A note on the paper "Smoothness and the property of Kelley"

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Abstract. Let X be a continuum. In Proposition 31 of J.J. Charatonik and W.J. Charatonik, *Smoothness and the property of Kelley*, Comment. Math. Univ. Carolin. **41** (2000), no. 1, 123–132, it is claimed that $L(X) = \bigcap_{p \in X} S(p)$, where $L(X)$ is the set of points at which X is locally connected and, for $p \in X$, $a \in S(p)$ if and only if X is smooth at p with respect to a. In this paper we show that such equality is incorrect and that the correct equality is $P(X) = \bigcap_{p \in X} S(p)$, where $P(X)$ is the set of points at which X is connected im kleinen. We also use the correct equality to obtain some results concerning the property of Kelley.

Keywords: connectedness im kleinen, continuum, hyperspace, local connectedness, property of Kelley, smoothness

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1. Introduction

The purpose of this paper is to correct an inconsistency made in [3]. Namely, in that paper it is claimed that, for a continuum X,

$$L(X) = \bigcap_{p \in X} S(p),$$

where $L(X)$ is the set of points at which X is locally connected and, for $p \in X$, $a \in S(p)$ if and only if X is smooth at p with respect to a. As we show in Theorem 3.3, the correct equality is the following one:

$$P(X) = \bigcap_{p \in X} S(p),$$

where $P(X)$ is the set of points at which X is connected im kleinen. In this paper, we also present consequences of the previous equality that involve conditions, under which, the union of two continua with the property of Kelley has the property of Kelley.

2. General notions and results

All spaces considered in this paper are assumed to be metric. For a space X, a point $x \in X$ and a positive number ε, we denote by $B_X(x,\varepsilon)$ the open ball in X centered at x and having radius ε. If A is a subset of a space X, we define $N_X(A,\varepsilon) = \bigcup_{a\in A} B_X(a,\varepsilon)$. We use the symbols $\mathrm{cl}_X(A), \mathrm{int}_X(A)$ and $\mathrm{bd}_X(A)$ to denote the closure, the interior and the boundary of A in X, respectively. The letter I stands for the unit interval $[0,1]$ in the real line $\mathbb{R}$, and the letter $\mathbb{N}$ represents the set of positive integers.

A space X is *connected im kleinen at* $p \in X$ (*cik at* p) if for any open set U of X such that $p \in U$, there is a connected subset V of X such that $p \in \mathrm{int}_X(V) \subset V \subset U$.

A *continuum* is a nonempty, compact, connected, metric space. The *hyperspace of subcontinua* of a given continuum X is denoted by $C(X)$. We consider that $C(X)$ is metrized by the *Hausdorff metric* H ([5, Definition 0.1]). If $A, B \in C(X)$ and $\varepsilon > 0$, then it is not difficult to see that $H(A,B) < \varepsilon$ if and only if $A \subset N_X(B,\varepsilon)$ and $B \subset N_X(A,\varepsilon)$.

If $A, B \in C(X)$ are such that $A \subsetneq B$, then an *order arc from* A *to* B *in* $C(X)$ is a continuous function $\lambda\colon I \to C(X)$ such that $\lambda(0) = A$, $\lambda(1) = B$ and $\lambda(s) \subsetneq \lambda(t)$ if $s < t$ ([5, Definitions 1.2 and 1.7]). For a sequence $(A_n)_n$ in $C(X)$, the symbol $A_n \to A$ means that $(A_n)_n$ converges to A (in the Hausdorff metric). If $P \in C(X)$ we put

$$C(P,X) = \{A \in C(X) : P \subset A\}.$$

If $P = \{p\}$ is a one-point set we write $C(p,X)$ instead of $C(\{p\},X)$.

3. Smoothness and the property of Kelley

A continuum X has the *property of Kelley at a point* $a \in X$ if for each sequence $(a_n)_n$ in X such that $a_n \to a$ and each $A \in C(a,X)$, there is a sequence $(A_n)_n$ in $C(X)$ such that $A_n \to A$ and $a_n \in A_n$, for each $n \in \mathbb{N}$. We say that X has the *property of Kelley* if it has this property at each of its points. It is well known that locally connected continua have the property of Kelley. Moreover, if X is a continuum and X is cik at $p \in X$, then X has the property of Kelley at p. As a kind of converse of this result we have the following theorem.

