Quasi-concave copulas, asymmetry and transformations

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Abstract. In this paper we consider a class of copulas, called quasi-concave; we compare them with other classes of copulas and we study conditions implying symmetry for them.

Recently, a measure of asymmetry for copulas has been introduced and the maximum degree of asymmetry for them in this sense has been computed: see Nelsen R.B., *Extremes of nonexchangeability*, Statist. Papers **48** (2007), 329–336; Klement E.P., Mesiar R., *How non-symmetric can a copula be*?, Comment. Math. Univ. Carolin. **47** (2006), 141–148. Here we compute the maximum degree of asymmetry that quasi-concave copulas can have; we prove that the supremum of $\{|C(x,y) - C(y,x)|; x, y$ in $[0,1]$; C is quasi-concave$\}$ is $\frac{1}{5}$. Also, we show by suitable examples that such supremum is a maximum and we indicate copulas for which the maximum is achieved.

Moreover, we show that the class of quasi-concave copulas is preserved by simple transformations, often considered in the literature.

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1. Copulas and asymmetry

As known, copulas link the joint distribution function of a random vector to the corresponding marginal distribution functions. Moreover, from some years, in Finance, Statistics and Probability there is a growing interest on nonexchangeability of random variables, and this can be studied in terms of non-symmetric copulas.

We recall some definitions.

A (bivariate) *copula* is a function C: $[0,1]^2 \to [0,1]$ satisfying:

$$C(1,y) = y,\ C(x,1) = x,\ \text{for } 0 \le x,\ y \le 1, \tag{1}$$

$$C(x',y') - C(x,y') \ge C(x',y) - C(x,y) \text{ for } 0 \le x \le x' \le 1,\ 0 \le y \le y' \le 1. \tag{2}$$

In particular, condition (2), usually called 2-*increasingness*, together with (1) implies:

$$C(x,y) \ \text{ is increasing in each variable} \tag{3}$$

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and

$$C(0,y)=0,\ C(x,0)=0, \quad \text{for} \quad 0\le x,\ y\le 1. \tag{4}$$

Also, we obtain from (2) (set $y'=1$ or $x'=1$):

$$C(x,y) \ \text{ is 1-Lipschitz in each variable.} \tag{5}$$

A copula is the restriction to the unit square of a distribution function with uniform marginals on [0,1]. We refer to [5] for general results on copulas.

A copula $C(x,y)$ is *commutative* or *symmetric*, if

$$C(x,y)=C(y,x) \ \text{ for all } x,\ y \text{ in } \ [0,1]. \tag{6}$$

If a copula is not commutative, it can be interesting to know how large the difference between $C(x,y)$ and $C(y,x)$ can be.

According to [6], we set, for a copula C:

$$\beta_C=\sup\{|C(x,y)-C(y,x)|;\ x,y\in[0,1]\}. \tag{7}$$

As proved in [6, Theorem 2.2] and in [3], we have:

$$\sup\{\beta_C;\ \ C \text{ is a copula}\} \ = \frac{1}{3}\,; \tag{8}$$

due to this fact, it was suggested to use $3\beta_C$ as a normalized measure of asymmetry for copulas.

Moreover, the supremum is achieved: the set of copulas for which such value is attained, was characterized in [6]; their elements were called *maximally nonexchangeable copulas*. These and other results on asymmetry have been considered also in [3].

To see how and where the interest in asymmetry can arise, we recall that for example in [2] it was explained why it can be suitable to change symmetric copulas into asymmetric ones.

2. Quasi-concave copulas, symmetry and other related classes of copulas

We define a class of copulas, described in [5, Section 3.4.3].

