

## Further properties of 1-sequence-covering maps

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*Abstract.* Some relationships between 1-sequence-covering maps and weak-open maps or sequence-covering  $s$ -maps are discussed. These results are used to generalize a result from Lin S., Yan P., *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001), 301–314.

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### 1. Introduction

To find internal characterizations of certain images of metric spaces is one of central problems in general topology. Arhangel'skii [1] showed that a space is an open compact image of a metric space if and only if it has a development consisting of point-finite open covers. In 1996, Lin [14] introduced the notion of 1-sequence-covering maps and proved that a space is a 1-sequence-covering and  $s$ -image of a metric space if and only if it has a point-countable  $sn$ -network, and a space is a 1-sequence-covering, quotient and  $s$ -image of a metric space if and only if it has a point-countable weak base. Then Lin and Yan [16] proved that every sequence-covering, quotient and  $s$ -image of a locally separable metric space is a local  $\aleph_0$ -space. In that paper they also show the following

**Theorem 1.1.** *Every sequence-covering and compact map of a metric space is a 1-sequence-covering map.*

Recently, Xia [25] introduced the concept of weak-open maps, and by using it, certain  $gf$ -countable spaces are characterized as images of metric spaces under various weak-open maps.  $\pi$ -map is another important map which was introduced by Ponomarev in 1960, and  $\pi$ -images of metric spaces attract attention again in [7], [11], [23].

The purpose of this paper is to establish some relationships between 1-sequence-covering maps and weak-open maps or sequence-covering  $s$ -maps, and also to give a generalization of Theorem 1.1.

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We assume that all spaces are  $T_2$ , all maps are continuous and onto,  $\mathbb{N}$  denotes the set of all natural numbers,  $\omega = \mathbb{N} \cup \{0\}$ , and any convergent sequence includes its limit point. Let  $f : X \rightarrow Y$  be a map and  $\mathcal{P}$  be a collection of subsets of  $X$ . We denote  $\text{st}(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$ ,  $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$ ,  $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$ ,  $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$ .

**Definition 1.2.** Let  $X$  be a space and  $P \subset X$ .

- (1) A sequence  $\{x_n\}$  in  $X$  is called *eventually* in  $P$ , if  $\{x_n\}$  converges to  $x$ , and there exists  $m \in \mathbb{N}$  such that  $\{x\} \cup \{x_n : n \geq m\} \subset P$ .
- (2)  $P$  is called a *sequential neighborhood* of  $x$  in  $X$  [4], if whenever  $\{x_n\}$  is a sequence converging to  $x$  in  $X$ , then  $\{x_n\}$  is eventually in  $P$ .
- (3)  $X$  is a *sequential space* [4], if whenever  $A$  is a non closed subset of  $X$ , then there is a sequence in  $A$  converging to a point not in  $A$ .

**Definition 1.3.** Let  $\mathcal{P}$  be a collection of subsets of  $X$ .

- (1)  $\mathcal{P}$  is *point-countable*, if each point  $x \in X$  belongs to only countably many members of  $\mathcal{P}$ .
- (2)  $\mathcal{P}$  is a *network at  $x$  in  $X$* , if  $x \in P$  for every  $P \in \mathcal{P}$ , and whenever  $x \in U$  with  $U$  open in  $X$ , then  $x \in P \subset U$  for some  $P \in \mathcal{P}$ .

**Definition 1.4.** Let  $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$  be a cover of a space  $X$ . Assume that  $\mathcal{P}$  satisfies the following (a) and (b) for every  $x \in X$ .

