Product of vector measures on topological spaces

Surjit Singh Khurana

Abstract. For $i = (1, 2)$, let X_i be completely regular Hausdorff spaces, E_i quasi-complete locally convex spaces, $E = E_1\breve{\otimes}E_2$, the completion of the their injective tensor product, $C_b(X_i)$ the spaces of all bounded, scalar-valued continuous functions on X_i, and μ_i E_i-valued Baire measures on X_i. Under certain conditions we determine the existence of the E-valued product measure $\mu_1 \otimes \mu_2$ and prove some properties of these measures.

Keywords: injective tensor product, product of measures, tight measures, τ-smooth measures, separable measures, Fubini theorem

Classification: Primary 46E10, 28C05, 28C15, 46G10, 60B05; Secondary 46A08, 28B05

1. Introduction and notations

In this paper, all vector spaces are taken on K (we will call them scalars), the field of real or complex numbers ($\mathbb{R}$ will denote the field of real numbers). For a Hausdorff completely regular space X, $C(X)$ (resp. $C_b(X)$) are the spaces of all scalar-valued continuous (continuous and bounded) functions on X, $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ are the classes of Borel and Baire subsets of X, $M_\sigma(X)$, $M_\infty(X)$, $M_\tau(X)$, $M_t(X)$ are resp. σ-smooth, separable, τ-smooth and tight scalar measures on X. The elements of $M_\tau(X)$ and $M_t(X)$ extend to Borel measures ([8], [16], [17]); also there are locally convex topologies β_σ, β_∞, β_τ, β_t on $C_b(X)$ which give as their duals $M_\sigma(X)$, $M_\infty(X)$, $M_\tau(X)$, $M_t(X)$ ([8], [17], [16]). $\tilde{X}$ will denote the Stone-Čech compactification of X and for an $f \in C_b(X)$, $\tilde{f}$ will be its continuous extension to $\tilde{X}$.

For $i = (1, 2)$, X_i will always denote Hausdorff completely regular spaces, E_i Hausdorff locally convex spaces, P_i all continuous seminorms on E_i, and $E = E_1\breve{\otimes}E_2$, the completion of the injective tensor product of E_1 and E_2. For a $p_i \in P_i$ and $f \in E_i'$, we will say $f \le p_i$ if $f \in S_i$ where $S_i = \{h \in E_i' : |h(x)| \le p_i(x)\ \forall\, x \in E_i\}$; S_i is an equicontinuous, convex and $\sigma(E_i', E_i)$-compact subset of E_i'. With the norm topology on $C(S_1 \times S_2)$, the topology on E is the one induced by $\prod_{(S_1,S_2)} C(S_1 \times S_2)$; to prove convergence in E, many times the problem boils down to $C(S_1 \times S_2)$ and we will say that E can be considered as a subspace of $C(S_1 \times S_2)$. For a locally convex space F with its dual F' and $(x, y) \in F \times F'$, $\langle x, y\rangle$ will denote $y(x)$; also for a continuous seminorm p on F, $V_p = \{x \in F : p(x) \le 1\}$. $\mathbb{N}$ will denote the set of natural numbers.

Now we come to vector-valued measures. Let F be a locally convex space with P the family of all continuous semi-norms on F, $\mathcal{A}$ is a σ-algebra of subsets of a set Y, $\mu : \mathcal{A} \to F$ a countably additive vector measure. For a $p \in P$, we denote the p-semi-variation of μ by $\bar{\mu}_p$,

$$\bar{\mu}_p(A) = \sup \left\{ |g \circ \mu|(A) : g \in V_p^0 \right\}$$

(here V_p^0 is the polar of V_p in the duality $\langle F, F' \rangle$) [15]. Also we can select a control measure, λ_p, for $\bar{\mu}_p$ which has the properties:

(i) with norm topology on measures, λ_p is in the closed convex hull of $\{|g \circ \mu| : g \in V_p^0\}$ ([12, p. 20, proof of Theorem 1]);
(ii) $|f \circ \mu| \ll \lambda_p$ for every f in F' with $\|f\|_p \leq 1$ (note that $\|f\|_p = \sup\{|f(x)| : x \in V_p\}$);
(iii) if $\lambda_p(A) = 0$ then $\bar{\mu}_p(A) = 0$;
(iv) $\lim_{\lambda_p(A) \to 0} \bar{\mu}_p(A) = 0$;
(v) $\lambda_p \leq \bar{\mu}_p$.

