

Spectrum of twisted Dirac operators on the complex projective space $\mathbb{P}^{2q+1}(\mathbb{C})$

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Abstract. In this paper, we explicitly determine the spectrum of Dirac operators acting on smooth sections of twisted spinor bundles over the complex projective space $\mathbb{P}^{2q+1}(\mathbb{C})$ for $q \geq 1$.

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1. Introduction

For spin Riemannian symmetric spaces, the spectrum of the Dirac operator is explicitly known only in few cases (see, e.g., [6], [7], [9], [10]).

Consider the complex projective space $\mathbb{P}^n(\mathbb{C}) = SU(n+1)/S(U(n) \times U(1))$ with $n = 2q + 1$, equipped with the metric induced by the negative of the Killing form of $SU(n + 1)$. As a compact simply connected Riemannian symmetric space, the manifold $M = \mathbb{P}^n(\mathbb{C})$ admits a unique homogeneous spin structure (see, e.g., [1]). Let $L = \{([z], w) \in M \times \mathbb{C}^{n+1}; w \in [z]\}$ be the tautological bundle over M . Recall that each complex line bundle over M is (up to isomorphism) of the form $L^m := L^{\otimes m}$ for some $m \in \mathbb{Z}$. Let $S \rightarrow M$ be the spinor bundle, and let ∇^S be the spinor connection. We fix $m \in \mathbb{Z}$ and consider the vector bundle $S \otimes L^m$ endowed with the connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^{L^m}$, where ∇^{L^m} denotes the Levi-Civita connection on L^m . Let $\Gamma^\infty(S \otimes L^m)$ be the space of smooth sections of the bundle $S \otimes L^m$. The twisted Dirac operator D_m acting on $\Gamma^\infty(S \otimes L^m)$ is defined by $D_m = \tilde{\mu} \circ \nabla$, where $\tilde{\mu} : T^*M \otimes S \otimes L^m \rightarrow S \otimes L^m$ is the bundle homomorphism induced by the Clifford multiplication. Being an elliptic operator, D_m has discrete (real) eigenvalues with finite multiplicities. The aim of this paper is to establish the following result:

Theorem 1. *On the complex projective space of dimension $n = 2q + 1$ ($q \geq 1$), the spectrum of the twisted Dirac operator D_m acting on smooth sections of the spinor bundle tensored with L^m ($m \in \mathbb{Z}$) is the union of the following sets:*

- (1) $\left\{ \pm \sqrt{a_{k,m}(0,0)}; k \geq \max \left\{ 0, -\frac{n+1}{2} - m \right\} \right\};$
- (2) $\left\{ \pm \sqrt{a_{k,m}(\varepsilon,l)}; \varepsilon \in \{0,1\}, 1 \leq l \leq n-1, k \geq \max \left\{ \varepsilon, l - m - \frac{n-1}{2} \right\} \right\};$

$$(3) \left\{ \pm \sqrt{a_{k,m}(1, n)}; k \geq \max \left\{ 0, \frac{n+1}{2} - m \right\} \right\},$$

where

$$a_{k,m}(\varepsilon, l) = \frac{1}{2(n+1)} \{(2k + 2m - 2\varepsilon + n + 1)(k + n - l)\}$$

for $\varepsilon \in \{0, 1\}$ and $0 \leq l \leq n$.

For $m = 0$, the above theorem gives the spectrum of the classical Dirac operator on $\mathbb{P}^{2q+1}(\mathbb{C})$ which has already been computed by Seifarth and Semmelmann [9]. Observing that each $\text{Spin}^{\mathbb{C}}$ -bundle over $\mathbb{P}^{2q+1}(\mathbb{C})$ is (up to isomorphism) of the form $S \otimes L^m$, $m \in \mathbb{Z}$, with S and L being as above (see [4] for generalities on $\text{Spin}^{\mathbb{C}}$ Riemannian manifolds), one obtains from Theorem 1 the spectrum of each $\text{Spin}^{\mathbb{C}}$ -Dirac operator on $\mathbb{P}^{2q+1}(\mathbb{C})$, $q \geq 1$.