Theorem 3.1 ([2, Theorem 2.1]). *Let X be a continuum with the property of Kelley at $p \in X$. If p is a cut point of X, then X is cik at p.*

A continuum X is *smooth at a point* $p \in X$ *with respect to a point* $a \in X$ provided that for each sequence $(a_n)_n$ in X such that $a_n \to a$ and any $A \in C(X)$ such that $a, p \in A$, there is a sequence $(A_n)_n$ in $C(X)$ such that $A_n \to A$ and $a_n, p \in A_n$, for each $n \in \mathbb{N}$. We say that X is *smooth at* $p \in X$ if X is smooth at

p with respect to any point $a \in X$. For a continuum X consider the sets:

$$\begin{aligned} I(X) &= \{p \in X \colon X \text{ is smooth at } p\}, \\ L(X) &= \{p \in X \colon X \text{ is locally connected at } p\}, \\ P(X) &= \{p \in X \colon X \text{ is cik at } p\}, \\ K(X) &= \{p \in X \colon X \text{ has the property of Kelley at } p\}. \end{aligned}$$

If $p \in X$ we also consider the set

$$S(p) = \{a \in X \colon X \text{ is smooth at } p \text{ with respect to } a\}.$$

Note that $I(X) \subset L(X) \subset P(X) \subset K(X)$. Note also that $I(X) = X$ if and only if X is locally connected [4, Corollary 3.3], and $p \in I(X)$ if and only if $S(p) = X$.

Theorem 3.2. *Let X be a continuum. If $p \in S(p)$, then $p \in P(X)$.*

Proof: If $p \notin P(X)$ then there is an open subset U of X with the following properties: $p \in U$ and no connected neighborhood of p is contained in U. Thus if C is the component of U that contains p, then $p \notin \mathrm{int}_X(C)$. Then $p \in \mathrm{bd}_X(C)$, so there is a sequence $(x_n)_n$ in $X - C$ such that $x_n \to p$. Since X is smooth at p with respect to p, there is a sequence $(K_n)_n$ in $C(X)$ such that $K_n \to \{p\}$ and $p, x_n \in K_n$, for any $n \in \mathbb{N}$. Let $\varepsilon > 0$ be such that $B_X(p, \varepsilon) \subset U$. Since $K_n \to \{p\}$, there is $m \in \mathbb{N}$ such that $H(K_m, \{p\}) < \varepsilon$. Then $p \in K_m \subset N_X(\{p\}, \varepsilon) = B_X(p, \varepsilon) \subset U$, so $K_m \subset C$ and then $x_m \in C$. This contradiction shows that $p \in P(X)$. □

For a continuum X put $C^2(X) = C(C(X))$. For a given $p \in X$ consider a function F_p defined on X by letting

$$F_p(a) = \{A \in C(X) : a, p \in A\}.$$

In [3, p. 124] it is shown that, for any $a \in X$, $F_p(a)$ is a closed and arcwise connected subset of $C(X)$. Thus $F_p(a) \in C^2(X)$ and we can write $F_p \colon X \to C^2(X)$. Some other properties of this function are discussed in [3]. For example, in [3, Corollary 8] it is shown that F_p is continuous at $a \in X$ if and only if $a \in S(p)$. Thus F_p is continuous if and only if $p \in I(X)$.

Theorem 3.3. *If X is a continuum, then*

$$P(X) = \bigcap_{p \in X} S(p). \tag{3.1}$$

Proof: Assume first that $x \in \bigcap_{p \in X} S(p)$. Then, in particular, $x \in S(x)$ so, by Theorem 3.2, $x \in P(X)$. Thus $\bigcap_{p \in X} S(p) \subset P(X)$.

Assume now that $x \in P(X)$. Take a point $p \in X$. In order to show that $x \in S(p)$, take a sequence $(x_n)_n$ in X such that $x_n \to x$ and $K \in C(X)$ such that $p, x \in K$. Given $n \in \mathbb{N}$

$$F_x(x_n) = \{M \in C(X) : x_n, x \in M\}$$

is a closed subset of $C(X)$, so there is $M_n \in F_x(x_n)$ such that

$$H(M_n, \{x\}) = \min\{H(A, \{x\}) : A \in F_x(x_n)\}.$$

Note that $(M_n)_n$ is a sequence in $C(X)$ such that $x, x_n \in M_n$, for any $n \in \mathbb{N}$. We claim that

1) $M_n \to \{x\}$.