Definition 1. We say that a copula is *quasi-concave* if for all (x,y), (x',y') in $[0,1]^2$ and all $\lambda\in[0,1]$, we have:

$$C(\lambda x+(1-\lambda)x',\ \lambda y+(1-\lambda)y')\ge\min\{C(x,y),\ C(x',y')\}. \tag{9}$$

Another, more popular, class of copulas can be described in the following way (see for example [5, Definition 3.4.6]):

Definition 2. We say that a copula is *Schur-concave* if for all x, y, λ in $[0,1]$, we have:

$$C(x,y) \leq C(\lambda x + (1-\lambda)y,\ \lambda y + (1-\lambda)x). \tag{10}$$

It is clear that a copula satisfying (10) is commutative.

A weakening of condition (10) has been considered, mainly in a context different from that of copulas (see [4, (4.1)]):

$$C(\frac{x+y}{2}, \frac{x+y}{2}) \geq C(x,y) \quad \text{for all} \quad x, y \in [0,1]. \tag{11}$$

For copulas, an "asymmetric" version of (11) was considered in [1].

The meaning of (11) and, respectively, (10), is the following. Consider the values of C along the line segment $x + y = 2\alpha$ $(0 \leq \alpha \leq 1)$; if (11) holds, then $C(x,y)$ takes the maximum value at the point (α,α); if (10) holds, then $C(x,y)$ is also increasing in the upper part of the line $x+y = 2\alpha$, from the border of the unit square to the diagonal, and decreasing along the lower part (from (α,α) to the border).

We recall (see [5, p. 104]) that quasi-concave copulas are also Schur-concave if they are symmetric (but not in general).

Now we prove that also quasi-concavity together with (11) implies Schur-concavity. Thus we obtain a description of symmetric quasi-concave copulas.

Theorem 1. *If a quasi-concave copula satisfies* (11)*, then it is Schur-concave.*

Proof: Let $C(x,y)$ be quasi-concave and satisfy (11); assume, by contradiction that C is not Schur-concave, and let be (x,y), (x',y') points along the segment $x + y = 2\alpha$ $(0 \leq \alpha \leq 1)$ such that:

$$0 \leq x < x' \leq \frac{x+y}{2}; \quad C(x,y) > C(x',y'). \tag{$*$}$$

Since, according to (11):

$$C(\frac{x+y}{2},\ \frac{x+y}{2}) \geq C(x,y),$$

the quasi-concavity of C implies

$$C(x',y') \geq \min\{C(x,y),\ C(\frac{x+y}{2},\ \frac{x+y}{2})\} = C(x,y),$$

against $(*)$; so we have a contradiction.

Analogously, we obtain a contradiction starting from $\frac{x+y}{2} \leq x < x' \leq 1$. This concludes the proof of the theorem. □

We have immediately the following consequence.

Corollary. *For a quasi-concave copula C the following are equivalent:*

(i) C *is symmetric,*
(ii) C *satisfies* (11),
(iii) C *is Schur-concave.*

Remark. For an example of a (symmetric) Schur-concave copula which is not quasi-concave see [5, Example 3.28(a)]; so condition (11) does not imply quasi-concavity. Example 2 in [1] describes a symmetric copula satisfying (11), which is not Schur-concave (so neither quasi-concave).

We can ask for some other condition implying the quasi-concavity of a copula. We give below one possible answer.

We consider another class of copulas, satisfying a condition which also has a statistical meaning (see [5, Definition 5.2.9 and Corollary 5.2.11]).

Definition 3. We say that a copula is *stochastically increasing* in x and y, (SI) for short, if it is concave in each variable; namely:

$$\begin{aligned} &C(x,y) \ \text{ is a concave function of } y \text{ for any fixed } x, \\ &\text{and a concave function of } x \text{ for any fixed } \ y \quad (x,y\in[0,1]). \end{aligned} \tag{12}$$

We have the following result.

Theorem 2. (*SI*) *copulas are quasi-concave.*

Proof: We recall that, since we are dealing with continuous functions, quasi-concavity for copulas is equivalent to Jensen (midpoint) quasi-concavity, that is to

$$C(\frac{x+x'}{2},\ \frac{y+y'}{2}) \geq \min\{C(x,y),\ C(x',y')\}. \tag{9'}$$

We prove now a simple claim.