2. Preliminaries

2.1 Twisted Dirac operator on a Riemannian symmetric space

Let $M = G/K$ be a compact, simply connected n -dimensional irreducible Riemannian symmetric space with G compact and simply connected. We assume that G and K have the same rank and that M has a spin structure (which is necessarily unique). Let \mathfrak{g} and \mathfrak{k} be the respective Lie algebras of G and K . We denote by \mathfrak{m} the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to the Killing form B of \mathfrak{g} . We identify canonically \mathfrak{m} with the tangent space to M at the point eK , e being the neutral element of G . The fixed Riemannian metric on M is the metric induced by the restriction to \mathfrak{m} of the scalar product $\langle \cdot, \cdot \rangle := -B$ on \mathfrak{g} .

The isotropy representation $\text{Ad} : K \rightarrow SO(\mathfrak{m})$ lifts to a homomorphism $\widetilde{\text{Ad}} : K \rightarrow \text{Spin}(\mathfrak{m})$ satisfying $\widetilde{\text{Ad}} \circ \lambda = \text{Ad}$, where $\lambda : \text{Spin}(\mathfrak{m}) \rightarrow SO(\mathfrak{m})$ is the usual 2-fold covering. Let $\kappa : \text{Spin}(\mathfrak{m}) \rightarrow \text{Aut}(\Delta)$ be the spin representation. The spinor bundle S over M is naturally identified with the homogeneous vector bundle $G \times_{K, \chi} \Delta$, where $\chi := \kappa \circ \widetilde{\text{Ad}}$.

Let τ be an irreducible unitary representation of K on a finite dimensional complex vector space V . Let us set $E = G \times_{K, \tau} V$, the associated homogeneous vector bundle over M , and denote by ∇^E the Levi-Civita connection on E . We consider the spinor bundle twisted by E , i.e. the vector bundle $S \otimes E$, endowed with the connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^E$, where ∇^S denotes the spinor connection. The Clifford multiplication $\mu : \text{Cliff}(\mathfrak{m}) \rightarrow \text{End}(\Delta)$ induces a bundle homomorphism $\widetilde{\mu} : T^*M \otimes S \otimes E \rightarrow S \otimes E$. Let $\Gamma^\infty(S \otimes E)$ be the space of smooth sections of $S \otimes E$. The twisted Dirac operator acting on $\Gamma^\infty(S \otimes E)$ is defined by $D_E = \widetilde{\mu} \circ \nabla$.

The main goal of this section is to describe the spectrum of D_E . To this end, we next recall a remarkable formula of its square D_E^2 which is essentially due to Parthasarathy [8].

2.2 A formula for the square of D_E

Let us introduce the following vector space:

$$(C^\infty(G) \otimes \Delta \otimes V)^K := \{ \varphi \in C^\infty(G) \otimes \Delta \otimes V; (r \otimes \chi \otimes \tau)(k)\varphi = \varphi, \forall k \in K \},$$

where r denotes the right regular representation of G on $C^\infty(G)$, the space of complex valued smooth functions on G . Obviously, $(C^\infty(G) \otimes \Delta \otimes V)^K$ is isomorphic to $\Gamma^\infty(S \otimes E)$. In what follows, we will identify these two spaces in the canonical fashion.

Fix an orthonormal basis $\{X_1, \dots, X_d, Y_1, \dots, Y_{d'}\}$ of \mathfrak{g} with respect to the scalar product $\langle \cdot, \cdot \rangle$ in such a way that $\{X_1, \dots, X_d\}$ forms a basis of \mathfrak{m} and $\{Y_1, \dots, Y_{d'}\}$ a basis of \mathfrak{k} . The Casimir operators of G and K (relative to $\langle \cdot, \cdot \rangle$) are respectively:

$$\Omega_G = - \sum_{i=1}^d X_i^2 - \sum_{p=1}^{d'} Y_p^2 \quad \text{and} \quad \Omega_K = - \sum_{p=1}^{d'} Y_p^2.$$

