Directoids with an antitone involution

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Abstract. We investigate $\sqcap$-directoids which are bounded and equipped by a unary operation which is an antitone involution. Hence, a new operation $\sqcup$ can be introduced via De Morgan laws. Basic properties of these algebras are established. On every such an algebra a ring-like structure can be derived whose axioms are similar to that of a generalized boolean quasiring. We introduce a concept of symmetrical difference and prove its basic properties. Finally, we study conditions of direct decomposability of directoids with an antitone involution.

Keywords: directoid, antitone involution, D-quasiring, symmetrical difference, direct decomposition

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1. Bounded directoids with an antitone involution

The concept of directoid was introduced by J. Ježek and R. Quackenbush [6] and independently by V.M. Kopytov and Z.I. Dimitrov [7] and B.J. Gardner and M.M. Parmenter [5]. Recall that a *directoid* is an algebra $\mathcal{D} = (D; \sqcap)$ of type (2) satisfying the identities

(D1) $x \sqcap x = x$;

(D2) $(x \sqcap y) \sqcap x = x \sqcap y$;

(D3) $y \sqcap (x \sqcap y) = x \sqcap y$;

(D4) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

Putting $x \leq y$ if and only if $x \sqcap y = x$, the relation $\leq$ is an order on D, the so-called *induced order* of directoid $\mathcal{D}$. It was shown in [6] that $x \sqcap y$ is a common lower bound of x, y. Also conversely, if $(D; \leq)$ is an ordered set where for each $x, y \in D$ their lower bound set $L(x, y) = \{d \in D; d \leq x \text{ and } d \leq y\}$ is non-void, one can pick up freely an element $d \in L(x, y)$ with only one constrain: if $x \leq y$ then d must be equal to x. Then, putting $x \sqcap y = d$, the algebra $(D; \sqcap)$ is a directoid. We do not assume the commutativity $x \sqcap y = y \sqcap x$ throughout the paper.

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Lemma 1. *A directoid $\mathcal{D} = (D; \sqcap)$ is a semilattice if and only if it satisfies the condition*

$$\text{(S)} \qquad (x \leq a \ \text{ and } \ x \leq b) \ \Rightarrow \ x \leq a \sqcap b.$$

Proof: Of course, (S) is satisfied in every $\wedge$-semilattice. Conversely, let a directoid $\mathcal{D} = (D; \sqcap)$ satisfy (S), let $a, b \in D$ and $x \in L(a, b)$. Then, by (S), $x \leq a \sqcap b$ and hence, $a \sqcap b$ is the greatest lower bound of a, b, i.e. $a \sqcap b = \inf(a, b)$. Thus $(D; \sqcap)$ is a $\wedge$-semilattice. □

In what follows, we will deal with directoids having a least element 0 and a greatest element 1. This fact will be expressed by the notation $\mathcal{D} = (D; \sqcap, 0, 1)$. By an *antitone involution* on $\mathcal{D} = (D; \sqcap, 0, 1)$ is meant a mapping $x \mapsto x'$ of $D \to D$ such that $x'' = x$ and $x \leq y \Rightarrow y' \leq x'$ where $\leq$ is the induced order of $\mathcal{D}$. If $\mathcal{D} = (D; \sqcap, 0, 1)$ has an antitone involution, we will write $\mathcal{D} = (D; \sqcap, ', 0, 1)$. Of course, $0' = 1$ and $1' = 0$ is valid in every bounded directoid with an antitone involution. Due to [7], the operations $\sqcup$ and $\sqcap$ are connected by the absorption laws.

Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$. The term operation $\sqcup$ defined via $x \sqcup y = (x' \sqcap y')'$ will be called an *assigned operation* of $\mathcal{D}$.

Theorem 1. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution, let $\sqcup$ be the assigned operation. Then:*

(i) $x \sqcap y = (x' \sqcup y')'$;

(ii) $x \sqcup x = x$,
$(x \sqcup y) \sqcup x = x \sqcup y$,
$y \sqcup (x \sqcup y) = x \sqcup y$,
$x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$;

(iii) $x \sqcap (x \sqcup y) = x, \quad x \sqcup (x \sqcap y) = x, \quad x \sqcap (y \sqcup x) = x, \quad x \sqcup (y \sqcap x) = x,$
$(x \sqcup y) \sqcap x = x, \quad (x \sqcap y) \sqcup x = x, \quad (y \sqcup x) \sqcap x = x, \quad (y \sqcap x) \sqcup x = x.$