- (a)  $\mathcal{P}_x$  is a network at  $x$ .
  - (b) If  $P_1, P_2 \in \mathcal{P}_x$ , then there exists  $P \in \mathcal{P}_x$  such that  $P \subset P_1 \cap P_2$ .
- (1)  $\mathcal{P}$  is a *weak base* of  $X$  [1], if for  $G \subset X$ ,  $G$  is open in  $X$  if and only if for every  $x \in G$ , there exists  $P \in \mathcal{P}_x$  such that  $P \subset G$ ;  $\mathcal{P}_x$  is said to be a weak neighborhood base at  $x$ .
  - (2)  $\mathcal{P}$  is an *sn-network* for  $X$  [14], if every element of  $\mathcal{P}_x$  is a sequential neighborhood of  $x$  for every  $x \in X$ ;  $\mathcal{P}_x$  is said to be an *sn-network* at  $x$ .
  - (3) A space  $X$  is *gf-countable* [1] (resp., *snf-countable* [6]), if  $X$  has a weak base (resp., *sn-network*)  $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$  such that  $\mathcal{P}_x$  is countable for every  $x \in X$ .

*Remark 1.5* ([5]). (1) Weak base  $\implies$  *sn-network*, so *gf-countable*  $\implies$  *snf-countable*.

- (2) In a sequential space, weak bases  $\iff$  *sn-networks*, so *gf-countable*  $\iff$  sequential and *snf-countable*.

**Definition 1.6.** Let  $f : X \rightarrow Y$  be a map.

- (1)  $f$  is a *weak-open* map [25], if there exists a weak base  $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$  for  $Y$ , and for  $y \in Y$ , there exists  $x_y \in f^{-1}(y)$  such that for each open neighborhood  $U$  of  $x_y$ ,  $B_y \subset f(U)$  for some  $B_y \in \mathcal{B}_y$ .

- (2)  $f$  is a 1-sequence-covering map [14], if for each  $y \in Y$ , there is  $x_y \in f^{-1}(y)$  such that whenever  $\{y_n\}$  is a sequence converging to  $y$  in  $Y$ , there is a sequence  $\{x_n\}$  converging to  $x_y$  in  $X$  with  $x_n \in f^{-1}(y_n)$  for every  $n \in \mathbb{N}$ .
- (3)  $f$  is a sequence-covering map [20], if every convergent sequence of  $Y$  is the image of some convergent sequence of  $X$ .
- (4)  $f$  is a quotient map [3], if whenever  $f^{-1}(U)$  is open in  $X$ , then  $U$  is open in  $Y$ .
- (5)  $f$  is an  $s$ -map [8] (resp., a Lindelöf map [22], a compact map), if  $f^{-1}(y)$  is separable (resp., Lindelöf, compact) for each  $y \in Y$ .

**Definition 1.7.** Let  $f : (X, d) \rightarrow Y$  be a map, and  $(X, d)$  be a metric space.

- (1)  $f$  is a  $\pi$ -map [1], if for each  $y \in Y$  and its open neighborhood  $U$  in  $Y$ ,  $d(f^{-1}(y), X \setminus f^{-1}(U)) > 0$ .
- (2)  $f$  is a  $\pi$ - $s$ -map, if  $f$  is both  $\pi$ -map and  $s$ -map.

*Remark 1.8.*

- (1) 1-sequence-covering map  $\implies$  sequence-covering map.
- (2) Sequence-covering map  $\implies$  quotient if  $Y$  is a sequential space.
- (3) Weak-open map  $\implies$  quotient map.

## 2. Main results

**Lemma 2.1.** Let  $f : X \rightarrow Y$  be a weak-open map. If  $X$  is first countable, then  $Y$  is  $gf$ -countable.

PROOF: Let  $f$  be a weak-open map, and  $X$  be first countable. Since  $f$  is a weak-open map, there exists a weak base  $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$  for  $Y$  such that for each  $y \in Y$  there exists  $x_y \in f^{-1}(y)$  with the property that for every open neighborhood  $U$  of  $x_y$ , there exists  $B_y \in \mathcal{B}_y$  such that  $B_y \subset f(U)$ . Because  $X$  is first countable, there is a countable neighborhood base  $\mathcal{P}_x$  for every  $x \in X$ . Now, for each  $y \in Y$ , we put

$$\mathcal{F}_y = \{f(P) : P \in \mathcal{P}_{x_y}\}, \quad \mathcal{F} = \bigcup\{\mathcal{F}_y : y \in Y\}.$$

Since for each  $y \in Y$ ,  $x_y$  belongs to only countably many members of  $\mathcal{P}_{x_y}$ . It implies that  $\mathcal{F}_y$  is countable for every  $y \in Y$ . So, we only need to prove that  $\mathcal{F}$  is a weak base  $Y$ . Indeed,

(1) Note that  $f$  is a continuous map, and  $\mathcal{P}_{x_y}$  is a neighborhood base at  $x_y$ , it follows that  $\mathcal{F}_y$  is a network at  $y$  in  $Y$ .