To show 1) let $\varepsilon > 0$ and U be an open subset of X such that $x \in U \subset \mathrm{cl}_X(U) \subset B_X(x, \varepsilon)$. Since X is cik at x, there is a connected subset V of X such that $x \in \mathrm{int}_X(V) \subset V \subset U$. Put $A = \mathrm{cl}_X(V)$ and note that $A \subset \mathrm{cl}_X(U) \subset B_X(x, \varepsilon) = N_X(\{x\}, \varepsilon)$. Since the inclusion $\{x\} \subset N_X(A, \varepsilon)$ also holds, we have $H(A, \{x\}) < \varepsilon$. Now, since $x_n \to x$, there is $N \in \mathbb{N}$ such that $x_n \in \mathrm{int}_X(V)$ for any $n \geq N$. Thus $A \in F_x(x_n)$ for any $n \geq N$, so $H(M_n, \{x\}) \leq H(A_n, \{x\}) < \varepsilon$, for any $n \geq N$. This shows 1).

Given $n \in \mathbb{N}$, put $K_n = M_n \cup K$. Note that $(K_n)_n$ is a sequence in $C(X)$ such that $p, x_n \in K_n$, for any $n \in \mathbb{N}$. Moreover, by 1), $K_n = M_n \cup K \to \{x\} \cup K = K$. This shows that $p \in S(p)$. Thus $P(X) \subset \bigcap_{p \in X} S(p)$. □

Corollary 3.4. *Let X be a continuum and $p \in X$. Then $p \in P(X)$ if and only if $p \in S(p)$.*

Proof: If $p \in P(X)$ then, by equality $P(X) = \bigcap_{p \in X} S(p)$, we have $p \in S(p)$. On the other hand, if $p \in S(p)$ then, by Theorem 3.2, $p \in P(X)$. □

Let X be a continuum. In [3, Proposition 31] it is claimed that

$$L(X) = \bigcap_{p \in X} S(p). \tag{3.2}$$

Using equation (3.2), in [3, Corollary 32] it is claimed that

(*) $p \in L(X)$ if and only if $p \in S(p)$.

Note that if equation (3.2) is correct then, using equation (3.1) it follows that $P(X) = L(X)$, for any continuum X. This is a contradiction, since there exists a continuum X which is cik at some point $p \in X$ and it is not locally connected at p (see Figure 5.22 of [6] on page 84). Thus equation (3.2) is wrong. The right way of calculating $\bigcap_{p \in X} S(p)$ is the one presented in Theorem 3.3. By the same reasons, claim (*) is wrong. The right claim is the one presented in Corollary 3.4.

With respect to the set $K(X)$ defined for a continuum X, in [3, Observation 35] it is observed that

$$\{p \in X : p \in S(p)\} \subset K(X).$$

Combining this with Corollary 3.4, we have $P(X) \subset K(X)$ (an assertion that can be shown without using Corollary 3.4).

In [3, Proposition 39] it is claimed that, for a continuum X such that $L(X) \neq \emptyset$, we have

$$K(X) \subset \bigcap_{p \in L(X)} S(p). \tag{3.3}$$

In this paper we show the following result.