Denote by ν the representation of \mathfrak{g} on $C^\infty(G)$ given by

$$\nu(X)f = Xf \quad (X \in \mathfrak{g} : f \in C^\infty(G)).$$

We have ([8, Proposition 1.1])

$$D_E = \sum_{i=1}^d \nu(X_i) \otimes \mu(X_i) \otimes 1.$$

Thus the square of D_E satisfies

$$\begin{aligned} D_E^2 &= \left(\sum_i \nu(X_i) \otimes \mu(X_i) \otimes 1 \right)^2 \\ &= - \sum_i \nu(X_i)^2 \otimes 1 \otimes 1 + \frac{1}{2} \sum_{i,j} \nu([X_i, X_j]) \otimes \mu(X_i)\mu(X_j) \otimes 1. \end{aligned}$$

Using the facts that $[X_i, X_j] = \sum_p \langle [X_i, X_j], Y_p \rangle Y_p$ for $1 \leq i, j \leq d$, and that the action of $Y_p \in \mathfrak{k}$ on Δ is given by $\chi(Y_p) = \frac{1}{4} \sum_{i,j} \langle [Y_p, X_i], X_j \rangle \mu(X_i)\mu(X_j)$, we obtain

$$D_E^2 = - \sum_i \nu(X_i)^2 \otimes 1 \otimes 1 + 2 \sum_p \nu(Y_p) \otimes \chi(Y_p) \otimes 1.$$

By a direct computation, one has

$$\begin{aligned} \sum_p (\nu \otimes \chi)(Y_p)^2 \otimes 1 - \sum_p \nu(Y_p)^2 \otimes 1 \otimes 1 - \sum_p 1 \otimes \chi(Y_p)^2 \otimes 1 \\ = 2 \sum_p \nu(Y_p) \otimes \chi(Y_p) \otimes 1. \end{aligned}$$

Since $(\nu \otimes \chi)(\Omega_K) \otimes 1 = 1 \otimes 1 \otimes \tau(\Omega_K)$, it follows that

$$D_E^2 = \nu(\Omega_G) \otimes 1 \otimes 1 + 1 \otimes \chi(\Omega_K) \otimes 1 - 1 \otimes 1 \otimes \tau(\Omega_K).$$

Let T be a common maximal torus of G and K , and let \mathfrak{h} be its Lie algebra. The Killing form of \mathfrak{g} induces a natural scalar product on the dual $(\mathfrak{h}_{\mathbb{R}})^* := (i\mathfrak{h})^*$ which we also denote by $\langle \cdot, \cdot \rangle$. Let R_G be the root system of G with respect to T . Let R_G^+ be the system of positive roots of G , R_K^+ be the system of positive roots of K , with respect to a fixed lexicographic ordering in R_G . Let δ_G (resp. δ_K) be the half-sum of the positive roots of G (resp. K).

Note that $\chi(\Omega_K) = \frac{R}{8} \text{Id}$ with R the (constant) scalar curvature of the Riemannian space M (see [4, Chapter 3] for details). Furthermore, if η is the highest weight of the representation τ , then we have $\tau(\Omega_K) = \langle \eta + 2\delta_K, \eta \rangle \text{Id}$ (see, e.g., [11, Lemma 5.6.4]). Consequently, we get

$$D_E^2 = \nu(\Omega_G) \otimes 1 \otimes 1 + \left(\frac{R}{8} - \langle \eta + 2\delta_K, \eta \rangle \right) \cdot 1.$$

For simplicity, we write

$$D_E^2 = \Omega_G + \frac{R}{8} - \langle \eta + 2\delta_K, \eta \rangle.$$

2.3 The spectrum of D_E

Let us fix a K -invariant Hermitian scalar product (\cdot, \cdot) on the space $\Delta \otimes V$. Let $L^2(G, \Delta \otimes V)^K$ denote the completion of the vector space

$$\begin{aligned} C^\infty(G, \Delta \otimes V)^K := \{ \psi \in C^\infty(G, \Delta \otimes V); \psi(gk) = (\chi \otimes \tau)(k^{-1})\psi(g), \\ \forall g \in G, k \in K \} \end{aligned}$$

with respect to the scalar product $(\cdot, \cdot)_{L^2}$ defined by

$$(\psi, \psi')_{L^2} = \int_G (\psi(g), \psi'(g)) dg,$$

where dg is a Haar measure on G .