Proof: (i) $(x' \sqcup y')' = (x'' \sqcap y'')'' = x \sqcap y$.
(ii) $x \sqcup x = (x' \sqcap x')' = x'' = x$,
$(x \sqcup y) \sqcup x = (x' \sqcap y')' \sqcup x = ((x' \sqcap y') \sqcap x')' = (x' \sqcap y')' = x \sqcup y$,
$y \sqcup (x \sqcup y) = y \sqcup (x' \sqcap y')' = (y' \sqcap (x' \sqcap y'))' = (x' \sqcap y')' = x \sqcup y$,
$x \sqcup ((x \sqcup y) \sqcup z) = (x' \sqcap ((x' \sqcap y') \sqcap z')) = ((x' \sqcap y') \sqcap z')' = (x \sqcup y) \sqcup z$.
(iii) The absorption laws were proved in [7]. For the reader's convenience, we present an easy proof as follows. By using (ii), we compute

$$x \sqcup (x \sqcup y) = x \sqcup ((x \sqcup y) \sqcup x) = (x \sqcup y) \sqcup x = x \sqcup y$$

thus $x \leq x \sqcup y$ whence $x \sqcap (x \sqcup y) = x$. Similarly we can prove the remaining absorption laws. □

Remark 1. The identities $x \sqcup y = (x' \sqcap y')'$ and $x \sqcap y = (x' \sqcup y')'$ will be referred under the name De Morgan laws because they are formally the same as De Morgan laws in lattices.

Due to De Morgan laws, $(D; \sqcup)$ is a directoid again for any $\mathcal{D} = (D; \sqcap, ', 0, 1)$ with the assigned operation $\sqcup$. Clearly $x \leq y$ if and only if $x \sqcup y = y$.

Example 1. Consider the directed set whose diagram is drawn in Figure 1

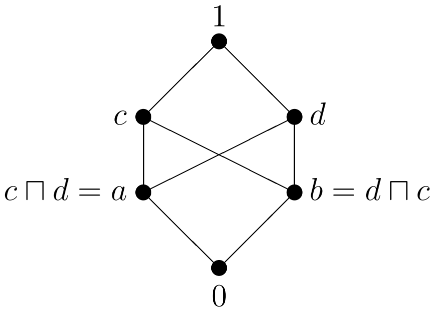

FIGURE 1

Let us pick up $c \sqcap d = a$ and $d \sqcap c = b$. Then $\mathcal{D} = (D; \sqcap, 0, 1)$ for $D = \{0, a, b, c, d, 1\}$ is a bounded $\sqcap$-directoid. Further, define $x \mapsto x'$ on D as follows

x	0	a	b	c	d	1
x'	1	d	c	b	a	0

.

It is clearly an antitone involution on D. For the assigned operation $\sqcup$ we have:

$$a \sqcup b = (a' \sqcap b')' = (d \sqcap c)' = b' = c,$$
$$b \sqcup a = (b' \sqcap a')' = (c \sqcap d)' = a' = d.$$

◊

The following example gives an answer to the question whether is it possible to define an antitone involution on every $\sqcap$-directoid:

Example 2. Consider the $\sqcap$-directoid $\mathcal{D} = (\{0, x, y, z, 1\}; \sqcap)$ depicted in Figure 2 where for binary operation $\sqcap$ we have: $x \sqcap y = 0$, $y \sqcap x = z$ (and trivially for comparable elements).

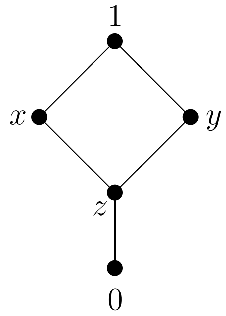

Figure 2

We show that on this $\sqcap$-directoid it is not possible to define an antitone involution $'$: Clearly, $0' = 1$ and $1' = 0$. If we put $x' = z$, then y' must be equal to y but $z \leq y$ implies $y = y' \leq z' = x$, a contradiction. If we pick $x' = y$, then $z' = z$ and $z \leq x$ implies $y = x' \leq z' = z$, a contradiction. Finally, if $x' = x$ then for $z' = z$ or $z' = y$ we have $x \leq z$ or $x \leq y$ which is a contradiction again.

Note, that if a $\sqcap$-directoid is not commutative, it needs to have at least 2 non-comparable elements x, y such that $|L(x, y)| \geq 2$. Thus, the directoid from Figure 2 is the smallest one which cannot have an antitone involution and hence also the assigned operation $\sqcup$. $\Diamond$

It can be proved dually as in Lemma 1 that a $\sqcup$-directoid $(D; \sqcup)$ is a $\vee$-semilattice if and only if it satisfies the condition

(S') $$(a \leq x \ \text{ and } \ b \leq x) \ \Rightarrow \ a \sqcup b \leq x.$$

Lemma 1 enables us to show that when $\sqcup$ and $\sqcap$ are connected by a stronger identity such as modularity or distributivity then the resulting structure is a lattice. A similar result was already shown by J. Nieminen [9] for the so-called χ-lattices.

Theorem 2. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. If $\mathcal{D}$ satisfies the modularity laws*

$$x \sqcup (y \sqcap (x \sqcup z)) = (x \sqcup y) \sqcap (x \sqcup z),$$
$$x \sqcap (y \sqcup (x \sqcap z)) = (x \sqcap y) \sqcup (x \sqcap z)$$

then $(D; \sqcup, \sqcap)$ is a lattice.