(2) For each  $y \in Y$  and  $U, V \in \mathcal{F}_y$ , there exist  $P_1, P_2 \in \mathcal{P}_{x_y}$  such that  $x_y \in P_1 \cap P_2$ , and  $f(P_1) = U$ ,  $f(P_2) = V$ . Since  $\mathcal{P}_{x_y}$  is a neighborhood base at  $x_y$ , there exists  $P \in \mathcal{P}_{x_y}$  such that  $x_y \in P \subset P_1 \cap P_2$ . This implies that  $f(P) \in \mathcal{F}_y$ , and  $f(P) \subset f(P_1 \cap P_2) \subset U \cap V$ .

(3) Suppose that  $G$  is open in  $Y$ . Then for each  $y \in G$ ,  $x_y \in f^{-1}(G)$ . Since  $\mathcal{P}_{x_y}$  is a neighborhood base at  $x_y$ , there exists  $P \in \mathcal{P}_{x_y}$  such that  $x_y \in P \subset f^{-1}(G)$ . Thus,  $f(P) \in \mathcal{F}_y$ , and  $f(P) \subset G$ .

Conversely, suppose that  $G \subset Y$  is such that for each  $y \in G$ , there exists  $F \in \mathcal{F}_y$  satisfying  $F \subset G$ . Then there exists  $P \in \mathcal{P}_{x_y}$  such that  $x_y \in P$ , and  $F = f(P)$ . Since  $P$  is an open neighborhood of  $x_y$ , and  $f$  is a weak-open map, there exists  $B_y \in \mathcal{B}_y$  such that  $B_y \subset f(P)$ . So, for each  $y \in G$ , there exists  $B_y \in \mathcal{B}_y$  such that  $B_y \subset G$ . Because  $\mathcal{B}$  is a weak base,  $G$  is an open subset of  $Y$ .

By (1), (2), and (3),  $\mathcal{F}$  is a weak base for  $Y$ . Therefore  $Y$  is  $gf$ -countable.  $\square$

**Proposition 2.2.** *Let  $f : X \rightarrow Y$  be a sequence-covering map, and  $Y$  be  $snf$ -countable. If (1) or (2) holds, then  $f$  is a 1-sequence-covering map.*

- (1)  $f$  is an  $s$ -map, and  $X$  has a point-countable base.
- (2)  $f$  is a Lindelöf map, and  $X$  is first countable.

PROOF: Let  $\mathcal{P} = \bigcup\{\mathcal{P}_y : y \in Y\}$  be an  $sn$ -network for  $Y$  such that  $\mathcal{P}_y$  is countable for every  $y \in Y$ . We can suppose that  $\mathcal{P}_y$  is closed under finite intersections for every  $y \in Y$ .

(1) Let  $f$  be an  $s$ -map, and  $\mathcal{B}$  a point-countable base of  $X$ .