Let \widehat{G} be the unitary dual of G . For $\gamma \in \widehat{G}$, let (π_γ, V_γ) be a fixed representative of the equivalence class γ . Applying the Peter-Weyl theorem, we obtain

$$L^2(G, \Delta \otimes V)^K \cong \widehat{\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \text{Hom}_K(V_\gamma, \Delta \otimes V)},$$

where $\text{Hom}_K(V_\gamma, \Delta \otimes V)$ is the vector space of K -equivariant linear homomorphisms from V_γ to $\Delta \otimes V$. Let $\Delta \otimes V = \bigoplus_{\delta \in \Lambda} m_\delta W_\delta$ be the irreducible decomposition of the K -module $\Delta \otimes V$ into irreducible K -submodules, where $\Lambda \subset \widehat{K}$ and $m_\delta \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ for each $\delta \in \Lambda$. Then

$$\begin{aligned} L^2(G, \Delta \otimes V)^K &\cong \widehat{\bigoplus_{\gamma \in \widehat{G}} \bigoplus_{\delta \in \Lambda} m_\delta (V_\gamma \otimes \text{Hom}_K(V_\gamma, W_\delta))} \\ &\cong \widehat{\bigoplus_{\gamma \in \widehat{G}} \bigoplus_{\delta \in \Lambda} m_\delta m_{\gamma|_K}(\delta) V_\gamma}, \end{aligned}$$

where $m_{\gamma|_K}(\delta) := \dim_{\mathbb{C}} \text{Hom}_K(V_\gamma, W_\delta)$.

The Casimir operator Ω_G acts on the G -module V_γ as a scalar multiple of the identity:

$$\Omega_G|_{V_\gamma} = \langle \lambda_\gamma + 2\delta_G, \lambda_\gamma \rangle \text{Id},$$

where λ_γ is the highest weight of the representation π_γ . Using the identifications $\Gamma^\infty(S \otimes E) = (C^\infty(G) \otimes \Delta \otimes V)^K = C^\infty(G, \Delta \otimes V)^K$, one can deduce that the spectrum of the operator D_E^2 is given by

$$\begin{aligned} \text{Spec}(D_E^2, M) &= \left\{ \langle \lambda_\gamma + 2\delta_G, \lambda_\gamma \rangle - \langle \eta + 2\delta_K, \eta \rangle + \frac{R}{8}; \right. \\ &\quad \left. \gamma \in \widehat{G}, \delta \in \Lambda, m_{\gamma|_K}(\delta) \neq 0 \right\}. \end{aligned}$$

Since $M = G/K$ is a Riemannian symmetric space, the spectrum of the twisted Dirac operator D_E is symmetric with respect to the origin (see, e.g., [3]). Hence it is completely determined by the spectrum of D_E^2 .

3. Spectrum of twisted Dirac operators on $\mathbb{P}^{2q+1}(\mathbb{C})$

Throughout this section, let $G = SU(n + 1)$ and $K = S(U(n) \times U(1))$ with $n \geq 2$. The Riemannian symmetric space $G/K = \mathbb{P}^n(\mathbb{C})$ admits a spin structure (which is, of course, unique) if and only if n is odd. In the sequel, we assume that $n = 2q + 1$ with $q \geq 1$. Let S be the spinor bundle associated to the homogeneous

spin structure of $M = G/K$. Since M is a spin Kähler manifold, one has ([4]) the following isomorphism:

$$S \cong S_0 \oplus \cdots \oplus S_n,$$

where S_n is a complex line bundle satisfying $S_n^2 = K_M := \bigwedge^{n,0}(T^*M)^\mathbb{C}$, and $S_{n-l} = \bigwedge^{0,l}(T^*M)^\mathbb{C} \otimes S_n$ for $l = 0, \dots, n$.

