S$, by 2.5(a) they are in a connected subset of $X \setminus P_S$.

Case 2) If $x \in P_S \setminus S$ and $y \notin P_S$, choose x' and $C(x, x')$ as in (ii). Now either $x' = y$ or $x' \neq y$:

If $x' = y$ then the connected subset $C(x, x')$ contains both of x and y.

If $x' \neq y$ then they are distinct elements of F, so there is some connected subset $C(x', y)$ of F containing both of x' and y. Now the connected subset $C(x, x') \cup C(x', y)$ contains both of x and y and is contained in $X \setminus S$.

Case 3) If $x \notin P_S$ and $y \in P_S$, reverse the role of x and y in Case 2).

Case 4) If $x \in P_S$ and $y \in P_S$, by (ii) there is connected subsets $C(x, x') \subset X \setminus S$ containing both x and a point $x' \in X \setminus P_S$ and $C(y, y') \subset X \setminus S$ containing both of y and a point $y' \in X \setminus P_S$. Again either $x' = y'$ or $x' \neq y'$:

If $x' = y'$, then $C(x, x') \cup C(y', y)$ is the union of two connected sets with the point x' in common and hence is a connected subset of $X \setminus S$ containing both of x and y.

If $x' \neq y'$ then the set $C(x, x') \cup C(x', y') \cup C(y', y)$ is a connected subset of $X \setminus S$ containing both of x and y.

Thus (i) holds.

(d) It will be shown that the hypothesis of (d) implies that of (b). For otherwise, for each $i \in D$, there is an $a_i \in S_i$ with an open neighborhood U_i contained in S_i. If $p = (a_d, a_{d'})$, then $U = U_d \times U_{d'}$ is an open neighborhood of p contained in $S_d \times S_{d'}$. By assumption, there is a connected subset C of $X_d \times X_{d'}$ containing p and a point $p' \notin S_d \times S_{d'}$. If $C \cap U = \{p\}$, then $\{p\}$ is both a closed and an open subset of the connected set C. This contradiction shows that the hypothesis of (d) fails to hold and we may conclude from (b) that $X \setminus S$ is connected.

(e) We ape the proof of (d). As in the proof of (d), it will be shown that the hypothesis of (e) implies that of (b). For otherwise, for each $i \in D$, there is an $a_i \in S_i$ with an open neighborhood U_i contained in S_i. If $p = a_d$, then $U = U_d$ is an open neighborhood of p contained in S_d. By assumption, there is an connected subset C of X_d containing p and a point $p' \notin S_d$. If $C \cap U = \{p\}$, then $\{p\}$ is both a closed and an open subset of the connected set C. This contradiction shows that the hypothesis of (e) fails to hold and we may conclude from (b) that $X \setminus S$ is connected.

(f) If $X_d \setminus S_d$ is not dense in X_d, then there is some $p \in S_d$ and some open neighborhood U_d of p contained completely in S_d. Because X_d is locally connected, U_d contains a connected open neighborhood of p, so the component of p in U_d contains more than one point, contrary to the assumption that X_d is totally disconnected. So $X_d \setminus S_d$ is dense in X_d and the desired conclusion follows from 2.8(b). □

2.9 Theorem. *If* $\mathbf{X}$ *is the product of a family* $\{X_i\}_{i\in I}$ *of more than one infinite connected (Tychonoff) spaces,* $S \subset X$ *is nonempty,* $\mathrm{Card}(D) > 1$, *and there is some* $d \in D$ *such that* $\mathrm{Card}(S_d) < 2^\omega$ *(in particular if* $\mathrm{Card}(S) < 2^\omega$*), then* $X \setminus S$ *is connected.*

PROOF: Use will be made of 2.8(b).