Firstly, we prove that for each  $y \in Y$ , there exists a point  $x_y \in f^{-1}(y)$  such that whenever  $U$  is an open neighborhood of  $x_y$ , there exists  $P \in \mathcal{P}_y$  satisfying  $P \subset f(U)$ . Otherwise, there exists  $y \in Y$  so that for every  $x \in f^{-1}(y)$ , there is an open neighborhood  $U_x$  of  $x$  such that  $P \not\subset f(U_x)$  for every  $P \in \mathcal{P}_y$ . Since  $\mathcal{B}$  is a base of  $X$ , for each  $x$ , there exists  $B_x \in \mathcal{B}$  such that  $x \in B_x \subset U_x$ . This implies that for every  $x \in f^{-1}(y)$ ,  $P \not\subset f(B_x)$  whenever  $P \in \mathcal{P}_y$ . On the other hand, because  $\mathcal{B}$  is a point-countable base and  $f^{-1}(y)$  is a separable subset of  $X$ , it follows that  $\{B_x : x \in f^{-1}(y)\}$  is countable. Assume that  $\{B_x : x \in f^{-1}(y)\} = \{B_m : m \in \mathbb{N}\}$ , and  $\mathcal{P}_y = \{F_n : n \in \mathbb{N}\}$ . Put  $\mathcal{R}_y = \{P_n = \bigcap_{i=1}^n F_i : n \in \mathbb{N}\}$ . It is easy to see that  $\mathcal{R}_y \subset \mathcal{P}_y$ , and  $P_{n+1} \subset P_n$ , for every  $n \in \mathbb{N}$ . Hence, for each  $m, n \in \mathbb{N}$ , there exists  $x_{n,m} \in P_n \setminus f(B_m)$ . For  $n \geq m$ , we denote  $y_k = x_{n,m}$  with  $k = m + n(n-1)/2$ . Since  $\mathcal{P}_y$  is a network at  $y$  and  $P_{n+1} \subset P_n$  for each  $n \in \mathbb{N}$ ,  $\{y_k\}$  is a sequence converging to  $y$  in  $Y$ . Because  $f$  is a sequence-covering map,  $\{y_k\}$  is an image of some sequence  $\{x_n\}$  converging to  $x \in f^{-1}(y)$  in  $X$ . Since  $x \in f^{-1}(y) \subset \bigcup\{B_m : m \in \mathbb{N}\}$ , there exists  $m_0 \in \mathbb{N}$  such that  $x \in B_{m_0}$ . So  $\{x\} \cup \{x_k : k \geq k_0\} \subset B_{m_0}$  for some  $k_0 \in \mathbb{N}$ . Thus,  $\{y\} \cup \{y_k : k \geq k_0\} \subset f(B_{m_0})$ . But if we take  $k \geq k_0$ , then there exists  $n \geq m_0$  such that  $y_k = x_{n,m_0}$ , and it implies that  $x_{n,m_0} \in f(B_{m_0})$ . This contradicts to  $x_{n,m_0} \in P_n \setminus f(B_{m_0})$ .

We now prove that  $f$  is a 1-sequence-covering map. Suppose that  $y \in Y$ . By the above proof there is  $x_y \in f^{-1}(y)$  such that whenever  $U$  is an open neighborhood of  $x_y$ , there exists  $P \in \mathcal{P}_y$  satisfying  $P \subset f(U)$ . Denote  $\{B_n : n \in \mathbb{N}\}$  a countable neighborhood base at  $x_y$  such that  $B_{n+1} \subset B_n$  for every  $n \in \mathbb{N}$ . Let  $\{y_n\}$  be any sequence in  $Y$ , which converges to  $y$ . Now, we choose a sequence  $\{z_n\}$  in  $X$  as follows.

Since  $B_n$  is an open neighborhood of  $x_y$ , by the above argument, there exists  $P_{k_n} \in \mathcal{P}_y$  satisfying  $P_{k_n} \subset f(B_n)$  for every  $n \in \mathbb{N}$ , and by assumption every  $P \in \mathcal{P}_y$  is a sequential neighborhood. It follows that for each  $n \in \mathbb{N}$ ,  $f(B_n)$  is a sequential neighborhood of  $y$  in  $Y$ . Hence, for each  $n \in \mathbb{N}$ , there exists  $i_n \in \mathbb{N}$  such that  $y_i \in f(B_n)$  for every  $i \geq i_n$ . Assume that  $1 < i_n < i_{n+1}$  for each  $n \in \mathbb{N}$ . Then for each  $j \in \mathbb{N}$ , we take

$$z_j = \begin{cases} z_j \in f^{-1}(y_j) & \text{if } j < i_1, \\ z_{j,n} \in f^{-1}(y_j) \cap B_n & \text{if } i_n \leq j < i_{n+1}. \end{cases}$$

Denote  $S = \{z_j : j \geq 1\}$ . Then  $S$  converges to  $x_y$  in  $X$  and  $f(S) = \{y_n\}$ .