We also know that if $f : Y \to K$ is a measurable function, $B \in \mathcal{A}$ and $|f| \leq c$ on B, then $\| \int_B f \, d\mu \|_p \leq c \bar{\mu}_p(B)$.

$L^1(\mu)$ will denote the space of all μ-integrable functions ([12]). For any $f \in L^1(\mu)$, we define $\bar{\mu}_p(f) = \sup\{|g \circ \mu|(|f|) : g \in V_p^0\}$ ([12, Lemma 2, p. 23]).

2. Integration of vector-valued functions with respect to vector-valued measures

Let $\mathcal{A}$ be a σ-algebra of subsets of a set Y and $\mu : \mathcal{A} \to E_1$ a countably additive measure. A function $f : Y \to E_2$ will be called *μ-integrable* if $g_2 \circ f \in L^1(\mu)$ for every $g_2 \in E_2'$ and for every $A \in \mathcal{A}$, there exists a $z \in E$ such that $\int_A g_2 \circ f \, d(g_1 \circ \mu) = \langle g_1 \otimes g_2, z \rangle \; \forall \, (g_1, g_2) \in E_1' \times E_2'$. We write $\int_A f \, d\mu = z$. The collection of all μ-integrable $f : Y \to E_2$ will be denoted by $L^1(\mu, E_2)$. It is easily verified that $L^1(\mu, E_2)$ is a vector space and for every $f \in L^1(\mu, E_2)$ and for every $A \in \mathcal{A}$, $f \chi_A \in L^1(\mu, E_2)$; also $\mu : L^1(\mu, E_2) \to E$, $\mu(f) = \int f \, d\mu$, is linear. For $i = (1, 2)$, for a function $f : Y \to E_i$ and for a $p_i \in P_i$, the function $\|f\|_{p_i} : Y \to [0, \infty)$ is defined by $\|f\|_{p_i}(y) = \|f(y)\|_{p_i}$.

We first prove the following result.

Theorem 1. *Let $\mu : \mathcal{A} \to E_1$ be countably additive and $f : Y \to E_2$ be μ-integrable. Then $\nu(A) = \int_A f \, d\mu$ is countably additive.*

Proof: For $i = (1, 2)$, fix $p_i \in P_i$ and let

$$S_i = \left\{ g \in E_i' : \sup(|g(p_i^{-1}([0, 1]))|) \leq 1 \right\}.$$

E can be considered as a subspace of $C(S_1 \times S_2)$. Suppose that $\{A_n\} \subset \mathcal{A}$ and that the sets in $\mathcal{A}_n$ are pairwise disjoint. For for any $(g_1, g_2) \in S_1 \times S_2$ and for any $M \subset \mathbb{N}$, we have

$$\Big\langle g_1 \otimes g_2, \nu\Big(\bigcup_{n\in M} A_n\Big)\Big\rangle = \int_{\bigcup_{n\in M} A_n} (g_2 \circ f)\, d(g_1 \circ \mu) = \sum_{n\in M} \int_{A_n} (g_2 \circ f)\, d(g_1 \circ \mu).$$

Thus the mapping $\lambda : 2^{\mathbb{N}} \to C(S_1 \times S_2)$, $\lambda(M) = \nu(\bigcup_{n\in M} A_n)$, is countably additive for pointwise topology on $C(S_1 \times S_2)$. By ([9, Theorem 2.1, p. 163]), it is countably additive with norm topology on $C(S_1 \times S_2)$. This proves the result. $\square$

Theorem 2. *Suppose $\{f_n\}$ is a sequence in $L^1(\mu, E_2)$, $f : Y \to E_2$ and $f_n \to f$, in E_2, pointwise a.e. $[\mu]$. Assume that for any $p_2 \in P_2$, there is $\phi_{p_2} \in L^1(\mu)$ such that $\|f_n\|_{p_2} \le |\phi_{p_2}|$, a.e. $[\mu]$ for all n. Then $f \in L^1(\mu, E_2)$ and $\int f_n\, d\mu \to \int f\, d\mu$, in E.*

