# On the lattices of quasivarieties of differential groupoids

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*Abstract.* The main result of Romanowska A., Roszkowska B., *On some groupoid modes*, Demonstratio Math. **20** (1987), no. 1–2, 277–290, provides us with an explicit description of the lattice of varieties of differential groupoids. In the present article, we show that this variety is $\mathcal{Q}$-universal, which means that there is no convenient explicit description for the lattice of quasivarieties of differential groupoids. We also find an example of a subvariety of differential groupoids with a finite number of subquasivarieties.

*Keywords:* mode, differential groupoid, lattice of subquasivarieties, $\mathcal{Q}$-universal quasivariety

*Classification:* 08C15, 20N02

## Introduction

A *differential groupoid* is a structure with one fundamental binary operation satisfying the identities

(I) $$x \cdot x = x,$$

(E) $$(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t),$$

(D) $$x \cdot (x \cdot y) = x.$$

Let **Dm** denote the variety of differential groupoids.

Many authors use the term *medial* groupoid instead of *entropic*, i.e., satisfying (E), see [3]. Differential groupoids were studied in [5]–[7], where they were called LIR-*groupoids* (*left normal, idempotent, and reductive groupoids*) and a different basis for identities was used. The term *differential groupoid* appeared in [8]. For more information, the reader is referred to the monograph [9].

For $i \geqslant 0$ and $n > 0$, let $\mathbf{D}_{i,n}$ denote the subvariety of **Dm** defined by the identity

(1) $$xy^{i+n} = xy^i,$$

where $xy^k = (\ldots((x \cdot \underbrace{y) \cdot y) \ldots) \cdot y}_{k \text{ times}}$. The structure of the lattice $\mathrm{L}_{\mathcal{V}}(\mathbf{Dm})$ of subvarieties of **Dm** is described by [6, Theorem 5.3], cf. also [9, Theorem 8.4.14].

The work was partially supported by INTAS (grant 03-51-4110) and the Russian Council for Support of Leading Scientific Schools (grant NSh-4787.2006.1).

**Proposition 1.** *Let $\mathbb{N}_c$ denote the lattice of natural numbers with the usual order and let $\mathbb{N}_d$ denote the lattice of positive integers ordered by the divisibility relation.*

*Proper subvarieties of* **Dm** *form a lattice which is isomorphic to the direct product $\mathbb{N}_c \times \mathbb{N}_d$. Moreover, a pair $(i, n)$ corresponds to the variety $\mathbf{D}_{i,n}$.*

A quasivariety **K** of groupoids is said to be $\mathcal{Q}$-*universal* if, for every quasivariety $\mathbf{K}'$ of structures of finite type, the lattice $\mathsf{L}_\mathsf{q}(\mathbf{K}')$ of subquasivarieties of $\mathbf{K}'$ is a homomorphic image of some sublattice of the lattice $\mathsf{L}_\mathsf{q}(\mathbf{K})$ of subquasivarieties of **K**. For every $\mathcal{Q}$-universal quasivariety **K**, the lattice $\mathsf{L}_\mathsf{q}(\mathbf{K})$ is highly complicated. Namely, $|\mathsf{L}_\mathsf{q}(\mathbf{K})| = 2^\omega$; moreover, this lattice satisfies no nontrivial lattice identity and contains a sublattice that is isomorphic to the ideal lattice of a free $\omega$-generated lattice.

In Section 1, we prove that the variety **Dm** is $\mathcal{Q}$-universal. This shows that there is no convenient description for the lattice $\mathsf{L}_\mathsf{q}(\mathbf{Dm})$. The following question naturally arises: Which proper subvarieties of differential groupoids are $\mathcal{Q}$-universal? In Section 2, we show that $\mathbf{D}_{1,1}$ is not $\mathcal{Q}$-universal.

## 1. The variety Dm is $\mathcal{Q}$-universal

We use the standard notation for class operators. Namely, **Q** stands for taking the least quasivariety containing a given class, while $\mathbf{P_s}$, **S**, and **H** stand for formation of subdirect products, subgroupoids, and homomorphic images, respectively. For every class operator **O** and classes **X** and **K**, we denote by $(\mathbf{O} \cap \mathbf{K})(\mathbf{X})$ the class $\mathbf{O}(\mathbf{X}) \cap \mathbf{K}$.