It will be shown that the hypothesis of 2.8(b) is satisfied, i.e., $X_d \setminus S_d$ is dense in X_d. This is clear because if $x_d \in X_d$ and U_d is any open neighborhood of x_d, then by 2.7(b) U_d is uncountable while $\mathrm{Card}(S_d) < 2^\omega$. Hence U_d contains some element of $X_d \setminus S_d$, so x_d belongs the closure of $X_d \setminus S_d$ in X_d. Because x_d is an arbitrary element of X_d, we conclude that the closure of $X_d \setminus S_d$ equals X_d. Hence $X_d \setminus S_d$ is dense in X_d and the desired conclusion holds. □

3. Some solutions of problem ($\mathcal{P}$) for arcwise connected products

In this section we examine the impact of adding more stringent connectivity assumptions.

3.1 Theorem. *If* X *is a product of a family* $\{X_i\}_{i\in I}$ *of more than one infinite arcwise connected spaces,* $S \subset X$ *is nonempty, and* $\mathrm{Card}(D) > 1$, *then:*

(a) *The following assertions are equivalent.*

(i) $X \setminus S$ *is arcwise connected.*

(ii) *For every* $x \in P_S \setminus S$ *there is an* $x' \in X \setminus P_S$ *and some arc* A *in* $x \cup x'$ *lying in* $X \setminus S$.

(b) *If for some* $d \neq d' \in D$ *and any* $p \in S_d \times S_{d'}$, *there is some arc* A_p *of* $X_d \times X_{d'}$ *connecting* p *and a point* $p' \in X_d \times X_{d'} \setminus S_d \times S_{d'}$ *such that* $A_p \cap S_d \times S_{d'} = \{p\}$, *then* $X \setminus S$ *is arcwise connected.*

(c) *If there is some* $d \in D$ *such that for every element* $x_d \in S_d$, *there is some arc* A_d *of* X_d *connecting* x_d *and a point* $z_d \in X_d \setminus S_d$, *such that* $A_d \cap S_d = \{x_d\}$ *then* $X \setminus S$ *is arcwise connected.*

PROOF: (a) If (i) holds and $x = \{x_i\}_{i\in I} \in P_S \setminus S$, by assumption there is a $d \in D$ for which there is a $z_d \in X_d \setminus S_d$. If we let $x'_d = z_d$ and let $x'_i = x_i$ when $i \neq d$ is in I, then $x' \in X \setminus P_S$, so by hypothesis there is some arc in $x \cup x'$ lying in $X \setminus S$. So (ii) holds.

Suppose (ii) holds. To show (i), it suffices to show that whenever x and y are distinct elements of $X \setminus S$, there is an arc in $x \cup y$ lying in $X \setminus S$. We consider 4 cases.

Case 1) If $x \notin P_S$ and $y \notin P_S$, then since x and y are distinct elements of $F = X \setminus P_S$, by 2.5(b) there is some arc in $x \cup x'$ lying in $X \setminus P_S$.

Case 2) If $x \in P_S \setminus S$ and $y \notin P_S$, choose x' and an arc A_1 in $x \cup x'$ as in (ii). Now either $x' = y$ or $x' \neq y$:

If $x' = y$ then the arc A_1 connects x to y.

If $x' \neq y$ then they are distinct elements of F, so there is some arc A_2 in $x' \cup y$ lying in F. Now the set $A_1 \cup A_2$ is the union of two arcwise connected subsets of $X \setminus S$ with the point x' in common, and hence it is an arcwise connected subset of $X \setminus S$ containing both x and y. So there is some arc A in $A_1 \cup A_2 \subset X \setminus S$ which connects x to y.

Case 3) If $x \notin P_S$ and $y \in P_S$, reverse the role of x and y in Case 2).

Case 4) If $x \in P_S$ and $y \in P_S$, by (ii) there is an arc A_1 in $x \cup x' \subset X \setminus S$ connecting x to a point $x' \in X \setminus P_S$ and there is an arc A_2 in $y \cup y' \subset X \setminus S$ connecting y to a point $y' \in X \setminus P_S$. Again either $x' = y'$ or $x' \neq y'$:

If $x' = y'$, then as in case (2) the set $A_1 \cup A_2$ is the union of two arcwise connected sets with the point x' in common and hence is an arcwise connected subset of $X \setminus S$ containing both of x and y.