PROOF: Take a $p_1 \in P_1$, a $p_2 \in P_2$ and an $A \in \mathcal{A}$. For any $g_2 \le p_2$, $|g_2 \circ f| \le |\phi_{p_2}|$, *a.e.* $[\mu]$ and so $g_2 \circ f \in L^1(\mu)$. We first prove that $\bar{\mu}_{p_1}(g_2 \circ (f_n - f)) \to 0$, uniformly for $g_2 \le p_2$. If this is not true then, by taking a subsequence of $\{f_n\}$, if necessary, and again denoting it by $\{f_n\}$, there is a $c > 0$ and a sequence $\{g_2^n\} \subset E_2'$, $g_2^n \le p_2$ for all n, such that $\bar{\mu}_{p_1}(g_2^n \circ (f_n - f)) > c$ for all n. But $|g_2^n \circ (f_n - f)| \le 2\phi_{p_2}$ *a.e.* $[\mu]$ for all n, and $g_2^n \circ (f_n - f) \to 0$, *a.e.* $[\mu]$. By the dominated convergence theorem ([12, Theorem 2, p. 30]), this is a contradiction. This implies $\bar{\mu}_{p_1}(g_2 \circ (\chi_A(f_n - f))) \to 0$, uniformly for $g_2 \le p_2$.

Now take a $g_1 \le p_1$ and $g_2 \le p_2$. We have

$$\begin{aligned}
&\Big|\Big\langle g_1 \otimes g_2, \int_A (f_n - f_m)\, d\mu\Big\rangle\Big| = \Big|\int_A g_2 \circ (f_n - f_m)\, d(g_1 \circ \mu)\Big| \\
&\le \int_A |g_2 \circ (f_n - f)|\, d(|g_1 \circ \mu|) + \int_A |g_2 \circ (f_m - f)|\, d(|g_1 \circ \mu|) \\
&\le \bar{\mu}_{p_1}(g_2 \circ (f_n - f)) + \bar{\mu}_{p_1}(g_2 \circ (f_m - f))
\end{aligned}$$

which goes to 0 uniformly for $g_2 \le p_2$. If $z = \lim \int_A f_n\, d\mu$, then it is a simple verification that $\int_A f\, d\mu = z$, $f \in L^1(\mu, E_2)$ and $\int f_n\, d\mu \to \int f\, d\mu$, in E. This proves the result. $\square$

Corollary 3. *E_2-valued simple functions are in $L^1(\mu, E_2)$. If an $f : Y \to E_2$ is the pointwise limit, a.e. $[\mu]$, of a sequence of uniformly bounded simple functions in $L^1(\mu, E_2)$, then $f \in L^1(\mu, E_2)$.*

PROOF: Obviously every E_2-valued simple function is in $L^1(\mu, E_2)$. Take a $p_2 \in P_2$. There exists an $M > 0$ such that $\|f_n\|_{p_2} \le M$ for all n. By Theorem 1, the result follows. $\square$

Before the next theorem, we set some notations. For $i = (1,2)$, let Y_i be some sets, $\mathcal{A}_i$ be σ-algebras of subsets of Y_i and $\mu_i : \mathcal{A}_i \to E_i$ be countably additive measures. It is well-known ([3]) that there is a unique countably additive product measure $\mu : \mathcal{A}_1 \times \mathcal{A}_2 \to E_1 \check{\otimes} E_2$ such that $\mu(A_1 \times A_2) = \mu_1(A_1) \otimes \mu_2(A_2)$ for every $A_i \in \mathcal{A}_i$ for $i = (1,2)$ (we will derive this result as a consequence of one of our theorems). An example is given in [5, Theorem 12, p. 336] which shows that the classical Fubini theorem does not work for the injective tensor product $\mu_1 \otimes \mu_2$. With these notations, the following weak form of Fubini theorem is easy to prove.