Our proof is based on the following sufficient condition for $\mathcal{Q}$-universality (cf. [2, Theorem 5.4.26]).

**Proposition 2.** *A quasivariety* **K** *of groupoids is $\mathcal{Q}$-universal if there exist a subclass* **B** *of* **K** *and a family $(\mathcal{A}_i)_{i<\omega}$ of finite groupoids in* **B** *such that the following conditions are satisfied.*

(Q1) *For every $n < \omega$ and* **B***-congruences $\theta$ and $\theta'$ on $\mathcal{A}_n$, if $\mathcal{A}_n/\theta'$ is embeddable into $\mathcal{A}_n/\theta$ then either $\theta = \theta'$ or $\mathcal{A}_n/\theta'$ is a trivial groupoid.*

(Q2) *For every $n < \omega$, the meet semilattice $L_n$ of* **B***-congruences on $\mathcal{A}_n$ is a subsemilattice of the meet semilattice of congruences on $\mathcal{A}_n$. Moreover, the meet semilattice of subsets of an $n$-element set is embeddable into $L_n$.*

(Q3) *If $m \neq n$ then the class $\mathbf{A}_n \cap \mathbf{S}(\mathbf{A}_m)$, where $\mathbf{A}_n = \mathbf{H}(\mathcal{A}_n) \cap \mathbf{B}$, consists of trivial groupoids only.*

(Q4) *For every $\mathbf{X} \subseteq \mathbf{K}$ and $n < \omega$, we have*

$$\mathbf{Q}(\mathbf{X}) \cap \mathbf{A}_n = (\mathbf{P_s} \cap \mathbf{A}_n)(\mathbf{S} \cap \mathbf{A}_n)(\mathbf{X}).$$

For more information on $\mathcal{Q}$-universal quasivarieties, the reader is referred to [1, Section 5].

Recall that a groupoid $G$ is called a *left zero band* if $G$ satisfies the identity $x \cdot y = x$, i.e., if $G \in \mathbf{D}_{0,1}$. We say that a groupoid $G$ is an **Lz**-**Lz**-*sum* (of left zero bands $G_i$ over a left zero band $I$) *satisfying the left normal law* if there exists a partition $G = \bigcup_{i \in I} G_i$ and, for every pair $(i,j) \in I^2$, there exists a map $h_{ij} : G_i \to G_i$ such that the following conditions are satisfied:

(i) $h_{ii}$ is the identity map for every $i \in I$,
(ii) $h_{ij}(h_{ik}(x)) = h_{ik}(h_{ij}(x))$ for all $i,j,k \in I$ and $x \in G_i$,
(iii) $a_i \cdot a_j = h_{ij}(a_i)$ for all $i,j \in I$, $a_i \in G_i$, and $a_j \in G_j$.

The structure of differential groupoids was completely described in [6, Section 2], cf. also [4, 5, 7]. Namely, we have $G \in \mathbf{Dm}$ if and only if $G$ is an **Lz**-**Lz**-sum satisfying the left normal law.

Let $\mathcal{C}_0$ denote the trivial groupoid whose universe is $\{\infty\}$. For every $n > 0$, let $\mathcal{C}_n$ denote the **Lz**-**Lz**-sum of $G_1 = \{0, 1, \dots, n-1\}$ and $G_2 = \{\infty\}$, where $h_{12}(k) \equiv k + 1 \pmod{n}$ and $h_{21}$ is the identity map. We have $\mathcal{C}_n \in \mathbf{Dm}$ for each $n \geqslant 0$.

We describe congruences on the constructed groupoids. Let $m$ divide $n$. For every $k < n$, let $r_k$ denote the remainder in the division of $k$ by $m$. It is easy to see that the map defined by the rule

$$\infty \mapsto \infty, \quad k \mapsto r_k$$

is a homomorphism from $\mathcal{C}_n$ onto $\mathcal{C}_m$. Let $\theta_m$ denote the kernel of this homomorphism.