Proof: Suppose $x, y, a \in D$, $x, y \leq a$. Then $x = a \sqcap x$, $y = a \sqcap y$ and hence $x \sqcup y = (a \sqcap x) \sqcup (a \sqcap y) = a \sqcap (x \sqcup (a \sqcap y)) = a \sqcap (x \sqcup y)$ thus $x \sqcup y \leq a$. In other words, it satisfies (S') and hence $(D; \sqcup)$ is a $\vee$-semilattice. Dually it can be shown that also $(D;\sqcap)$ is a $\wedge$-semilattice. Due to Theorem 1, $\sqcap$ and $\sqcup$ are connected with the absorption laws, i.e. $(D;\sqcup,\sqcap)$ is a lattice. $\square$

Let $\mathcal{D}=(D;\sqcap,',0,1)$ be a directoid with an antitone involution and $\sqcup$ its assigned operation. If $\sqcap$ is commutative, i.e. $x\sqcap y=y\sqcap x$ then also $\sqcup$ is commutative and $(D;\sqcup,\sqcap)$ is the so-called λ-lattice as defined in [10]. Moreover, every χ-lattice (defined in [9], [8]) is a particular case of λ-lattice. In our investigation we do not assume commutativity of $\sqcap$ and hence our algebras are more general. Nevertheless, we are still able to prove a result which holds for lattices, i.e.:

Theorem 3. *Let $\mathcal{D}=(D;\sqcap,',0,1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Then*

$$m(x,y,z)=((x\sqcap y)\sqcup(z\sqcap y))\sqcup(x\sqcap z)$$

is the majority term on $\mathcal{D}$ and hence the congruence lattice $\operatorname{Con}\mathcal{D}$ *is distributive.*

PROOF: $m(x,x,y)=((x\sqcap x)\sqcup(y\sqcap x))\sqcup(x\sqcap y)=(x\sqcup(y\sqcap x))\sqcup(x\sqcap y)=x\sqcup(x\sqcap y)=x$,
$m(x,y,x)=((x\sqcap y)\sqcup(x\sqcap y))\sqcup(x\sqcap x)=(x\sqcap y)\sqcup x=x$,
$m(y,x,x)=((y\sqcap x)\sqcup(x\sqcap x))\sqcup(y\sqcap x)=((y\sqcap x)\sqcup x)\sqcup(y\sqcap x)=x\sqcup(y\sqcap x)=x$. $\square$

2. Derived quasirings

The concept of a (boolean) quasiring was introduced firstly for orthomodular lattices and ortholattices and then for bounded lattices with an antitone involution in [4], [1], [2]. We are going to introduce similar ring-like structures for directoids with an antitone involution.

By a *D-quasiring* is meant an algebra $\mathcal{R}=(R;+,\cdot,0,1)$ of type $(2,2,0,0)$ satisfying the dentities

(Q1) $(x\cdot y)\cdot x=x\cdot y$;

(Q2) $y\cdot(x\cdot y)=x\cdot y$;

(Q3) $x\cdot((x\cdot y)\cdot z)=(x\cdot y)\cdot z$;

(Q4) $x\cdot 0=0$;

(Q5) $x\cdot 1=x$;

(Q6) $x+0=x$;

(Q7) $1+(1+x\cdot y)\cdot(1+y)=y$.

Remark 2. Due to (Q3) with $y=z=1$ and (Q5), we obtain immediately that a D-quasiring satisfies the identity

(I) $$x\cdot x=x.$$

Hence, for every D-quasiring $\mathcal{R} = (R; +, \cdot, 0, 1)$, $(R; \cdot, 0, 1)$ is a bounded directoid with 0 and 1, thus $\mathcal{R}$ may be considered as a partially ordered set $(R; \leq)$ with smallest element 0 and greatest element 1 where $\leq$ is the induced order of $(R; \cdot, 0, 1,)$ i.e. for every $x, y \in R$, the order $\leq$ is defined by $x \leq y$ if and only if $x \cdot y = x$.

Lemma 2. *Let $(R; +, \cdot, 0, 1)$ be a D-quasiring. Then $x \mapsto 1 + x$ is an antitone involution on R.*

Proof: Denote by $x' = x + 1$. If we put $x = y$ in (Q7) and apply (I), we obtain the identity

$$1 + (1 + x) = x \tag{N}$$

proving that $x'' = x$. Suppose $x \leq y$, i.e. $x = x \cdot y$. Then, from (Q7), we have

$$1 + (1 + x) \cdot (1 + y) = y,$$

whence

$$(1 + (1 + x) \cdot (1 + y))' = y',$$

i.e.

$$1 + (1 + (1 + x) \cdot (1 + y)) = 1 + y.$$

By (N) we obtain

$$(1 + x) \cdot (1 + y) = 1 + y$$

which yields $(1 + y) \leq (1 + x)$, i.e. $y' \leq x'$. Thus the operation $'$ is an antitone involution on R. $\square$

Theorem 4. *Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a D-quasiring. Define*

$$x \sqcap y = x \cdot y, \quad x' = 1 + x \quad \textit{and} \quad x \sqcup y = 1 + (1 + x) \cdot (1 + y).$$

Then $\mathcal{D}(R) = (R; \sqcap, ', 0, 1)$ is a bounded directoid with an antitone involution where $\sqcup$ is the assigned operation.