If $x' \neq y'$, then x' and y' are distinct elements of the arcwise connected space F. So there is an arc A_3 in $x' \cup y'$ lying in F. Now, because $A_1 \cap A_3$ contains $\{x'\}$ and $(A_1 \cup A_3) \cap A_2$ contains $\{y'\}$, the set $A_1 \cup A_2 \cup A_3$ is an arcwise connected subset of $X \setminus S$ containing both x and y and hence is an arc in $x \cup y$ lying in $X \setminus S$.

Thus (i) holds.

(b) It will be shown that the hypothesis of (b) implies (ii) of (a). That is:

$(*)$ If $x = \{x_i\}_{i \in I} \in P_S \setminus S$, then there is some arc A lying in $X \setminus S$ connecting x and a point $x' \in X \setminus P_S$.

To see this, note first that if $x \in P_S$, then $x_i \in S_i$ for each $i \in I$. In particular, if $d \neq d'$ are in D, then $p = (x_d, x_{d'}) \in S_d \times S_{d'}$. So by hypothesis there is some arc A_p lying in $X_d \times X_{d'}$ connecting p and a point $p' = (z_d, z_{d'}) \in X_d \times X_{d'} \setminus S_d \times S_{d'}$ such that $A_p \cap S_d \times S_{d'} = \{p\}$.

We consider two cases:

Case 1) If $\mathrm{Card}(I) = 2$ then $x = p$. So, if we let $x' = p'$ and $A = A_p$, then $(*)$ is satisfied.

Case 2) If $\mathrm{Card}(I) > 2$ and we let $J = I \setminus \{d, d'\}$, $x'_d = z_d$, $x'_{d'} = z_{d'}$, and $x'_i = x_i$ for each $i \in J$, then clearly $x' \in X \setminus P_S$. Also, if we let $E = \{y \in X : (y_d, y_{d'}) \in A_p$ and $y_i = x_i$ if $i \in J\}$, then both x and x' are in E. We will show next that $E \subset X \setminus S$.

To see this, note first that if $y \in E$, and $(y_d, y_{d'}) \in S_d \times S_{d'}$, then $(y_d, y_{d'}) = (x_d, x_{d'}) = p$ because $A_p \cap S_d \times S_{d'} = p = (x_d, x_{d'})$, then since for each $j \in J$, $y_j = x_j$, we conclude that $y = x$, but $x \notin S$ so neither is y.

Now for each $y \in E \setminus \{x\}$, there is some $i \in I$ such that $y_i \notin S_i$. It is clear that for any $y \in E \setminus \{x\}$, it is the case that $(y_d, y_{d'}) \in (A_p \setminus (x_d, x_{d'})) \subset X_d \times X_{d'} \setminus S_d \times S_{d'}$. So either $y_d \notin S_d$ or $y_{d'} \notin S_{d'}$. Hence $E \subset X \setminus S$.

To show that E is an arc, we show that E and A_p are homeomorphic: To do so, in the hypothesis of 2.4(d) replace H by A_p and M by E since in this case $\mathrm{Card}(I) > 2$, and $(x_d, x_{d'}) \in A_p = H$, it turns out the hypothesis of 2.4(d) holds and the desired conclusion follows. This completes the proof of (b).

(c) It suffices to show that the condition (ii) of (a) is satisfied.

Let $x = \{x_i\}_{i \in I} \in P_S \setminus S$, we show that there is some A lying in $X \setminus S$ connecting x and a point $x' \in X \setminus P_S$.