**Lemma 3.** *Let $n > 0$ and let $\theta$ be a congruence on $\mathcal{C}_n$. Then either $\mathcal{C}_n/\theta$ is a trivial groupoid or $\theta = \theta_m$ for some divisor $m$ of $n$.*

Proof: If $(\infty, k) \in \theta$, where $0 \leqslant k < n$, then, as in [4, p. 378], we find that $\mathcal{C}_n/\theta$ is a trivial groupoid. If $(\infty, k) \notin \theta$ for all $k$ with $0 \leqslant k < n$ then $\theta \leqslant \theta_1$. By [7, Propositions 2.2 and 2.5], we conclude that the restriction of $\theta$ to $G_1$ is a congruence on a cyclic abelian group of order $n$. Hence, $\theta = \theta_m$ for some $m$ dividing $n$. □

Let $\mathbf{B}$ denote the subclass of $\mathbf{Dm}$ consisting of trivial groupoids and differential groupoids that are not left zero bands. We have $\mathcal{C}_n \in \mathbf{B}$ if and only if $n \neq 1$.

Let $\mathbb{P}$ denote the set of prime numbers. Consider a partition $\mathbb{P} = \bigcup_{i<\omega} P_i$ with $|P_i| = i + 1$ for all $i < \omega$ and $P_i \cap P_j = \emptyset$ for all $i \neq j$. Let $k_i = \prod_{p \in P_i} p$. Put $\mathcal{A}_i = \mathcal{C}_{k_i}$ for $i < \omega$.

**Theorem 4.** *The class $\mathbf{B}$ and the family $(\mathcal{A}_i)_{i<\omega}$ satisfy conditions* (Q1)–(Q4) *of Proposition* 2. *Hence,* $\mathbf{Dm}$ *is a $\mathcal{Q}$-universal quasivariety.*

Proof: We have $(\mathcal{A}_i)_{i<\omega} \subseteq \mathbf{B}$. It is easy to see that, for $i,j < \omega$, the groupoid $\mathcal{C}_i$ is embeddable into the groupoid $\mathcal{C}_j$ if and only if $i = j$. By Lemma 3, this immediately implies (Q1) and (Q3). Since $L_i$ is obtained from the meet semilattice of congruences on $\mathcal{A}_i$ by removing the congruence $\theta_1$, we also obtain (Q2).

We prove (Q4). Let $\mathbf{X} \subseteq \mathbf{Dm}$ and let $n < \omega$. The inclusion $\mathbf{Q}(\mathbf{X}) \cap \mathbf{A}_n \supseteq (\mathbf{P_s} \cap \mathbf{A}_n)(\mathbf{S} \cap \mathbf{A}_n)(\mathbf{X})$ is obvious.

Consider a nontrivial groupoid $\mathcal{B} \in \mathbf{Q}(\mathbf{X}) \cap \mathbf{A}_n$. By [2, Corollary 2.3.4], we have $\mathbf{Q}(\mathbf{X}) = \mathbf{SP_uP}(\mathbf{X})$, where $\mathbf{P}$ and $\mathbf{P_u}$ are the class operators for formation of direct products and ultraproducts. Hence, there exists a family $(\mathcal{B}_i)_{i \in I}$ of groupoids and an ultrafilter $U$ over $I$ such that $\mathcal{B}$ is a subgroupoid of the ultraproduct $\prod_{i \in I} \mathcal{B}_i/U$. Moreover, each $\mathcal{B}_i$ is the direct product of a family $(\mathcal{B}_{ij})_{j \in I_i}$ of groupoids in $\mathbf{X}$.

Since $\mathcal{B}$ is a homomorphic image of the finite groupoid $\mathcal{A}_n$, we conclude that $\mathcal{B}$ is a finite groupoid too. There exists a first-order sentence $\varphi$ such that, for every groupoid $\mathcal{X}$, the following two conditions are equivalent: (a) $\mathcal{X}$ satisfies $\varphi$; (b) $\mathcal{B}$ is embeddable into $\mathcal{X}$. In particular, $\prod_{i \in I} \mathcal{B}_i/U$ satisfies $\varphi$. By the Łoś Theorem, there exists an $i \in I$ such that $\mathcal{B}_i$ satisfies $\varphi$. Hence, there exists an embedding $\alpha : \mathcal{B} \to \mathcal{B}_i$.