Proof: As mentioned in Remark 2, $(R; \sqcap, 0, 1)$ is a bounded directoid. By Lemma 2, $'$ is an antitone involution on R. Further, using (N), we compute

$$x' \sqcup y' = 1 + (1 + x') \cdot (1 + y') = 1 + x \cdot y = (x \sqcap y)'$$

and

$$x' \sqcap y' = (1 + x) \cdot (1 + y) = 1 + (1 + (1 + x) \cdot (1 + y)) = (x \sqcup y)',$$

thus $\mathcal{D}(R)$ satisfies De Morgan laws and hence $\sqcup$ is the assigned operation. $\square$

Theorem 5. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Define*

$$x + y = (x \sqcup y) \sqcap (x \sqcap y)' \quad \text{and} \quad x \cdot y = x \sqcap y.$$

Then $\mathcal{R}(D) = (D; +, \cdot, 0, 1)$ is a D-quasiring. Moreover, $\mathcal{R}(D)$ satisfies the following correspondence identity

(Cor1) $$(1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = x + y.$$

Proof: Since $(D; \sqcap, 0, 1)$ is a bounded $\sqcap$-directoid, the identities (Q1)–(Q5) hold. The identity (Q6) is evident. Evidently, $1 + x = (1 \sqcup x) \sqcap (1 \sqcap x)' = 1 \sqcap x' = x'$. For (Q7) we use the properties of an antitone involution to compute

$$1 + (1 + x \cdot y) \cdot (1 + y) = ((x \sqcap y)' \sqcap y')' = y'' = y.$$

Using the De Morgan laws we obtain

$$\begin{aligned}(1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) &= (x' \sqcap y')' \sqcap (x \sqcap y)' \\ &= (x \sqcup y) \sqcap (x \sqcap y)' = x + y\end{aligned}$$

which is just the identity (Cor1). □

Theorem 6. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a bounded directoid with an antitone involution. Then $\mathcal{D}(\mathcal{R}(D)) = \mathcal{D}$.*
Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a D-quasiring satisfying the correspondence identity (Cor1). *Then $\mathcal{R}(\mathcal{D}(R)) = \mathcal{R}$.*

Proof: Evidently, the operation meet coincides in both $\mathcal{D}(\mathcal{R}(D))$ and $\mathcal{D}$. Hence, it remains to prove $\cup = \sqcup$ and $x^\star = x'$ where $\cup$ is the binary operation and * the antitone involution of $\mathcal{D}(\mathcal{R}(D))$. We have

$$x^\star = 1 + x = (1 \sqcup x) \sqcap (1 \sqcap x)' = 1 \sqcap x' = x'$$

and

$$x \cup y = 1 + (1 + x) \cdot (1 + y) = (x' \sqcap y')' = x \sqcup y.$$

Analogously, the multiplicative operations coincide in the both $\mathcal{R}(\mathcal{D}(R))$ and $\mathcal{R}$. To prove $\mathcal{R}(\mathcal{D}(R)) = \mathcal{R}$ we need only to show that also $\oplus = +$ where $\oplus$ is the additive operation in $\mathcal{R}(\mathcal{D}(R))$. Applying (Cor1) we compute

$$x \oplus y = (x \sqcup y) \sqcap (x \sqcap y)' = (1 + (1 + x) \cdot (1 + y)) \cdot (1 + x \cdot y) = x + y.$$

□

Example 3. Consider the $\sqcap$ directoid $\mathcal{D}$ with an antitone involution $'$ and assigned operation $\sqcup$ from Example 1 (see Figure 1).

The operation tables of the D-quasiring $\mathcal{R}(D)$ corresponding to $\mathcal{D}$ are as follows (see Theorem 5):

$\cdot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	a	c
d	0	a	b	b	d	d
1	0	a	b	c	d	1

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	a	d	d
b	b	d	b	c	b	c
c	c	a	c	b	d	b
d	d	d	b	c	a	a
1	1	d	c	b	a	0

Note that $\cdot$ and $+$ are not commutative. $\Diamond$

Remark 3. Let us consider the directoid $\mathcal{D} = (D; \sqcap, ', 0, 1)$ of Example 1. One can pick $a \sqcup b = d$ and $b \sqcup a = c$ (and trivially for comparable elements). The resulting structure $(D; \sqcup)$ is clearly a $\sqcup$-directoid again but $\sqcup$ is not the assigned operation of $\mathcal{D}$. Evidently, the De Morgan laws are not satisfied. On the contrary the structure $\mathcal{L} = (D; \sqcup, \sqcap, ', 0, 1)$ still induces a D-quasiring $\mathcal{R}(\mathcal{L})$ via $x \cdot y = x \sqcap y$ and $x + y = (x \sqcup y) \sqcap (x \sqcap y)'$. However, (Cor1) is not satisfied and hence $\mathcal{R} \neq \mathcal{R}(\mathcal{L}(R))$.