Claim. If a copula $C(x,y)$ is not quasi-concave, then (9′) is violated by a pair of points (x,y), (x',y') such that the line joining them has a negative slope.

Proof of the claim: Let $C(x,y)$ be a copula. Let the points (x,y), (x',y') be such that the line joining them has a non-negative slope; if $x \leq x'$ and $y \leq y'$, then (3) implies: $\min\{C(x,y), C(x',y')\} = C(x,y) \leq C(x',y')$; moreover $C(\lambda x + (1-\lambda)x',\ \lambda y + (1-\lambda)y')$, $\lambda \in [0,1]$, is an increasing function of λ.

Thus in this case (9′) is satisfied; this proves the claim. □

Proof of Theorem 2: We deal with Jensen concavity. Let $C(x,y)$ be an (SI) copula; assume, by contradiction, that $C(x,y)$ is not quasi-concave: according to the claim, there are two points (x, y), (x', y') such that the line joining them has a negative slope and moreover:

$$\min\{C(x,y),\ C(x',y')\} > C(\frac{x+x'}{2},\ \frac{y+y'}{2}).$$

Assume that, for example, $x < x'$ and $y > y'$ (a similar reasoning applies in the case $x > x'$ and $y < y'$).
According to (12), we have

$$C(\frac{x+x'}{2},\ \frac{y+y'}{2}) \geq \frac{1}{2}(C(x,\ \frac{y+y'}{2}) + C(x', \frac{y+y'}{2}));$$

therefore:

$$\begin{aligned}\frac{1}{2}(C(x,\ \frac{y+y'}{2}) + C(x', \frac{y+y'}{2})) &< \min\{C(x,y),\ C(x',y')\}\\ &\leq \frac{1}{2}(C(x,y) + C(x',y')).\end{aligned}$$

Since (by (12))

$$C(x',y) - C(x', \frac{y+y'}{2}) \leq C(x', \frac{y+y'}{2}) - C(x',y'),$$

we obtain

$$C(x',y) - C(x', \frac{y+y'}{2}) < C(x,y) - C(x, \frac{y+y'}{2})$$

or

$$C(x, \frac{y+y'}{2}) + C(x',y) < C(x,y) + C(x', \frac{y+y'}{2}).$$

The last inequality contradicts 2-increasingness. This completes the proof of the theorem. □

Remark 1. Following the lines of the now given proof, also the following fact can be proved:

If a copula is (SI), then it is also concave along lines with a negative slope.

Recall that (SI) does not imply concavity of a copula (the definition of concavity being the usual one, which implies (SI)): in fact, there exists a unique concave copula (which is the greatest copula: see [5, Example 3.26.(a)]). Also: Schur concavity and concavity are independent notions for functions (see [7, p .258]); but the unique concave copula is Schur-concave.

Remark 2. Note that in general (SI) copulas are not symmetric (or equivalently, according to Theorem 2 and the Corollary to Theorem 1, they do not satisfy (10) or (11)). An example of an asymmetric, (SI) copula, is the following:

Example 1. Consider the following copula:

$$C(x,y)=\begin{cases} xy^{3/4} & \text{if } \ x\le y^{1/2} \\ yx^{1/2} & \text{if } \ x>y^{1/2}. \end{cases}$$

Remark 3. It is also possible to see that symmetric, quasi-concave copulas (see the Corollary) are not in general (SI) copulas: consider for example as $C(x,y)$ a copula whose level lines, which are broken lines, join $(x^2,1)$, (x,x), $(1,x^2)$, $x\in[0,1]$.

Remark 4. Recall that a copula is *associative* if for all $x,y,z\in[0,1]$ we have:

$$C(C(x,y),z)=C(x,C(y,z)) \ \text{ for all } \ x,y,z\in[0,1].$$

The following copula is (SI), symmetric but not associative:

$$C(x,y)=\begin{cases} x\sqrt{y} & \text{if } \ x\le y, \\ y\sqrt{x} & \text{if } \ x\ge y; \end{cases}$$

to see this, it is enough to consider for example in the above definition $x=y=\frac{1}{2}$; $z=\frac{1}{4}$.