Since $x \in P_S$ for every $i \in I$, $x_i \in S_i$, in particular $x_d \in S_d$, by hypothesis there is some arc A_d lying in X_d connecting x_d and a point $z_d \in X_d \setminus S_d$ such that $A_d \cap S_d = \{x_d\}$. We define a point x' as follows:

Let $x'_d = z_d$ and for $i \neq d$, let $x'_i = x_i$, and note that $x' \in X \setminus P_S$. Again as in the proof of part (b) let $E = \{y \in X : y_d \in A_d \text{ and } y_i = x_i \text{ if } i \neq d\}$. We will show that $E \subset X \setminus S$. Suppose y is in E. We consider 2 cases.

If $y_d \neq x_d$, it is in A_d which meets S_d only in x_d, so $y_d \notin S_d$, hence $y \notin S$.

If $y_d = x_d$, then $y = x$ and we know that $x \notin S$ so neither is y.

To see that E is an arc, in the hypothesis of 2.4(a) replace r with d, A_r with A_d and A with E to conclude that E and A_d are homeomorphic, so E is an arc and (c) holds. □

3.2 Theorem. *If $\mathbf{X} = \prod_{i \in I} X_i$ is the product of a family of more than one infinite arcwise connected spaces, and x, y are distinct points of X, then there is a family of 2^ω pairwise disjoint arcs in $x \cup y$.*

Proof: First we claim that there are arcs A_x and A_y so that $A_x \times A_y$ lies in X and contains both x and y: Let $K = \{r \in I : x_r \neq y_r\}$. Since x and y are distinct, $\mathrm{Card}(K) > 0$. We consider two cases:

If $\mathrm{Card}(K) > 1$, there are distinct r and s in K, and we may find an arc A_x connecting x_r to y_r lying in X_r. To construct A_y, we begin by excluding X_r from the product and denote the product of the remaining spaces $\{X_i : i \in I, i \neq r\}$ by X'. Since X' is a product of arcwise connected spaces, it is arcwise connected by 2.3. Denoting by x' and y' respectively the distinct points that are the restriction of x and y to X', we see that there is an arc A_y connecting x' to y' lying in X'. Clearly in this case, $A_x \times A_y$ contains both of x and y.

If r is the unique member of K, then we choose the arc A_x and the space $\mathbf{X}'$ as above. By assumption, there is an $s \neq r$ in I and a $z' \in \mathbf{X}'$ such that $z'_s \neq x_s$. Because $\mathbf{X}'$ is arcwise connected, there is an arc A_y connecting x' to z', and again clearly $A_x \times A_y$ contains both x and y.

Having established this claim in both cases, we note that since there are homeomorphisms $h_1 : [0,1] \to A_x$ and $h_2 : [0,1] \to A_y$, there is a homeomorphism $h : [0,1] \times [0,1] \to A_x \times A_y$ given by the rule $h(t_1, t_2) = (h_1(t_1), h_2(t_2))$, where t_1,

t_2 range over $[0,1]$. So there are distinct points a_x and b_y in $U = [0,1] \times [0,1]$ such that $h(a_x) = x$, $h(b_y) = y$. In U there are 2^ω pairwise disjoint arcs connecting a_x to b_y, and their images under h provide the same number of pair wise disjoint arcs in X connecting x to y. □

3.3 Lemma. *If T is an infinite point arcwise connected Tychonoff space and $H \subset T$ is discrete, then for each $p \in H$, there is an arc E containing p such that $E \cap H = \{p\}$.*

Proof: Because T is point arcwise connected, there is a $q \neq p$ and an arc A in $p \cup q$ lying in T. Since H is discrete, there is an open neighborhood U of p such that $U \cap H = \{p\}$. Then $C = U \cap A$ is open in A. We will show that C contains an arc containing p and some $q\prime \neq p$ in A. We claim that C contains some arc that connects p to some $q' \in A$. For if $h : [0,1] \to A$ is a homeomorphism with $h(0) = p$, then because C is open in A it turns out that $V = h^{-1}(C)$ is an open in $[0,1]$. Clearly V contains 0. Note that every open set in $[0,1]$ is as a union of intervals, so there is some $t \in (0,1]$ such that the interval $[0,t)$ which is open in $[0,1]$ and contains 0 is completely contained in V. If $E = h([0,t/2])$, then E is an arc in $p \cup h(t/2)$ lying in U, and therefore $E \cap S = \{p\}$. □