3. Symmetrical difference

Definition 1. Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Let $a, b \in D$. The element a is called a *complement* of b if $a \sqcap b = 0$ and $a \sqcup b = 1$.

Remark 4. If a is a complement of b then b need not be a complement of a; see the following

Example 4. A bounded $\sqcap$-directoid with an antitone involution $'$ is depicted in Figure 3 where $c \sqcap d = a$, $d \sqcap c = 0$ and $0' = 1$, $a' = d$, $b' = c$.
Then a is a complement of b but b is not a complement of a since

$$a \sqcup b = (a' \sqcap b')' = (d \sqcap c)' = 0' = 1,$$

but

$$b \sqcup a = (b' \sqcap a')' = (c \sqcap d)' = a' = d.$$

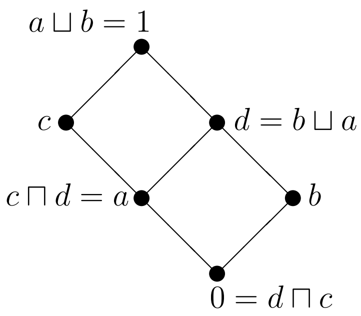

FIGURE 3

Analogously, d is a complement of c but not vice versa. On the other hand, b is a complement of c and c is a complement of b. Of course, 0 is a complement of 1 and 1 is a complement of 0. $\Diamond$

Lemma 3. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Let $\mathcal{R}(D) = (D; +, \cdot, 0, 1)$ be the induced D-quasiring. Then*

(a) $a + b = 1$ *if and only if* a *is a complement of* b*;*
(b) $a + b = a \sqcup b$ *if and only if* $a \sqcup b \leq a' \sqcup b'$*;*
(c) *if* $a \leq b$ *then* $a + b = b \sqcap a'$.

PROOF: (a) Assume $a + b = 1$. Then $(a \sqcup b) \sqcap (a \sqcap b)' = 1$, i.e. $a \sqcup b = 1$ and $(a \sqcap b)' = 1$, hence $a \sqcap b = 0$ thus a is a complement of b. The converse is trivial.

(b) If $a \sqcup b = a + b = (a \sqcup b) \sqcap (a \sqcap b)'$ then $a \sqcup b \leq (a \sqcap b)' = a' \sqcup b'$. The converse is evident.

(c) If $a \leq b$ then $a + b = (a \sqcup b) \sqcap (a \sqcap b)' = b \sqcap a'$. $\square$

Definition 2. Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. By a *symmetrical difference* of x, y is meant the term function

$$x \triangle y = (x' \sqcap y) \sqcup (x \sqcap y').$$

We can get a mutual relationship between the symmetrical difference and the operation $+$ of the induced D-quasiring as follows:

Lemma 4. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Then $x \triangle y = (x + y')'$ and $x + y = (x \triangle y')'$.*

PROOF: Using the De Morgan laws, we infer directly

$$(x \triangle y')' = ((x' \sqcap y') \sqcup (x \sqcap y))' = (x \sqcup y) \sqcap (x \sqcap y)' = x + y$$

and

$$(x+y')' = ((x \sqcup y') \sqcap (x \sqcap y')')' = (x \sqcup y')' \sqcup (x \sqcap y')$$
$$= (x' \sqcap y) \sqcup (x \sqcap y') = x \triangle y.$$

□

Lemma 5. *Let $\mathcal{D} = (D; \sqcap,', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Then*

(a) *$x \triangle y = 0$ if and only if x' is a complement of y;*
(b) *$x \triangle x = 0$ if and only if $x' \triangle x' = 0$ if and only if x' is a complement of x;*
(c) *$1 \triangle x = x \triangle 1 = x'$.*

Proof: (a) Assume $x \triangle y = 0$. Then $(x' \sqcap y) \sqcup (x \sqcap y') = 0$ thus also $x' \sqcap y = 0$ and $x \sqcap y' = 0$, whence $x' \sqcup y = (x \sqcap y')' = 0' = 1$, i.e. x' is a complement of y. Conversely, if x' is a complement of y then $x' \sqcap y = 0$ and $x' \sqcup y = 1$, i.e. $x \sqcap y' = (x' \sqcup y)' = 1' = 0$ and hence $x \triangle y = 0$.

(b) The first implication follows directly from the definition of symmetrical difference and (a) immediately yields the second.

(c) $1 \triangle x = (1' \sqcap x) \sqcup (1 \sqcap x') = x'$; analogously $x \triangle 1 = x'$. □

We are able to show that the symmetrical difference can also serve as an additive operation in a certain induced D-quasiring.