Theorem 4. *Let $f(y_1, y_2) \in L^1(\mu)$ ($\mu = \mu_1 \otimes \mu_2$) and suppose, for $i = (1,2)$, that there are $\phi_i(y_i) \in L^1(\mu_i)$ such that $|f(y_1, y_2)| \leq |\phi_1(y_1)||\phi_2(y_2)|$ on $Y_1 \times Y_2$. Then*

(i) *for every $y_1 \in Y_1$, $h_2(y_1) = \int f(y_1, \cdot)\, d\mu_2$ is in $L^1(\mu_1, E_2)$ and for every $y_2 \in Y_2$, $h_1(y_2) = \int f(\cdot, y_2)\, d\mu_1$ is in $L^1(\mu, E_1)$;*
(ii) *$\int h_2\, d\mu_1 = \int h_1\, d\mu_2 = \int f\, d(\mu_1 \otimes \mu_2)$.*

Proof: First we will prove that $h_2(y_1)$ exists for every $y_1 \in Y_1$. As for every $y_1 \in Y_1$, $|f(y_1, \cdot)| \leq |\phi_1(y_1)||\phi_2(\cdot)|$ by ([12, Theorem 1, p. 27]), $f(y_1, \cdot)$ is μ_2-integrable and so for each $y_1 \in Y_1$, $h_2 : Y_1 \to E_2$, $h_2(y_1) = \int f(y_1, \cdot)\, d\mu_2$ is well-defined and for any $g_2 \in E_2'$, $g_2 \circ h_2(y_1) = \int f(y_1, \cdot)\, d(g_2 \circ \mu_2)$. Now we want to prove that $h_2 \in L^1(\mu_1, E_2)$.

Take an $A \in \mathcal{A}_1$. For any $(g_1, g_2) \in E_1' \times E_2'$, $(g_1, g_2) \circ \mu = (g_1 \circ \mu_1) \otimes (g_2 \circ \mu_2)$ on $A_1 \times A_2$ ($A_i \in \mathcal{A}_i$) and since both are countably additive, they are equal on $\mathcal{A}_1 \times \mathcal{A}_2$. Now $\chi_A f \in L^1(\mu)$ and so $\chi_A f$ is integrable relative to $(g_1 \circ \mu_1) \otimes (g_2 \circ \mu_2)$. Let $\int \chi_A f\, d\mu = z$.

$$\langle (g_1, g_2), z \rangle = \int \Big(\int f(y_1, \cdot)\, d(g_2 \circ \mu_2) \Big) \chi_A d(g_1 \circ \mu_1) = \int \chi_A (g_2 \circ h_2(y_1))\, d(g_1 \circ \mu_1).$$

So $h_2 \in L^1(\mu_1, E_2)$ and $\int f\, d\mu = \int h_2\, d\mu_1$. The case of h_1 can be dealt with in a similar way. □

Corollary 5. *Let $f(y_1, y_2) \in L^1(\mu_1 \otimes \mu_2)$ be bounded. Then for every $y_1 \in Y_1$, $h_2(y_1) = \int f(y_1, \cdot)\, d\mu_2$ is in $L^1(\mu, E_2)$ and for every $y_2 \in Y_2$, $h_1(y_2) = \int f(\cdot, y_2)\, d\mu_1$ is in $L^1(\mu, E_1)$ and $\int h_2\, d\mu_1 = \int h_1\, d\mu_2 = \int f\, d(\mu_1 \otimes \mu_2)$.*

Proof: The result follows from Theorem 4. □

3. Product of vector-valued measures on compact Hausdorff spaces

For a compact Hausdorff space X, $M(X)$ will denote all scalar-valued regular Borel measures on X and for a quasi-complete locally convex space F, $M(X, F)$ will denote all F-valued regular Borel measures on X. There is a one-to-one correspondence between $\mu \in M(X,F)$ and the weakly compact linear operator $\mu : C(X) \to F$ ([13]).

The proof of the following lemma is obvious and well-known.