(2) Let  $f$  be a Lindelöf map, and  $X$  be first countable. Firstly, we prove that for each  $y \in Y$ , there exists  $x_y \in f^{-1}(y)$  such that for every neighborhood  $U$  of  $x_y$ , there exists  $P \in \mathcal{P}_y$  satisfying  $P \subset f(U)$ . Indeed, if not, there exists  $y \in Y$  such that for each  $x \in f^{-1}(y)$ , there exists an open neighborhood  $U_x$  satisfying  $P \not\subset f(U_x)$  for every  $P \in \mathcal{P}_y$ . Since  $f$  is a Lindelöf map, and  $\{U_x : x \in f^{-1}(y)\}$  is an open cover of  $f^{-1}(y)$ , there exists a countable family  $\{U_n : n \in \mathbb{N}\} \subset \{U_x : x \in f^{-1}(y)\}$  such that  $f^{-1}(y) \subset \bigcup\{U_n : n \in \mathbb{N}\}$ . Now, using the argument from the proof of (1), this leads to a contradiction. Then, using again the proof of (1), we obtain that  $f$  is a 1-sequence-covering map.  $\square$

**Corollary 2.3.** *Let  $f : X \rightarrow Y$  be a map. If one of the following conditions is satisfied, then  $f$  is a 1-sequence-covering map.*

- (1)  $f$  is a sequence-covering  $s$ -map,  $X$  has a point-countable base and  $Y$  is  $gf$ -countable.
- (2)  $f$  is a sequence-covering Lindelöf map,  $X$  is first countable and  $Y$  is  $gf$ -countable.
- (3)  $f$  is a weak-open map and  $X$  is first countable.

PROOF: It follows from Proposition 2.2 and Remark 1.5 that both (1) and (2) imply that  $f$  is 1-sequence-covering. Assuming (3), because  $f$  is a weak-open map and  $X$  is first countable, it follows from Lemma 2.1 that  $Y$  is  $gf$ -countable. Since  $f$  is a weak-open map, for each  $y \in Y$ , there exists  $x_y \in f^{-1}(y)$  such that for every neighborhood  $U$  of  $x_y$ , there exists  $P \in \mathcal{P}_y$  satisfying  $P \subset f(U)$ . Then using the proof of Proposition 2.2(1), we have  $f$  is 1-sequence-covering.  $\square$

**Corollary 2.4** ([14]). *Every open map of a first countable space is 1-sequence-covering.*

**Theorem 2.5.** *If  $f : X \rightarrow Y$  is a sequence-covering  $\pi$ - $s$ -map, then  $f$  is a 1-sequence-covering map.*

PROOF: Firstly, we prove that  $Y$  is  $snf$ -countable. Let  $f : (X, d) \rightarrow Y$  be a sequence-covering  $\pi$ - $s$ -map, and  $(X, d)$  be a metric space. For each  $n \in \mathbb{N}$ , denote

$$\mathcal{F}_n = \left\{ f \left[ B \left( z, \frac{1}{n} \right) \right] : z \in X \right\}, \quad \text{with } B \left( z, \frac{1}{n} \right) = \left\{ y \in X : d(z, y) < \frac{1}{n} \right\}.$$

It is clear that  $\mathcal{F}_{n+1}$  refines  $\mathcal{F}_n$  for each  $n \in \mathbb{N}$ . Now, for each  $x \in Y$ , we put  $\mathcal{P}_x = \{\text{st}(x, \mathcal{F}_n) : n \in \mathbb{N}\}$ ,  $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in Y\}$ . Then,

(i) It is obvious that  $\mathcal{P}_x$  is countable for every  $x \in Y$ .