3. Quasi-concave copulas and asymmetry

In this section we want to study the quantity

$$\beta(Q)=\sup\{\beta_C; \quad C \text{ is a quasi-concave copula}\}. \tag{13}$$

We recall the following result (see [6, Lemma 2.1]).

Lemma. *For any copula C and any $x,y\in[0,1]$ we have:*

$$|C(x,y)-C(y,x)|\le\min\{x,y,1-x,1-y,|x-y|\}. \tag{14}$$

Now we prove the main result of this section.

Theorem 3. *We have:*

$$\beta(Q)=1/5. \tag{15}$$

Proof: Let $\beta(C)=\beta>0$ for some quasi-concave copula C and let $\beta=C(x,y)-C(y,x)$ for a pair x,y. It is not a restriction to assume $x<y$ (otherwise, by

symmetry, we may construct a copula C', with same asymmetry, for which this holds true).

Let $C(y,x) = \alpha$ and $C(x,y) = \alpha + \beta$. According to the Lemma, $P \equiv (x,y)$ belongs to the triangle $T : \{(x,y);\ x \geq \beta;\ y \leq 1-\beta;\ y \geq x+\beta\}$.

The points $(1, \alpha+\beta)$, $(\alpha+\beta, 1)$, (x,y) all belong to the level sets $L = \{(u,v);\ C(u,v) = \alpha+\beta\}$.

Recall that $C(y,x) = \alpha$; let (y,z) be the lower point of abscissa y that belongs to L, and (y,z') the point of abscissa y that belongs to the segment of extremes (x,y), $(1,x)$ $(x \geq \alpha+\beta)$. Considering that segment, if we write $y = \frac{1-y}{1-x}x + \frac{y-x}{1-x}1$, we see that

$$z' = \frac{1-y}{1-x}y + \frac{y-x}{1-x}x.$$

Now the quasi-concavity of C implies $C(y,z') \geq \alpha+\beta$, so $z' \geq z > x$, and then (by using (5)) $z' - x \geq z - x \geq C(y,z) - C(y,x) = \beta$; then

$$\frac{1-y}{1-x}y + \frac{y-x}{1-x}x - x \geq \beta; \quad \text{equivalently} \quad \frac{y - y^2 + yx - x}{1-x} \geq \beta.$$

Now consider the function $f(x,y) = \frac{-y^2+y(1+x)-x}{1-x}$ in the triangle T; simple computations show that it attains its maximum at the point $(\beta, \frac{1+\beta}{2})$. So we have

$$\frac{1-\beta}{4} \geq \beta, \quad \text{which is equivalent to} \quad \beta \leq 1/5.$$

We have proved that $1/5$ is an upper bound for $\beta(Q)$. To conclude the proof we must produce an example of a quasi-concave copula C such that

$$\beta_C = \sup\{|C(x,y) - C(y,x)|;\ x,y \in [0,1]\} = 1/5.$$

This is done by the example below.

Example 2. The above proof shows that the value $1/5$ for asymmetry can only be attained, in the upper triangle $y \geq x$ of the unit square, at the point $(\frac{1}{5}, \frac{3}{5})$.

Note that the copulas we are considering are related to examples in Section 3.2.1 of [5].