Theorem 3.5. *Let X be a continuum. If $P(X) \neq \emptyset$, then*

$$K(X) \subset \bigcap_{p \in P(X)} S(p), \tag{3.4}$$

and if $L(X) \neq \emptyset$, then inclusion (3.3) *holds.*

Proof: We will show, simultaneously, that inclusions (3.3) and (3.4) hold. To verify inclusion (3.3) we take a point $p \in L(X)$ and, to verify inclusion (3.4), we take a point $p \in P(X)$. Since $L(X) \subset P(X)$ in any case we have $p \in P(X)$ so, by Corollary 3.4, $p \in S(p)$. To show that $K(X) \subset S(p)$, consider a point $a \in K(X)$, a sequence $(a_n)_n$ in X such that $a_n \to a$ and a subcontinuum A of X such that $a, p \in A$. Since X has the property of Kelley at a, there is a sequence $(L_n)_n$ in $C(X)$ such that $L_n \to A$ and $a_n \in L_n$, for any $n \in \mathbb{N}$. Let $(p_n)_n$ be a sequence in X such that $p_n \to p$ and $p_n \in L_n$, for any $n \in \mathbb{N}$. Since $p \in S(p)$, there is a sequence $(M_n)_n$ in $C(X)$ such that $M_n \to \{p\}$ and $p, p_n \in M_n$, for any $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, let $A_n = L_n \cup M_n$. Then $(A_n)_n$ is a sequence in $C(X)$ such that $A_n \to A$ and $p, a_n \in A_n$, for any $n \in \mathbb{N}$. Thus $a \in S(p)$. □

Using inclusion (3.4) we can also prove the following result.

Theorem 3.6. *Let X be a continuum with the property of Kelley and $p \in X$. Then X is smooth at p if and only if X is cik at p.*

Proof: The first part follows from the fact that $I(X) \subset P(X)$. To show the second part, assume that X is cik at p. Then $p \in P(X)$ so, by Theorem 3.5, $X = K(X) \subset S(p)$. This implies that $S(p) = X$, so $p \in I(X)$. □

In [3, Proposition 42] it is claimed that

(**) a continuum X having the property of Kelley is smooth at a point $p \in X$ if and only if X is locally connected at p.

Assertion (**) is correct and the proof of it uses inclusion (3.3) and the fact that $I(X) \subset L(X)$. Thus, combining the previous results we obtain the following theorem.

Theorem 3.7. *Let X be a continuum with the property of Kelley and $p \in X$. Then the following assertions are equivalent:*

(1) *X is smooth at p;*
(2) *X is locally connected at p;*
(3) *X is cik at p.*

In other words, in the realm of continua with the property of Kelley, the pointwise versions of smoothness, local connectedness and connectedness im kleinen are all equivalent.

4. Property of Kelley and local connectedness at a point

On page 130 of [3] the following question if formulated.

Question 4.1. *For which continua X the property of Kelley of X implies the existence of a point at which X is locally connected?*

In this section we present some partial answers to this question. As mentioned in [3], for dendroids we have an affirmative answer to the previous question. As a consequence of Theorems 3.1 and 3.7, we have the following result.

Theorem 4.2. *Let X be a continuum with the property of Kelley. If X has a cut point p, then X is locally connected at p.*

For a continuum X, a point $p \in X$ is called an *end-point* of X if for any open subset U of X such that $p \in U$, there exists an open subset V of X such that $p \in V \subset U$ and $\mathrm{bd}_X(V)$ consists of precisely one point. It is known that every end-point of a continuum X is a non-cut point of X. It is also known that if p is an end-point of X, then X is cik at p. Using this and Theorem 3.7, we have the following result.

Theorem 4.3. *Let X be a continuum with the property of Kelley. If X has an end-point p, then X is locally connected at p.*

By Theorems 4.2 and 4.3, Question 4.1 can be reformulated as follows.

Problem 4.4. *Classify all continua X with the following properties:*

(a) *X has the property of Kelley;*
(b) *no point of X is a cut point of X;*
(c) *no point of X is an end-point of X;*
(d) *X has a point at which it is cik.*

In [3, Example 45] it is shown that there exists an arcwise connected continuum X with the property of Kelley and locally connected at none of its points. We will show that this is not the case if we add atriodicity. Recall that for $n \in \mathbb{N}$ a continuum X is an *n-od* if X contains a subcontinuum B such that $X - B$ has at least n components. Moreover, X is said to be *atriodic* if it contains no 3-ods.

Theorem 4.5. *Let X be an atriodic arcwise connected continuum with the property of Kelley. Then X is an arc or a simple closed curve.*

Proof: Since X has the property of Kelley and it is atriodic, by [2, Corollary 5.2], X has the property of Kelley hereditarily, i.e., any subcontinuum of X has the property of Kelley. Now, since X has the property of Kelley hereditarily and it is arcwise connected, by [2, Theorem 1.1], X is hereditarily locally connected. Thus X is atriodic and locally connected, so X is an arc or a simple closed curve (see (b) of [6, 8.40]). □

5. Union of continua with the property of Kelley

Easy examples show that the union of continua with the property of Kelley does not have the property of Kelley. However we have the following result.