Let $\pi_j : \prod_{j \in I_i} \mathcal{B}_{ij} \to \mathcal{B}_{ij}$ be the $j$th projection map. Denote by $\psi_j$ the composition $\pi_j \circ \alpha$ of homomorphisms. For every $j \in I_i$, let $\mathcal{G}_j$ be the homomorphic image of $\mathcal{B}$ with respect to $\psi_j$. Then $\mathcal{G}_j$ is a subgroupoid of $\mathcal{B}_{ij}$ and a homomorphic image of $\mathcal{A}_n$.

We show that $\mathcal{B}$ is a subdirect product of the family $(\mathcal{G}_j)_{j \in I_i}$, i.e., if $x, y \in B$ and $x \neq y$ then there exists a $j \in I_i$ such that $\psi_j(x) \neq \psi_j(y)$ (or, which is equivalent, $\bigcap_{j \in I_i} \ker \psi_j$ is the equality relation $\Delta_B$ on $B$). Indeed, since $\alpha$ is an embedding, we have $\alpha(x) \neq \alpha(y)$. Since each $\pi_j$, $j \in I_i$, is a projection, we have $\psi_j(x) = \pi_j(\alpha(x)) \neq \pi_j(\alpha(y)) = \psi_j(y)$ for at least one $j \in I_i$.

Let $J = \{j \in I_i : \mathcal{G}_j \notin \mathbf{D}_{0,1}\}$. If $J = \emptyset$ then $\mathcal{B}$ is a left zero band, a contradiction. By Lemma 3, we have $\ker \psi_j \subseteq \ker \psi_k$ for all $j \in J$ and $k \in I_i \setminus J$. Hence $\bigcap_{j \in J} \ker \psi_j = \bigcap_{j \in I_i} \ker \psi_j = \Delta_B$. Therefore, $\mathcal{B}$ is a subdirect product of the family $(\mathcal{G}_j)_{j \in J} \subseteq \mathbf{B}$. Consequently, $\mathcal{B} \in (\mathbf{P_s} \cap \mathbf{A}_n)(\mathbf{S} \cap \mathbf{A}_n)(\mathbf{X})$. $\square$

## 2. The variety $\mathbf{D}_{1,1}$ is not $\mathcal{Q}$-universal

In this section, we find subdirectly irreducible groupoids in $\mathbf{D}_{1,1}$ and show that the lattice $\mathrm{L_q}(\mathbf{D}_{1,1})$ is finite.

For $i = n = 1$, identity (1) has the following form:

$$xy^2 = xy. \tag{1'}$$

Define a relation $\leqslant$ on $G$ as follows:

$$a \leqslant b \iff b = ax_1 \dots x_n \text{ for some } x_1, \dots, x_n \in G,$$

where, $ax_1 \dots x_n = (\dots((a \cdot x_1) \cdot x_2) \dots \cdot x_n)$. Using the left normal law

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y \tag{L}$$

(see [9, Proposition 5.6.2]) and ($1'$), it is easy to check that the relation $\leqslant$ is a partial order on $G$ and

$$x \leqslant y \ \text{ implies } \ xz \leqslant yz \tag{2}$$

for all $x, y, z \in G$.

Assume that $G$ is a finite groupoid. Let $M$ denote the set of maximal elements with respect to the order $\leqslant$ and, for every $m \in M$, let $G_m$ denote the order ideal generated by $m$ (or the *orbit* of $m$). It is easy to see that $m_1 \neq m_2$ implies that $G_{m_1} \cap G_{m_2} = \emptyset$.

As in [9, p. 537] (cf. also [5]), let $\beta$ denote the congruence on $G$ defined as follows:

$$(a, b) \in \beta \iff a, b \in G_m \ \text{ for some } \ m \in M.$$

Then $G$ is an **Lz**-**Lz**-sum of its $\beta$-orbits.

Let $\mathcal{G}_0$ denote the two-element left zero band with the universe $\{0, 1\}$. Let $\mathcal{G}_1$ denote the **Lz**-**Lz**-sum of $\beta$-orbits $\{0, 1\}$ and $\{2\}$, where $0 < 1$, i.e., $0 \cdot 2 = 1$ and $x \cdot y = x$ if the pair $(x, y)$ is different from $(0, 2)$.