Theorem 7. *Let $\mathcal{D} = (D; \sqcap,', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Let $\triangle$ be the symmetric difference. Then $\mathcal{R}^*(D) = (D; \triangle, \sqcap, 0, 1)$ is a D-quasiring.*

Proof: It is trivial to verify the axioms (Q1)–(Q5). For (Q6) we have

$$x \triangle 0 = (x' \sqcap 0) \sqcup (x \sqcap 0') = 0 \sqcup x = x.$$

It remains to prove (Q7). By Lemma 5 (c) we have

$$1 \triangle (1 \triangle (x \sqcap y)) \sqcap (1 \triangle y) = ((x \sqcap y)' \sqcap y')' = y'' = y.$$

□

Lemma 6. *Let $\mathcal{D} = (D; \sqcap,', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. The D-quasiring $\mathcal{R}^*(D) = (D; \triangle, \cdot, 0, 1)$ with $x \cdot y = x \sqcap y$ satisfies the identity*

(Cor2) $$1 \triangle (1 \triangle (1 \triangle x) \cdot y) \cdot (1 \triangle x \cdot (1 \triangle y)) = x \triangle y.$$

Proof: By using Lemma 5 (c) and the De Morgan laws we compute

$$1\triangle(1\triangle(1\triangle x)\cdot y)\cdot(1\triangle x\cdot(1\triangle y)) = ((x'\sqcap y)'\sqcap(x\sqcap y')')'$$
$$= (x'\sqcap y)\sqcup(x\sqcap y') = x\triangle y.$$

□

The following result is a counterpart of Theorem 6 and can be proved analogously:

Theorem 8. *Let $\mathcal{D}=(D;\sqcap,',0,1)$ be a directoid with an antitone involution, $\sqcup$ the assigned operation and $\triangle$ the symmetrical difference. Then $\mathcal{D}(\mathcal{R}^*(D))=\mathcal{D}$. Let $\mathcal{R}=(R;\triangle,\cdot,0,1)$ be a D-quasiring satisfying* (Cor2). *Then $\mathcal{R}^*(\mathcal{D}(R))=\mathcal{R}$.*

4. A decompositions of directoids

Define $a\,C\,b$ if $b=(b\sqcap a)\sqcup(b\sqcap a')$. An element $a\in D$ is called **central** if $a\,C\,x$ and $a'\,C\,x$ for each $x\in D$. Denote by $C(D)$ the set of all central elements of a directoid $\mathcal{D}=(D;\sqcap,',0,1)$. Hence,

(C) $\quad a\in C(D)\quad$ iff $\quad x=(x\sqcap a)\sqcup(x\sqcap a')=(x\sqcap a')\sqcup(x\sqcap a)$

for each $x\in D$.

Lemma 7. *Let $\mathcal{D}=(D;\sqcap,',0,1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Then*

(a) *if $b\le a$ then $a\,C\,b$;*
(b) $0,1\in C(D)$*;*
(c) *if $a\in C(D)$ then a' is a complement of a and a is a complement of a';*
(d) *if $a\in C(D)$ then*

$$(x\sqcup a')\sqcap(x\sqcup a)=x=(x\sqcup a)\sqcap(x\sqcup a')$$

for each $x\in D$.

Proof: (a) If $b\le a$ then $(b\sqcap a)\sqcup(b\sqcap a')=b\sqcup(b\sqcap a')=b$.

(b) Of course, $x=(x\sqcap 1)\sqcup(x\sqcap 0)=(x\sqcap 0)\sqcup(x\sqcap 1)$ for each $x\in D$.

(c) Take $x=1$ in (C). Then

$$1=(1\sqcap a)\sqcup(1\sqcap a')=a\sqcup a'$$

and

$$1=(1\sqcap a')\sqcup(1\sqcap a)=a'\sqcup a.$$

Due to De Morgan laws, we have that a' is a complement of a and vice versa.

(d) We compute

$$(x\sqcup a')\sqcap(x\sqcup a)=(x'\sqcap a)'\sqcap(x'\sqcap a')'=((x'\sqcap a)\sqcup(x'\sqcap a'))'=x''=x.$$

The second equation can be shown analogously. □

Definition 3. Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Denote by $\mathrm{Is}\,(D)$ the set of all elements $a \in D$ such that

(i) $(x \sqcap y) \sqcap a = (x \sqcap a) \sqcap (y \sqcap a)$, $(x \sqcap y) \sqcap a' = (x \sqcap a') \sqcap (y \sqcap a')$;
(ii) $(x \sqcup y) \sqcap a = (x \sqcap a) \sqcup (y \sqcap a)$, $(x \sqcup y) \sqcap a' = (x \sqcap a') \sqcup (y \sqcap a')$.

It is clear that $0, 1 \in \mathrm{Is}\,(D)$ in any case.

Remark 5. It is immediate that $a \in \mathrm{Is}\,(D)$ if and only if $a' \in \mathrm{Is}\,(D)$ and $a \in C(D)$ if and only if $a' \in C(D)$.