Fix the maximal torus

$$T = \{A = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}, e^{-i\sum_{j=1}^n \theta_j}); \theta_j \in \mathbb{R} \text{ for } j = 1, \dots, n\}$$

of G and K , and denote by \mathfrak{h} its Lie algebra. For $j \in \{1, \dots, n+1\}$, we define the linear functional

$$e_j : \mathfrak{h}^\mathbb{C} \longrightarrow \mathbb{C}, \quad H = \text{diag}(h_1, \dots, h_{n+1}) \longmapsto h_j.$$

Considering the root spaces decomposition of the complexified Lie algebra $\mathfrak{g}^\mathbb{C}$ of G under the action of T , we can choose the following systems of positive roots

$$R_G^+ = \{e_i - e_j; 1 \leq i < j \leq n\} \cup \{e_1 + \cdots + 2e_i + \cdots + e_n; 1 \leq i \leq n\} \text{ and}$$

$$R_K^+ = \{e_i - e_j; 1 \leq i < j \leq n\},$$

respectively of G and K relative to T . Then

$$\delta_G = \sum_{j=1}^n (n+1-j)e_j \quad \text{and} \quad \delta_K = \frac{1}{2} \sum_{j=1}^n (n-2j+1)e_j.$$

Recall that finite-dimensional irreducible representations of K are classified by their highest weights which are of the form $\eta = \eta_1 e_1 + \cdots + \eta_n e_n$ with $\eta_j \in \mathbb{Z}$ for all $1 \leq j \leq n$, and $\eta_1 \geq \cdots \geq \eta_n$. Observe that $(S_n)_{e_K}$ is an irreducible (one-dimensional!) K -module with highest weight

$$\xi = \frac{n+1}{2} (e_1 + \cdots + e_n).$$

It follows that $(S_{n-l})_{e_K}$ is also an irreducible K -module whose highest weight ϑ_l is easily calculated from the observation that $(TM)_{e_K} \cong (\mathbb{C}^n)^* \otimes \mathbb{C}$ as a $U(n) \otimes U(1)$ -module:

$$\vartheta_l = \begin{cases} \frac{n+1}{2} (e_1 + \cdots + e_n) & \text{if } l = 0, \\ \left(\frac{n+1}{2} - l \right) (e_1 + \cdots + e_{n-l}) \\ \quad + \left(\frac{n-1}{2} - l \right) (e_{n-l+1} + \cdots + e_n) & \text{if } 1 \leq l \leq n-1, \\ -\frac{n+1}{2} (e_1 + \cdots + e_n) & \text{if } l = n. \end{cases}$$

Let $L = \{([z], w) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}; w \in [z]\}$ be the tautological bundle over $\mathbb{P}^n(\mathbb{C})$. For $m \in \mathbb{Z}$, we set $L^m := L^{\otimes m}$. As a homogeneous vector bundle, L^m is associated to the irreducible K -representation τ^m defined by

$$\tau^m \left(\begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \right) = (\det A)^m \quad (A \in U(n)),$$

whose highest weight is $\eta^m = m(e_1 + \dots + e_n)$. Consequently, $(S_{n-l} \otimes L^m)_{eK}$ is an irreducible K -module with highest weight

$$\sigma_{l,m} = \begin{cases} (\frac{n+1}{2} + m)(e_1 + \dots + e_n) & \text{if } l = 0, \\ (\frac{n+1}{2} - l + m)(e_1 + \dots + e_{n-l}) \\ + (\frac{n-1}{2} - l + m)(e_{n-l+1} + \dots + e_n) & \text{if } 1 \leq l \leq n-1, \\ (-\frac{n+1}{2} + m)(e_1 + \dots + e_n) & \text{if } l = n. \end{cases}$$

We shall denote by $\tau_{\sigma_{l,m}}$ the unique (up to equivalence) irreducible representation of K with highest weight $\sigma_{l,m}$.