3.4 Theorem. *Suppose $X = \prod_{i \in I} X_i$ is a product of more than one infinite arcwise connected (Tychonoff) spaces, and S is a nonempty proper subset of X such that* $\mathrm{Card}(D[S]) > 1$, *then:*

(a) *if there are $d \neq d' \in D[S]$ such that $\mathrm{Card}(S_d \times S_{d'}) < 2^\omega$ (in particular, if $\mathrm{Card}(S) < 2^\omega$), then $X \setminus S$ is arcwise connected;*
(b) *if there is some $d \in D$ such that S_d is discrete, then $X \setminus S$ is arcwise connected.*

Proof: (a) It is enough to show that the hypothesis of 3.1(b) holds by verifying that if $p \in S_d \times S_{d'}$, then there is a point $p' \in X_d \times X_{d'} \setminus S_d \times S_{d'}$ and some arc A_p in $p \cup p'$ on which $A_p \cap S_d \times S_{d'} = \{p\}$. The hypothesis of Lemma 3.2 holds, so for any $p' \in X_d \times X_{d'} \setminus S_d \times S_{d'}$ there is a family of 2^ω pairwise disjoint arcs in $p \cup p'$. Since $\mathrm{Card}(S_d \times S_{d'}) < 2^\omega$, there is some arc A_p in $p \cup p'$ lying in $X_d \times X_{d'}$ such that $A_p \cap S_d \times S_{d'} = \{p\}$.

(b) follows if we combine Lemma 3.3 and Theorem 3.1(c) replacing X_d by T, S_d by H, x_d by p, z_d by $q' = h(t/2)$, and A_d by E. □

3.5 Theorem. *Suppose X is the product of a family $\{X_i\}_{i \in I}$ of more than one infinite connected (Tychonoff) spaces, $S \subset X$ is nonempty, and* $\mathrm{Card}(D) > 1$.

(a) *If there are $d \neq d'$ in D such that each $p \in S_d \times S_{d'}$ is the end point of an arc A_p connecting p to a point p' in its compliment in $X_d \times X_{d'}$ such that $A_p \cap (S_d \times S_{d'}) = \{p\}$, then $X \setminus S$ is point arcwise connected.*
(b) *If there are $d \neq d'$ in D such that X_d and $X_{d'}$ are point arcwise connected and if $\mathrm{Card}(S_d \times S_{d'}) < 2^\omega$ (in particular if $\mathrm{Card}(S) < 2^\omega$), then $X \setminus S$ is point arcwise connected.*

(c) *If there is some $d \in D$ such that X_d is point arcwise connected and S_d is a discrete subspace of X_d, then $X \setminus S$ is point arcwise connected.*

Proof: (a) It will be shown that $X \setminus S$ is connected and for any $x = \{x_i\}_{i\in I} \in X \setminus S$, there is a point $z = \{z_i\}_{i\in I} \in X \setminus S$ and some arc in $x \cup z$ lying in $X \setminus S$. To see this, observe first that $F = X \setminus P_S$ is point arcwise connected by 2.5(c). So, if $x \in F$, then it is the end point of an arc lying in F connecting it to some $z \in F$. So we may assume that $x \in P_S \setminus S$ and hence that $x_i \in S_i$ for all $i \in I$.

If d and d', $p = (x_d, x_{d'})$, and $p' = (z_d, z_{d'})$ are chosen as in the hypothesis, and we let $z \in X$ be such that $(z_d, z_{d'}) = (x_d, x_{d'})$ and $z_i = x_i$ for all $i \in J = I\setminus\{d, d'\}$, then clearly $z \notin P_S$. Letting $E = \{y \in X : (y_d, y_{d'}) \in A_p$ and $y_j = x_j$ for all $j \in J\}$, then $E \subset X\setminus S$ is an arc that contains both x and z by the proof of 3.1(b). So E is the desired arc. Moreover, if in the proof of 2.8(c), if we replace $C(x, x')$ by E, it follows that $X \setminus S$ is connected and hence is point arcwise connected.