**Theorem 5.** *A finite groupoid $G$ is subdirectly irreducible in $\mathbf{D}_{1,1}$ if and only if $G$ is isomorphic to either $\mathcal{G}_0$ or $\mathcal{G}_1$.*

PROOF: It is easy to see that $\mathcal{G}_0$ and $\mathcal{G}_1$ are subdirectly irreducible in $\mathbf{D}_{1,1}$ because 0 and 1 cannot be separated by *proper* homomorphisms, i.e., homomorphisms that are not isomorphisms.

We prove the "only if" part.

(i) Let $G \in \mathbf{D}_{1,1}$ and let $J = \{m \in M : |G_m| > 1\}$. Notice that, for every groupoid $G$ that is subdirectly irreducible in $\mathbf{D}_{1,1}$, we have $|J| \leqslant 1$.

Indeed, let there exist $m_1, m_2 \in M$ such that $m_1 \neq m_2$ and $|G_{m_1}|, |G_{m_2}| > 1$. For $j = 1, 2$, consider the map $\psi_j$ defined by the rule

$$\psi_j(x) = \begin{cases} x, & x \notin G_{m_j}, \\ m_j, & x \in G_{m_j}. \end{cases} \tag{3}$$

Since $m_j$ is a maximal element and $G_{m_j}$ is a non-singleton orbit, $\psi_j$ is a proper homomorphism, $j = 1, 2$. It is easy to see that $\ker \psi_1 \cap \ker \psi_2$ is the equality relation $\Delta_G$, i.e., the homomorphisms $\psi_1$ and $\psi_2$ separate points of $G$. Therefore, if $|J| > 1$ then $G$ is not subdirectly irreducible.

(ii) If $J = \emptyset$ then $G \in \mathbf{D}_{0,1}$, i.e., $G$ is a left zero band. Each subdirectly irreducible groupoid in $\mathbf{D}_{0,1}$ is isomorphic to $\mathcal{G}_0$. In the sequel, we only consider subdirectly irreducible groupoids in $\mathbf{D}_{1,1}$ that are not left zero bands and assume that $|J| = 1$, i.e.,

$$G = \bigcup_{1 \leqslant i \leqslant n} G_i, \quad \text{where } |G_1| > 1 \text{ and } G_i = \{g_i\} \text{ for } i > 1.$$

(iii) Let $x, y \in G$ and let $x \neq y$. We show that $x$ and $y$ are separated by homomorphisms to $\mathcal{G}_1$.

If either $x = g_i$ or $y = g_i$, $2 \leqslant i \leqslant n$, then it suffices to consider the homomorphism $\psi_1$ from (3).

Assume that $x, y \in G_1$ and $y \nleqslant x$. Define a map $\varphi_{xy}$ as follows:

$$\varphi_{xy}(a) = \begin{cases} 0, & a \leqslant x, \\ 1, & \text{either } \ a \in G_1 \text{ with } a \nleqslant x \ \text{ or } \ a = g_k \ \text{ with } \ xg_k = x, \\ 2, & a = g_k \ \text{ with } \ xg_k \neq x. \end{cases}$$

It is clear that $\varphi_{xy}$ is a map from $G$ onto $\mathcal{G}_1$ and $\varphi_{xy}(x) = 0 \neq 1 = \varphi_{xy}(y)$. It remains to prove that $\varphi_{xy}$ is a homomorphism.

We show that $\varphi_{xy}(ab) = \varphi_{xy}(a)\varphi_{xy}(b)$. Three cases are possible.

(a) Let $\varphi_{xy}(a) = 0$, i.e., let $a \leqslant x$.

If $b \in G_1$ then $ab = a$ and $\varphi_{xy}(a)\varphi_{xy}(b) = 0 \cdot z = 0 = \varphi_{xy}(a) = \varphi_{xy}(ab)$, where $z \in \{0, 1\}$.

If $\varphi_{xy}(b) = 1$ and $b \notin G_1$ then $b = g_i$ with $xg_i = x$. Since $a \leqslant x$, we have $ab = ag_i \leqslant xg_i = x$ by (2). Hence, $\varphi_{xy}(ab) = 0 = 0 \cdot 1 = \varphi_{xy}(a)\varphi_{xy}(b)$.