To compute the spectrum of the twisted Dirac operator $D_m := D_{L^m}$, we need the following well known result (compare [2], [5]):

Proposition 1. *Let ρ_λ (resp. τ_σ) be an irreducible representation of $SU(n+1)$ (resp. $S(U(n) \times U(1))$) with highest weight $\lambda = \sum_{j=1}^n \lambda_j e_j$ (resp. $\sigma = \sum_{j=1}^n \sigma_j e_j$). Then the multiplicity $m(\lambda, \sigma) := m_{\rho_\lambda|_{S(U(n) \times U(1))}}(\tau_\sigma)$ is either 0 or 1, and is equal to 1 if and only if σ is of the form*

$$\sigma = \sum_{j=1}^n (\xi_j - a) e_j$$

with $\lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \dots \geq \lambda_n \geq \xi_n \geq 0$ and $a = \sum_{j=1}^n (\lambda_j - \xi_j)$.

Applying directly this proposition, we get:

Proposition 2. *Let ρ_λ be an irreducible representation of $SU(n+1)$ with highest weight $\lambda = \sum_{j=1}^n \lambda_j e_j$, and let $\tau_{\sigma_{l,m}}$ be as before with $0 \leq l \leq n$ and $m \in \mathbb{Z}$. Then the multiplicity $m(\lambda, \sigma_{l,m})$ is non-zero (and hence equal to 1) if and only if,*

- (i) for $l = 0$, λ is of the form

$$\lambda = \left(\frac{n+1}{2} + 2k + m \right) e_1 + \left(\frac{n+1}{2} + k + m \right) (e_2 + \dots + e_n)$$

with $k \geq \max\{0, -\frac{n+1}{2} - m\}$;

(ii) for $1 \leq l \leq n - 1$, λ is of the form

$$\begin{aligned} \lambda = & \left(\frac{n+1}{2} + 2k + m - l - \varepsilon \right) e_1 + \left(\frac{n+1}{2} + k + m - l \right) (e_2 + \cdots + e_{n-l}) \\ & + \left(\frac{n-1}{2} + k + m - l + \varepsilon \right) e_{n-l+1} \\ & + \left(\frac{n-1}{2} + k + m - l \right) (e_{n-l+2} + \cdots + e_n) \end{aligned}$$

with $\varepsilon \in \{0, 1\}$ and $k \geq \max\{\varepsilon, l - m - \frac{n-1}{2}\}$;

(iii) for $l = n$, λ is of the form

$$\lambda = \left(-\frac{n+1}{2} + 2k + m \right) e_1 + \left(-\frac{n+1}{2} + k + m \right) (e_2 + \cdots + e_n)$$

with $k \geq \max\{0, \frac{n+1}{2} - m\}$.

On the Lie algebra $\mathfrak{g} = \mathfrak{su}(n + 1)$, we fix the scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\langle X, Y \rangle = -B(X, Y) \text{ for } X, Y \in \mathfrak{g},$$

where B is the Killing form of \mathfrak{g} . As usual, we extend $\langle \cdot, \cdot \rangle$ to a scalar product on the vector space of real valued linear forms on $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$. Observe that

$$\langle e_i, e_i \rangle = \frac{n}{2(n+1)^2} \text{ and } \langle e_i, e_j \rangle = \frac{-1}{2(n+1)^2}$$

for $1 \leq i \neq j \leq n$.

The scalar curvature of the Riemannian space $\mathbb{P}^n(\mathbb{C}) = G/K$ is $R = n$. Thus the spectrum of the twisted Dirac operator D_m is given by

$$\begin{aligned} \text{Spec}(D_m, \mathbb{P}^n(\mathbb{C})) = & \{ \pm \sqrt{c(\lambda)}; \exists l \in \{0, \dots, n\}, : m(\lambda, \sigma_{l,m}) \neq 0 \} \text{ with} \\ c(\lambda) := & \langle \lambda + 2\delta_G, \lambda \rangle - \langle \eta^m + 2\delta_K, \eta^m \rangle + \frac{n}{8}. \end{aligned}$$

Computing the eigenvalues of D_m , we can easily derive Theorem 1.

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