(b) By 2.9, $X \setminus S$ is connected. By making use of (a), to complete the proof it must be shown that for any $x = \{x_i\}_{i\in I} \in X \setminus S$ there is some point $z = \{z_i\}_{i\in I} \in X \setminus S$ and some arc in $x\cup z$ lying in $X \setminus S$. To do so first we show that if $p = (x_d, x_{d'}) \in S_d\times S_{d'}$, then there is some arc A_p in $p\cup p'$ lying in $X_d \times X_{d'}$ which connects p and a point $p' = (y_d, y_{d'}) \setminus S_d \times S_{d'}$ such that $A_p \cap S_d \times S_{d'} = \{p\}$.

Since X_d is point arcwise connected, there is a point $z_d \neq x_d$ in X_d and some arc E_d in $x_d \cup z_d$ lying in X_d. Because $\mathrm{Card}(E_d) = 2^\omega$ and $\mathrm{Card}(S_d) < 2^\omega$, there is a $w_d \in E_d \setminus S_d$ and an arc lying in E_d containing both z_d and w_d if $z_d \in S_d$. We may conclude that E_d contains an arc A_d in $x_d \cup y_d$ for some $y_d \notin S_d$. Similarly for $x_{d'}$, we can find a $y_{d'} \in X_{d'} \setminus S_{d'}$ and an arc A' in $x_{d'} \cup y_{d'}$ lying in $X_{d'}$. Clearly $p' = (y_d, y_{d'}) \in X_d \times X_{d'} \setminus S_d \times S_{d'}$.

Since A_d and $A_{d'}$ are two arcs, their product $A_d \times A_{d'}$ is arcwise connected. Note that $S' = S_d \times S_{d'} \cap (A_d \times A_{d'} \setminus \{p\})$ has cardinality $< 2^\omega$. So if we let $X = A_d \times A_{d'}$ and replace S by S' in the hypothesis of Theorem 3.4(a), we may conclude that $A_d \times A_{d'} \setminus S'$ is arcwise connected. Moreover $p \notin S'$ and $p' \notin S_d \times S_{d'}$ and hence $A_d \times A_{d'} \setminus S'$ contains both of these points. Thus there is some arc A_p in $p \cup p'$ lying in $A_d \times A_{d'} \setminus S'$. Clearly $A_p \cap S_d \times S_{d'} = \{p\}$. We consider two cases:

Case 1) If $I = \{d, d'\}$ then $x = p$, $z = p'$ and A_p is an arc in $x \cup z$ lying in $X \setminus S$ as desired.

Case 2) If $\mathrm{Card}(I) > 2$ we let $J = I \setminus \{d, d'\}$, $z_d = y_d$, $z_{d'} = y_{d'}$ and for each $j \in J$, we let $z_j = x_j$. Clearly $z \notin S$. If we let $E = \{w \in X : (w_d, w_{d'}) \in A_p$ and for each $i \in J$, $w_j = x_j\}$, then clearly E contains both of x and z.

We will show that $E \subset X \setminus S$. To see this, note first that if $y \in E$ and $(y_d, y_{d'}) \in S_d \times S_{d'}$, then $(y_d, y_{d'}) = (x_d, x_{d'})$. Thus since $y_j = x_j$ for each $j \in J$, we conclude that $y = x$. So, since $x \notin S$, neither is y. Now for each $y \in E \setminus \{x\}$ there is some $i \in I$ such that $y_i \notin S_i$. It is clear since $(y_d, y_{d'}) \in (A_p \setminus (x_d, x_{d'})) \subset X_d \times X_{d'} \setminus S_d \times S_{d'}$, then either $y_d \notin S_d$ or $y_{d'} \notin S_{d'}$. Thus $E \subset X \setminus S$.