If $\varphi_{xy}(b) = 2$ then $b = g_i$ with $xg_i \neq x$. Assume that $ab = ag_i \leqslant x$. Since $a \leqslant x$, there exist $y_1, \dots, y_n \in G$ such that $ay_1 \dots y_n = x$. We obtain $xg_i = ay_1 \dots y_n g_i = ag_i y_1 \dots y_n \leqslant xy_1 \dots y_n = x$ by using (L), (2), and (1′). Hence, $xg_i \leqslant x$. By definition, $x \leqslant xg_i$, which implies $x = xg_i$, a contradiction. Thus, $ab \nleqslant x$ and $\varphi_{xy}(ab) = 1 = 0 \cdot 2 = \varphi_{xy}(a)\varphi_{xy}(b)$.

(b) Let $a \in G_1$ and let $a \nleqslant x$.

For every $b \in G$, we have $ab \in G_1$ and $ab \nleqslant x$. Since $1 \cdot z = 1$ in $\mathcal{G}_1$, we obtain $\varphi_{xy}(ab) = 1 = 1 \cdot z = \varphi_{xy}(a) \cdot \varphi_{xy}(b)$ for every $b \in G$.

(c) Let $a = g_i$.

For every $b \in G$, we have $ab = a$. Since $1 \cdot z = 1$ and $2 \cdot z = 2$ in $\mathcal{G}_1$, we obtain $\varphi_{xy}(ab) = t = t \cdot z = \varphi_{xy}(a) \cdot \varphi_{xy}(b)$ for every $b \in G$, where $t \in \{1, 2\}$.

Thus, if $|G| > 3$ then all points of $G$ are separated by proper homomorphisms to $\mathcal{G}_1$; hence, $G$ cannot be subdirectly irreducible in $\mathbf{D}_{1,1}$. □

**Lemma 6.** *If $G \in \mathbf{D}_{1,1} \setminus \mathbf{D}_{0,1}$ then $\mathcal{G}_1$ is embeddable into $G$.*

Proof: Since $G \notin \mathbf{D}_{0,1}$, there exist $a, b \in G$ such that $ab \neq a$. Define a map from $\mathcal{G}_1$ into $G$ as follows:

$$0 \mapsto a, \quad 1 \mapsto ab, \quad 2 \mapsto ba.$$

It is easy to see that this is the required embedding. □

**Theorem 7.** *The lattice* $\mathsf{L_q}(\mathbf{D}_{1,1})$ *is a three-element chain.*

Proof: Since $\mathbf{D}_{1,1}$ is locally finite and has finitely many finite subdirectly irreducible groupoids, there are no infinite subdirectly irreducible groupoids in $\mathbf{D}_{1,1}$. By the Birkhoff Subdirect Representation Theorem and Theorem 5, $\mathbf{D}_{1,1}$ is the quasivariety generated by $\mathcal{G}_1$. The lattice $\mathsf{L_q}(\mathbf{D}_{0,1})$ is a two-element chain. By Lemma 6, if a subquasivariety $\mathbf{K}$ of $\mathbf{D}_{1,1}$ contains a groupoid $G$ that is not a left zero band then $\mathbf{K} = \mathbf{D}_{1,1}$. $\square$

## 3. Concluding remarks

We have proven that the variety **Dm** is $\mathcal{Q}$-universal. It is easy to see that the method used in the proof of Theorem 4 does not allow us to prove that some subvariety of the form $\mathbf{D}_{i,n}$ is $\mathcal{Q}$-universal. Indeed, the family $(\mathcal{A}_i)_{i<\omega}$ does not belong to such a subvariety. We have also shown that the variety $\mathbf{D}_{1,1}$ is not $\mathcal{Q}$-universal. The following problem seems to be of an interest: Determine the borderline between simple and $\mathcal{Q}$-universal varieties of differential groupoids.

**Acknowledgment.** The author thanks Professor Anna Romanowska for introduction to classes of modes, useful discussions, and many valuable comments. The author also thanks the anonymous referee, whose comments helped to improve the text.

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(*Received* August 15, 2007, *revised* October 11, 2007)