Theorem 5.1 ([1, Theorem 3.1])**.** *Let X and D be continua such that $X \cap D \neq \emptyset$. Put $Y = X \cup D$. If both X and D have property of Kelley and Y is smooth at any point of $X \cap D$, then Y has the property of Kelley.*

Combining the previous results we obtain the following theorem.

Theorem 5.2. *Let X and D be continua such that $X \cap D = \{p\}$. Put $Y = X \cup D$. Then Y has the property of Kelley if and only if both X and D have the property of Kelley and Y is smooth at p.*

Proof: Note first that, for any $A \in C(Y)$ we have $A \cap X \in C(X)$ and $A \cap D \in C(D)$. Now assume that Y has the property of Kelley. Since p is a cut point of Y, by Theorem 3.1, Y is cik at p. Thus, by Theorem 3.6, Y is smooth at p. Now we show that X has the property of Kelley at p. Let $(p_n)_n$ be a sequence in X such that $p_n \to p$ and $A \in C(p, X)$. Since Y is smooth at p, there is a sequence $(A_n)_n$ in $C(Y)$ such that $A_n \to A$ and $p, p_n \in A_n$ for any $n \in \mathbb{N}$. Hence $A_n \cap X \in C(X)$ and $A_n \cap D \in C(D)$ for any $n \in \mathbb{N}$. Moreover $A_n \cap D \to \{p\}$ and $A_n \cap X \to A$. Thus X has the property of Kelley at p. Now we show that X has the property of Kelley at $a \in X - \{p\}$. Let $(a_n)_n$ be a sequence in X such that $a_n \to a$ and $A \in C(a, X)$. Since Y has the property of Kelley at a, there is a sequence $(A_n)_n$ in $C(Y)$ such that $A_n \to A$ and $a_n \in A_n$, for any $n \in \mathbb{N}$. Then $A_n \cap X \in C(X)$ for any $n \in \mathbb{N}$ and $A_n \cap X \to A$. This shows that X has the property of Kelley at a, so X has the property of Kelley. Similarly D has the property of Kelley. This completes the first part of the proof. The second part follows from Theorem 5.1. □

As we show in the following result, in the previous theorem the condition of Y being smooth at p can be replaced by the condition of Y being cik at p.

Theorem 5.3. *Let X and D be continua such that $X \cap D = \{p\}$. Put $Y = X \cup D$. Then Y has the property of Kelley if and only if both X and D have the property of Kelley and Y is cik at p.*

Proof: If Y has the property of Kelley then, by Theorem 5.2, both X and D have the property of Kelley and Y is smooth at p. Thus, by Theorem 3.6, Y is cik at p.

Assume now that both X and D have the property of Kelley and that Y is cik at p. By Theorem 3.3, $p \in S(p)$, so Y is smooth at p with respect to p. Take $a \in Y - \{p\}$. We will show that Y is smooth at p with respect to a, so take $A \in C(Y)$ such that $a, p \in A$ and a sequence $(a_n)_n$ in Y such that $a_n \to a$. Note that $A \cap X \in C(p, X)$ and $A \cap D \in C(p, D)$. Without loss of generality, we can assume that $a \in X - D$ and $a_n \in X - D$ for any $n \in \mathbb{N}$. Since X has the property of Kelley at a, there is a sequence $(A_n)_n$ in $C(X)$ such that $A_n \to A \cap X$ and $a_n \in A_n$, for any $n \in \mathbb{N}$. Let $(p_n)_n$ be a sequence in X such that $p_n \to p$ and $p_n \in A_n$, for any $n \in \mathbb{N}$. Since Y is smooth at p with respect to p, there is a sequence $(B_n)_n$ in $C(Y)$ such that $B_n \to A \cap D$ and $p, p_n \in B_n$ for any $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, put $C_n = A_n \cup B_n$. Then $(C_n)_n$ is a sequence in $C(Y)$ such that $C_n \to A$ and $a_n, p \in C_n$, for any $n \in \mathbb{N}$. This shows that Y is smooth at p so, by Theorem 5.1, Y has the property of Kelley. □

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