Now by letting $H = A_p$ and $M = E$ in 2.4(d), we may conclude that E is homeomorphic with A_p. Thus E is an arc in $x \cup z$ and the proof is complete.

(c) By Lemma 3.3, for any $p \in S_d$, there is an arc E_d containing p in X_d such that $E_d \cap S_d = \{p\}$. Proceeding in much the same way as in the proof of Theorem 3.4(b), we see that if in the hypothesis of 2.8(e) we replace x_d with p, z_d with q and $C(x_d, z_d)$ with E_d we conclude that $X \setminus S$ is connected.

To complete the proof it must be shown that for any $x = \{x_i\}_{i \in I} \in X \setminus S$ there is some point $z = \{z_i\}_{i \in I} \in X \setminus S$ and some arc in $x \cup z$ lying in $X \setminus S$. We consider two cases:

Case 1) If $x_d \in S_d$ then let $J = I \setminus \{d\}$. By the argument above we can find an arc A_d in $x_d \cup y_d$ lying in X_d. Now let $z \in X$ be such that $z_d = y_d$ and $z_j = x_j$ for each $j \in J$. Also let $A^* = \{w \in X : w_d \in A_d$, and $w_j = x_j$ for each $j \in J\}$. If in 2.4 we replace A_r with A_d and A with A^*, then since A_d is an arc in $z_d \cup x_d$, by using 2.4(b) we can conclude that A^* is an arc in $x \cup z$. Now as in the proof of part (e) of 2.8 it can be shown that $A^* \subset X \setminus S$. To do so, let the sets A_d and A^* have a same role of $C(x, x')$ and E of that proof.

Case 2) If $x_d \notin S_d$, then $x \notin P_S$ (otherwise we would have $x_i \in S_i$, for all $i \in I$). Hence $x \in F = X \setminus P_S$ which is point arcwise connected by 2.5(c) because as the hypothesis of 2.5(c) requires, $\mathrm{Card}(D) > 1$ and for $d \in I$, X_d is point arcwise connected. □

4. Instructive examples

Our concluding section is devoted to giving examples illustrating the strengths and weaknesses of our results and the interactions between them.

Perhaps the most interesting result of the previous sections of this paper is one cannot destroy the connectivity of a product of more than one infinite (Tychonoff) connected or arcwise connected spaces by removing a subset of cardinality less than 2^ω. In this section we give examples to show that our results can be used to show that sometimes sets of larger cardinality can be removed while preserving connectivity if the set being removed is positioned appropriately.

4.1 Example removing uncountable dense sets. Let $\mathbb{P}$ denote the subspace of irrational numbers in the real line $\mathbb{R}$ with their usual topologies. It follows immediately from 2.5(b) that for any integer $n > 1$, the spaces $\mathbb{R}^n \setminus \mathbb{P}^n$ and $\mathbb{R}^\omega \setminus \mathbb{P}^\omega$ are arcwise connected. Note that in each case the space being removed is dense in the product and has cardinality 2^ω.

4.2 Definitions and remarks. By the *long line* $L = L([0, \omega_1))$, we mean the connected linearly ordered space containing all countable ordinals constructed in 3.12.19 of [E89]. (See also p. 55 of [HY61].) The following properties of L are well known. It is locally connected, arcwise connected, each of its countable subsets is bounded, it fails to be metrizable, its one-point compactification $L^\# = L^\#([0, \omega_1])$

coincides with its Stone-Čech compactification, and there is no arc in $L^{\#}([0,\omega_1])$ that contains ω_1. So this latter space fails to be point arcwise connected.

Note also that since L is countably compact but not compact, it is not metrizable. Despite this, a product of copies of L is arcwise connected by 2.3.