(ii) Let  $U$  be an open neighborhood of  $x$ . Since  $f$  is a  $\pi$ -map, there exists  $n \in \mathbb{N}$  such that  $d(f^{-1}(x), X \setminus f^{-1}(U)) > \frac{1}{n}$ . Take  $m \in \mathbb{N}$  such that  $m \geq 2n$ . It is easy too see that if  $x \in f[B(z, \frac{1}{m})]$  for some  $z \in X$ , then  $B(z, \frac{1}{m}) \subset f^{-1}(U)$ , so  $f[B(z, \frac{1}{m})] \subset U$ . Since

$$\text{st}(x, \mathcal{F}_m) = \bigcup \left\{ f \left[ B \left( z, \frac{1}{m} \right) \right] : B \left( z, \frac{1}{m} \right) \cap f^{-1}(x) \neq \emptyset \right\},$$

this implies that  $\text{st}(x, \mathcal{F}_m) \subset U$ . Therefore  $\mathcal{P}_x$  is a network at  $x$ .

(iii) Let  $P_1, P_2 \in \mathcal{P}_x$ . Then there exist  $m, n \in \mathbb{N}$  such that  $P_1 = \text{st}(x, \mathcal{F}_m)$ , and  $P_2 = \text{st}(x, \mathcal{F}_n)$ . Pick  $k \in \mathbb{N}$  such that  $k > \max\{m, n\}$  and put  $P = \text{st}(x, \mathcal{F}_k)$ . It is obvious that  $P \in \mathcal{P}_x$ . Suppose  $y \in P = \text{st}(x, \mathcal{F}_k)$ ; then there exists  $z_1$  such that  $y \in f[B(z_1, \frac{1}{k})] \in \mathcal{F}_k$ . Since  $\mathcal{F}_k$  refines  $\mathcal{F}_m$  and  $\mathcal{F}_n$ , there exist  $f[B(z_2, \frac{1}{m})] \in \mathcal{F}_m$  and  $f[B(z_3, \frac{1}{n})] \in \mathcal{F}_n$  such that  $f[B(z_1, \frac{1}{k})] \subset f[B(z_2, \frac{1}{m})]$ , and  $f[B(z_1, \frac{1}{k})] \subset f[B(z_3, \frac{1}{n})]$ . Thus  $y \in f[B(z_2, \frac{1}{m})] \cap f[B(z_3, \frac{1}{n})]$ , so  $y \in \text{st}(x, \mathcal{F}_m) \cap \text{st}(x, \mathcal{F}_n)$ . Therefore  $P \subset P_1 \cap P_2$ .

(iv) Suppose  $P \in \mathcal{P}_x$  and let  $\{x_n\}$  be any sequence in  $Y$  which converges to  $x$  in  $Y$ . Because  $f$  is a sequence-covering map,  $\{x_n\}$  is the image of some sequence  $\{z_n\}$  converging to  $z \in f^{-1}(x)$  in  $X$ . Since  $P \in \mathcal{P}_x$ , there exists  $m \in \mathbb{N}$  such that

$$P = \text{st}(x, \mathcal{F}_m) = \bigcup \left\{ f \left[ B \left( z, \frac{1}{m} \right) \right] : f^{-1}(x) \cap B \left( z, \frac{1}{m} \right) \neq \emptyset \right\}.$$

Since  $\{z_n\}$  converges to  $z \in f^{-1}(x)$  and  $\bigcup\{B(z, \frac{1}{m}) : f^{-1}(x) \cap B(z, \frac{1}{m}) \neq \emptyset\}$  is a neighborhood of  $f^{-1}(x)$ ,  $\{z_n\}$  is eventually in  $\bigcup\{B(z, \frac{1}{m}) : f^{-1}(x) \cap B(z, \frac{1}{m}) \neq \emptyset\}$ . This implies that  $\{x_n\}$  is eventually in  $P$ . Hence,  $P$  is a sequential neighborhood of  $x$  in  $Y$ . Therefore,  $Y$  is *snf*-countable.