Lemma 8. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Then*

(a) *if $a \in \mathrm{Is}\,(D)$ then $x \leq y \Rightarrow x \sqcap a \leq y \sqcap a$;*
(b) *if $a \in C(D) \cap \mathrm{Is}\,(D)$ then*

$$(x \sqcap a)' \sqcap a = x' \sqcap a \quad \textit{and} \quad (x \sqcap a')' \sqcap a' = x' \sqcap a'.$$

Proof: (a) If $x \leq y$ then $x \sqcap y = x$ and, by (i) of Definition 3, $x \sqcap a = (x \sqcap y) \sqcap a = (x \sqcap a) \sqcap (y \sqcap a)$ thus $x \sqcap a \leq y \sqcap a$.

(b) Of course, $(x \sqcap a)' \sqcap a = (x' \sqcup a') \sqcap a$. By (ii) of Definition 3, we have $(x' \sqcup a') \sqcap a = (x' \sqcap a) \sqcup (a' \sqcap a)$ and, due to Lemma 7(c), $a' \sqcap a = 0$. Hence $(x \sqcap a)' \sqcap a = x' \sqcap a$. The second equality is established similarly. □

Theorem 9. *Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be a directoid with an antitone involution and $\sqcup$ the assigned operation. Let $a \in C(D) \cap \mathrm{Is}\,(D)$. Define*

$$x^* = x' \sqcap a \quad \textit{and} \quad x^+ = x' \sqcap a'.$$

Then $\mathcal{D}_1 = ((a]; \sqcap, ^*, 0, a)$ and $\mathcal{D}_2 = ((a']; \sqcap, ^+, 0, a')$ are bounded directoids with an antitone involution and $\mathcal{D}$ is isomorphic to $\mathcal{D}_1 \times \mathcal{D}_2$ where the isomorphism is defined by $\varphi(x) = (x \sqcap a, x \sqcap a')$.

Conversely, let $\mathcal{D}$ be isomorphic with $\mathcal{D}_1 \times \mathcal{D}_2$ where $\mathcal{D}_1$, $\mathcal{D}_2$ are directoids with an antitone involution. Then there exists $a \in C(D) \cap \mathrm{Is}\,(D)$ such that $\mathcal{D}_1 \cong ((a], \sqcap, ^*, 0, a)$ and $\mathcal{D}_2 \cong ((a'], \sqcap, ^+, 0, a')$.

Proof: Evidently, if $x, y \in (a]$ then $x \sqcap y \leq x \leq a$ thus also $x \sqcap y \in (a]$, i.e. $((a]; \sqcap)$ is a directoid as well as $((a']; \sqcap)$.

Let $x \in (a]$. Then $x \leq a$, i.e. $x \sqcup a = a$ and, by Lemma 7(d),

$$x^{**} = (x' \sqcap a)' \sqcap a = (x \sqcup a') \sqcap (x \sqcup a) = x.$$

Thus $\mathcal{D}_1 = ((a]; \sqcap, ^*, 0, a)$ is a bounded directoid with the involution *. Since $x \leq y$ implies $y' \leq x'$ and $a \in \mathrm{Is}\,(D)$, also

$$y^* = y' \sqcap a \leq x' \sqcap a = x^*$$

by (a) of Lemma 8, thus this involution is antitone. Similarly it can be shown for $\mathcal{D}_2 = ((a'];\sqcap,^+,0,a')$.

Now, define $\varphi : D \to D_1 \times D_2$ by $\varphi(x) = (x \sqcap a, x \sqcap a')$. Moreover, define $\psi : D_1 \times D_2 \to D$ by $\psi((x,y)) = x \sqcup y$. Since $a \in C(D)$, we infer

$$\psi(\varphi(x)) = (x \sqcap a) \sqcup (x \sqcap a') = x,$$

i.e., φ is an injective mapping. Suppose $(x,y) \in D_1 \times D_2$. Then $x \leq a$, $y \leq a'$ and by (ii) of Definition 3, we have

$$\begin{aligned}\varphi(\psi((x,y))) &= \varphi(x \sqcup y) = ((x \sqcup y) \sqcap a, (x \sqcup y) \sqcap a') \\ &= ((x \sqcap a) \sqcup (y \sqcap a), (x \sqcap a') \sqcup (y \sqcap a')) = (x \sqcup (y \sqcap a), (x \sqcap a') \sqcup y).\end{aligned}$$

Since $a, a' \in \mathrm{Is}(D)$, $y \leq a'$ we obtain (according to (a) of Lemma 8) that

$$y \sqcap a \leq a' \sqcap a = 0$$

and therefore $y \sqcap a = 0$. Analogously, $x \sqcap a' = 0$. Hence, $\varphi(\psi((x,y))) = (x \sqcup 0, 0 \sqcup y) = (x,y)$. Thus, φ is a bijection and $\psi = \varphi^{-1}$.