4.3 Example removing an uncountable set from a connected but not point arcwise connected space while preserving connectedness. Suppose $X = L^{\#} \times L^{\#}$, let $K = \{k_1, k_2\}$ be a subset of $L^{\#}$, let $E = L^{\#} \setminus \{a\}$ for some a in $L^{\#}$ such that $0 < a < \omega_1$, and let $S = K \times E \setminus \{b\}$ for some $b \in K \times E$. That $X \setminus S$ is connected follows immediately from 2.9.

We remark that the connectedness of the space $X \setminus S$ of Example 4.3 does not follow from 2.5(a), because S is not a Cartesian product of any two proper subsets of $L^{\#}$ since both K and E have cardinality at least two by the simple (and probably well known) lemma proved below.

4.4 Lemma. *If S_1, S_2 are two sets of cardinality at least two, then $S_1 \times S_2 \setminus \{a\}$ is not the Cartesian product of two sets for any $a = (x, y) \in S_1 \times S_2$.*

Proof: Suppose $S_1 \times S_2 \setminus \{a\} = B_1 \times B_2$ for sets B_1 and B_2. Since $\mathrm{Card}(S_1) > 1$, there is some $x' \in S_1 \setminus \{x\}$. Similarly there is some $y' \in S_2 \setminus \{y\}$. So all of the points $a = (x, y)$, $b = (x, y')$, $c = (x', y)$ are in $S_1 \times S_2$. By our assumed equality, the points b, c are in $B_1 \times B_2$. Since $b = (x, y') \in B_1 \times B_2$ it follows that $x \in B_1$. Similarly since $c \in B_1 \times B_2$, we may conclude that $y \in B_2$. Thus the point (x, y) must be in $B_1 \times B_2$ as well. □

The next example is similar to 4.3.

4.5 Example removing an uncountable set from the product of two point arcwise connected spaces without destroying point arcwise connectivity. Recall from Example 1.2 that the topologists sine curve Ts is point arcwise connected and let $K = \{k_1, k_2\}$ be a finite subset of it. If $E = \mathrm{Ts} \setminus \{a\}$ for some $a \in \mathrm{Ts}$ and $H = K \times E \setminus \{b\}$ for some $b \in K \times E$, then $\mathrm{Ts} \times \mathrm{Ts} \setminus H$ is point arcwise connected.

For if we let in 2.8 $X = X_1 \times X_2 = \mathrm{Ts} \times \mathrm{Ts}$ and $S = H$, then $\mathrm{Card}(D) = 2$ and argue again as in 4.3, then it follows from 3.5(c) that $X \setminus S$ is point arcwise connected.

We remark that the point arcwise connectedness of the space $X \setminus S$ of Example 4.5 does not follow from 2.5(c). By 4.4, 2.5(c) fails to do so because $\mathrm{Card}(K) > 1$ and $\mathrm{Card}(E) > 1$ and H is $K \times E$ minus one point.

One might hope that after removing a subset S from a product space X in such a way that $X \setminus S$ is still connected, that removing a finite set of additional points would not prevent the new remainder from being connected; especially if S satisfies the hypothesis of one of the theorems given above. This need not be the case as is noted next.

4.6 Example. Suppose $X = [0.2] \times [0,2]$ and $S = \{(x,1) : 0 \le x < 2\}$. Then $D[S] = 2$, the hypothesis of 2.8(b) and 2.8(d) holds, and $X \setminus S$ is connected, while $(1,2)$ is a cut point of the latter.

5. Epilogue

We hope that this is the first of a series of papers addressing Problem $(\mathcal{P})$. Much more remains to be done or appears in papers written much earlier concerning special cases such as what can be removed from $\mathbb{R} \times \mathbb{R}$ while preserving connectedness. For example, a line with one point deleted can be removed while still preserving arcwise connectedness. Reading Chapters 5 and 6 of [K68] will probably inspire many ideas, and studying which sets can be removed from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ while preserving (arcwise) connectedness will probably be a formidable task.

References

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Harvey Mudd College, Claremont, CA 91711, U.S.A.
E-mail: henriksen@hmc.edu

Chamran University, Ahvaz, Iran
E-mail: nikou_amir@yahoo.com

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