Then, it follows from Proposition 2.2(1) that  $f$  is a 1-sequence-covering map. □

*Remark 2.6.* Since every compact map of a metric space is a  $\pi$ -s-map, Theorem 2.5 is a generalization of Theorem 1.1. Furthermore, this generalization is proper.

**Example 2.7.** There is a sequence-covering  $\pi$ -s-map  $f : X \rightarrow Y$  of a metric space  $X$  which is not a sequence-covering compact map.

PROOF: Indeed, let  $\mathbb{R}$  be the real line with usual Euclidean topology. Denote  $X = \mathbb{R}$  and  $Y = (-\infty; 0]$ . Now, we define  $f : X \rightarrow Y$  by

$$f(t) = \begin{cases} t & \text{if } t \in (-\infty; 0] \\ 0 & \text{if } t \in [0; +\infty). \end{cases}$$

Then, we have

(1)  $f$  is a surjective, continuous  $s$ -map. It is obvious.

(2)  $f$  is a  $\pi$ -map. Indeed, let  $y \in (-\infty; 0]$  and  $U$  be an open neighborhood of  $y$ .

If  $y = 0$ , then  $f^{-1}(0) = [0, +\infty)$  and there is  $\varepsilon > 0$  such that  $0 \in (-\varepsilon; 0] \subset U$ . Hence, we have

$$\begin{aligned} d\left(f^{-1}(y), \mathbb{R} \setminus f^{-1}(U)\right) &\geq d\left([0, +\infty), \mathbb{R} \setminus f^{-1}[(-\varepsilon; 0]]\right) \\ &= d\left([0, +\infty), (-\infty; -\varepsilon]\right) \geq \varepsilon > 0. \end{aligned}$$

If  $y \neq 0$ , then  $f^{-1}(y) = \{y\}$  and there is  $\varepsilon > 0$  such that  $(y - \varepsilon, y + \varepsilon) \subset U$ . So, we have

$$\begin{aligned} d\left(f^{-1}(y), \mathbb{R} \setminus f^{-1}(U)\right) &\geq d\left(\{y\}, \mathbb{R} \setminus f^{-1}[(y - \varepsilon; y + \varepsilon)]\right) \\ &= d\left(y, (-\infty; y - \varepsilon] \cup [y + \varepsilon; 0]\right) \geq \varepsilon > 0. \end{aligned}$$

Therefore,  $f$  is a  $\pi$ -map.

(3)  $f$  is a sequence-covering map. In fact, suppose  $S = \{y_n : n \in \omega\}$  is any sequence converging to  $y_0$  in  $Y$ . Since  $Y \subset X$ ,  $S = \{y_n : n \in \omega\} \subset X$ . By definition of  $f$ , we have  $f(y_n) = y_n$  for every  $n \in \omega$ . Therefore  $f$  is a 1-sequence-covering map.

(4)  $f$  is not a compact map. Since  $f^{-1}(0) = [0, +\infty)$ , and  $[0, +\infty)$  is not compact in  $\mathbb{R}$ , it follows that  $f$  is not compact.

By Corollary 3.6 in [25], Remark 1.8 and Theorem 2.5, we get □

**Corollary 2.8.** *Let  $f : M \rightarrow X$  be a map. If  $M$  is a metric space, then the following are equivalent:*

- (1)  $f$  is a weak-open  $\pi$ - $s$ -map;
- (2)  $f$  is a 1-sequence-covering, quotient  $\pi$ - $s$ -map;
- (3)  $f$  is a sequence-covering, quotient  $\pi$ - $s$ -map.

**Corollary 2.9.** *Let  $f : M \rightarrow X$  be a map. If  $M$  is a metric space, then the following are equivalent:*

- (1)  $f$  is a weak-open compact map;
- (2)  $f$  is a 1-sequence-covering, quotient compact map;
- (3)  $f$  is a sequence-covering, quotient compact map.

PROOF: It follows immediately by Corollary 2.8. □

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