It remains to prove that φ is a homomorphism. Clearly,

$$\begin{aligned}&\varphi(b) \sqcap \varphi(c) = (b \sqcap a, b \sqcap a') \sqcap (c \sqcap a, c \sqcap a') \\ &= ((b \sqcap a) \sqcap (c \sqcap a), (b \sqcap a') \sqcap (c \sqcap a')) = ((b \sqcap c) \sqcap a, (b \sqcap c) \sqcap a') = \varphi(b \sqcap c)\end{aligned}$$

according to (i) of Definition 3. Further, using of Lemma 8(b), we obtain

$$\begin{aligned}\varphi(b)' &= (b \sqcap a, b \sqcap a')' = ((b \sqcap a)^*, (b \sqcap a')^+) \\ &= ((b \sqcap a)' \sqcap a, (b \sqcap a')' \sqcap a') = (b' \sqcap a, b' \sqcap a') = \varphi(b').\end{aligned}$$

Hence, φ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}_1 \times \mathcal{D}_2$.

Conversely, let $\mathcal{D}_1 = (D;\sqcap,^*,0_1,1_1)$ and $\mathcal{D}_2 = (D;\sqcap,^+,0_2,1_2)$ be directoids with antitone involutions and $\mathcal{D}$ is isomorphic to $\mathcal{D}_1 \times \mathcal{D}_2$. It is an easy exercise to verify that elements $a = (1_1, 0_2)$ and $(0_1, 1_2)$ belong to $C(D_1 \times D_2) \cap \mathrm{Is}(D_1 \times D_2)$ and $(0_1, 1_2) = a'$ in $\mathcal{D}_1 \times \mathcal{D}_2$. Of course, $\mathcal{D}_1 \cong \overline{\mathcal{D}}_1 = ((a];\sqcap,^*,(0_1,0_2),a)$ and $\mathcal{D}_2 \cong \overline{\mathcal{D}}_2 = ((a'];\sqcap,^+,(0_1,0_2),a')$ and hence also $\mathcal{D} \cong \overline{\mathcal{D}}_1 \times \overline{\mathcal{D}}_2$. $\square$

Remark 6. If $\mathcal{D} = (D;\sqcap,',0,1)$ is a semilattice with an antitone involution then every element satisfies (i) of Definition 3 and (a) of Lemma 8.

Example 5. Let $\mathcal{D} = (D; \sqcap, ', 0, 1)$ be the $\sqcap$-directoid with an antitone involution as shown in Example 4 (see Figure 3). Let $\sqcup$ be its assigned operation. Then $b \notin C(D)$ and $c \notin C(D)$, because

$$d \neq (d \sqcap b) \sqcup (d \sqcap b') = b \sqcup 0 = b$$

and

$$d \neq (d \sqcap c) \sqcup (d \sqcap c') = 0 \sqcup b = b.$$

Due to Lemma 7(c) also $a \notin C(D)$, $d \notin C(D)$. Further, elements c and d do not belongs to $\mathrm{Is}\,(D)$, since

$$a = a \sqcap c = (a \sqcap d) \sqcap c \neq (a \sqcap c) \sqcap (d \sqcap c) = a \sqcap 0 = 0$$

and

$$d = 1 \sqcap d = (a \sqcup b) \sqcap d \neq (a \sqcap d) \sqcup (b \sqcap d) = a \sqcup b = 1.$$

Hence also $b = c' \notin \mathrm{Is}\,(D)$ and $a = d' \notin \mathrm{Is}\,(D)$. Thus $C(D) = \mathrm{Is}\,(D) = \{0, 1\}$.

On the contrary, let Figure 3 be now the Hasse diagram of the lattice $\mathcal{L} = (L; \wedge, \vee)$ with a two binary operations join and meet. Then $\mathcal{L}$ is as a direct product of the two-element and three-element chains.

For the non-trivial decomposition of directoid let us see the following

Example 6. Consider the $\sqcap$-directoid $\mathcal{D} = (D; \sqcap)$ whose diagram is drawn in Figure 4 where $m \sqcap n = k$, $n \sqcap m = l$, $s \sqcap t = q$, $t \sqcap s = r$ and trivially for the other couples.

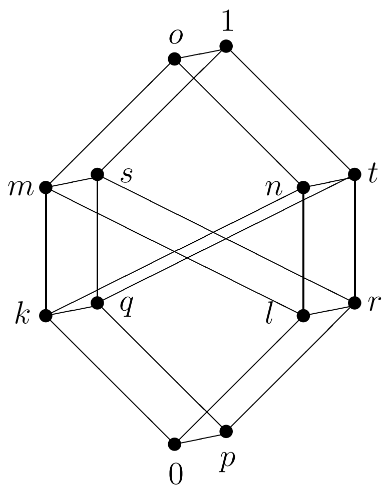

FIGURE 4

Define an antitone involution $x \mapsto x'$ on D as follows

$$\begin{array}{c||cccccc} x & 0 & k & l & p & q & r \\ \hline x' & 1 & t & s & o & n & m \end{array}.$$

One can easily check that $a = p$, $a' = o \in C(D) \cap \mathrm{Is}(D)$. Therefore, $\mathcal{D} \cong \mathcal{D}_1 \times \mathcal{D}_2$ for $\mathcal{D}_1 = ((a], \sqcap, ^*, 0, a)$ and $\mathcal{D}_2 = ((a'], \sqcap, ^+, 0, a')$.

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