We define a copula $C_1(x,y)$, whose asymmetry is $1/5$, in the following way:

$$C_1(x,y) = \begin{cases} \max\{y + (x-1)/2, 0\} & \text{if } 0 \leq y \leq \frac{x+1}{2}, \\ x & \text{if } \frac{x+1}{2} < y \leq 1. \end{cases}$$

The copula C_1 is quasi-concave (the upper boundary of level sets are convex: see [5, Theorem 3.4.5]). The support of C_1 consists of the two line segments in I^2:

$$\{(x,y) \in I^2;\ y = \frac{1+x}{2}\} \cup \{(x,y) \in I^2;\ y = \frac{1-x}{2}\}.$$

We can also consider a copula C_2, with the same asymmetry, whose support is distributed along some line segments: weight 4/5 spread along the line joining (1/5, 1) and (1, 1/5); weight 1/5 along the segment joining (0, 3/5) and (1/5, 2/5); weight 2/5 along the segment joining (1/5, 2/5) and (1, 0); finally, negative weight 2/5 spread along the segment of extremes (1/5, 3/5) and (1, 1/5).

The copulas C_1 and C_2 seem to be, respectively, the largest and the smallest quasi-concave copulas among of all quasi-concave copulas such that $C(3/5, 1/5) = 0$, $C(1/5, 3/5) = 1/5$.

Analogously we can construct, by symmetry, another pair of copulas C_3 and C_4 so that, by using these 4 copulas, we can indicate all quasi-concave copulas attaining the largest values for asymmetry. All of this can be done following the scheme of [6]. $\square$

Remark. Our last result also says how far a quasi-concave copula can be from being Schur-concave. For example, given any quasi-concave copula $C(x, y)$, the copula $C'(x, y) = \frac{1}{2}(C(x, y) + C(y, x))$ is a symmetric copula such that

$$|C(x, y) - C'(x, y)| \leq \frac{1}{10} \text{ for all } x, y.$$

But we can observe that in this way the copula we obtain is not in general a quasi-concave copula. This can be seen by starting, for example, from the copula in Exercise 3.8 in [5], with $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$.

4. Quasi-concave copulas and transformations

The following transformations have often been considered for aggregation operators, in particular for copulas.

Let φ be a strictly increasing bijection of [0,1].

Set

$$C_\varphi(x, y) = \varphi^{-1}(C(\varphi(x), \ \varphi(y))).$$

It is well known that only if φ is concave, C_φ is a copula whenever C is a copula.

Now we prove that if φ is concave, then also quasi-concavity of copulas is preserved.

Theorem 4. *If C is a quasi-concave copula and φ is concave, then C_φ is a quasi-concave copula.*

Proof: We already know that under our assumptions, C_φ is a copula. Assume, by contradiction, that C_φ is not quasi-concave. This means that there exist two pairs (x_1, y_1) and (x_2, y_2) and some $\lambda \in [0, 1]$ such that for the point (x_λ, y_λ), where $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$; $y_\lambda = \lambda y_1 + (1 - \lambda)y_2$, we have:

$$C_\varphi(x_\lambda, y_\lambda) < C_\varphi(x_1, y_1); \ C_\varphi(x_\lambda, y_\lambda) < C_\varphi(x_2, y_2);$$

since φ is increasing, these are equivalent to

$$C(\varphi(x_\lambda), \varphi(y_\lambda)) < C(\varphi(x_1), \varphi(y_1)); \; C(\varphi(x_\lambda), \varphi(y_\lambda)) < C(\varphi(x_2), \varphi(y_2)).$$

Now set, for $\lambda \in [0, 1]$:

$$x'_\lambda = \lambda\varphi(x_1) + (1 - \lambda)\varphi(x_2); \; y'_\lambda = \lambda\varphi(y_1) + (1 - \lambda)\varphi(y_2).$$

The fact that φ is concave implies

$$\varphi(x_\lambda) \geq x'_\lambda, \; \varphi(y_\lambda) \geq y'_\lambda;$$

therefore we obtain:

$$C(x'_\lambda, y'_\lambda) \leq C(\varphi(x_\lambda), \varphi(y_\lambda)) < C(\varphi(x_1), \varphi(y_1)),$$

and similarly,

$$C(x'_\lambda, y'_\lambda) < C(\varphi(x_2), \varphi(y_2)),$$

against the quasi-concavity of C. This contradiction proves the theorem. $\square$

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