Lemma 6. *For $i = (1,2)$, let X_i be compact Hausdorff spaces and $\mu_i \in M(X_i)$. Then, with injective tensor product topology on $C(X_1) \otimes C(X_2)$ (same as norm topology), the linear continuous mapping $\mu_1 \otimes \mu_2 : C(X_1) \otimes C(X_2) \to K$* ([7, p. 348]), *when uniquely, continuously extended to $\mu_1 \otimes \mu_2 : C(X_1 \times X_2) \to K$, is the product measure $\mu_1 \otimes \mu_2$.*

Theorem 7. *For $i = (1,2)$, let X_i be compact Hausdorff spaces and $\mu_i : C(X_i) \to E_i$ be weakly compact linear mappings. Then the linear mapping $\mu_1 \otimes \mu_2 : C(X_1) \otimes C(X_2) \to E$ is continuous (with respect to the norm topology on $C(X_1) \otimes C(X_2)$) and weakly compact. When extended to $C(X_1 \times X_2)$, the linear, weakly compact mapping $\mu_1 \otimes \mu_2 : C(X_1 \times X_2) \to E$ represents a regular Borel measure with the properties:*

(i) *$\mu(f_1 f_2) = \mu_1(f_1) \otimes \mu_2(f_2)$ for any $f_1 \in C(X_1)$ and any $f_2 \in C(X_2)$;*
(ii) *for Borel sets $B_i \subset X_i$ (for $i = (1,2)$), $\mu(B_1 \times B_2) = \mu_1(B_1) \otimes \mu_2(B_2)$;*
(iii) *for any $(g_1, g_2) \in E_1' \times E_2'$ and an $f \in C(X_1 \times X_2)$,*

$$\left\langle \int f \, d(\mu_1 \otimes \mu_2), (g_1 \otimes g_2) \right\rangle = \int f \, d((g_1 \circ \mu_1) \otimes (g_2 \circ \mu_2)),$$

where $((g_1 \circ \mu_1) \otimes (g_2 \circ \mu_2))$ is the usual product of the scalars measures $(g_1 \circ \mu_1)$ and $(g_2 \circ \mu_2)$.

Proof: The continuity follows from [7, p. 348]. For $i = (1,2)$, let S_i be equicontinuous, convex and $\sigma(E_i', E_i)$-closed subsets of E_i'. We consider E to be a subspace of $C(S_1 \times S_2)$. To prove weak compactness of the operator, take a uniformly bounded sequence $\{f_n\} \subset C(X_1) \otimes C(X_2)$ such that $f_n f_m = 0$ for every n and for every m with $n \neq m$ ([2, Corollary 17, p. 160]); we have to prove that $(\mu_1 \otimes \mu_2)(f_n) \to 0$. Suppose this is not true. This means, by taking a subsequence of $\{f_n\}$, if necessary, and again denoting it by $\{f_n\}$, that there are sequences $\{\phi_n^i\} \subset S_i$, $i = (1,2)$, and a $c > 0$ such that $((\phi_n^1 \circ \mu_1) \otimes (\phi_n^2 \circ \mu_2))(f_n) > c$ for all n. Putting $g_n(x_1) = (\phi_n^2 \circ \mu_2)(f_n(x_1, \cdot))$, we see that g_n is uniformly bounded and $g_n \to 0$ pointwise on X_1. Since the set $\{(\phi_n^1 \circ \mu_1)\}$ is relatively weakly compact in $M(X_1)$, we get $(\phi_n^1 \circ \mu_1)(g_n) \to 0$, which is a contradiction.

Considering $\mu_1 \otimes \mu_2$ as an E-valued regular Borel measure on $X_1 \times X_2$, proofs of the properties (i), (ii), (iii) are routine verifications ([11]). □

Now we derive from the above theorem the main result of ([3]).

Theorem 8 ([3])**.** *For $i = (1,2)$, let Y_i be some sets $\mathcal{A}_i$ be σ-algebras of subsets of Y_i and $\mu_i : \mathcal{A}_i \to E_i$ be countably additive measures. Then there is a unique* *countably additive product measure* $\mu : \mathcal{A}_1 \times \mathcal{A}_2 \to E_1 \breve{\otimes} E_2$ *such that* $\mu(A_1 \times A_2) = \mu_1(A_1) \otimes \mu_2(A_2)$ *for every* $A_i \in \mathcal{A}_i \quad (i = (1,2))$.

Proof: For $i = (1,2)$, let

$$B_i = \{f : Y_i \to K \mid f \text{ bounded and } \mathcal{A}_i\text{- measurable}\}.$$

As in [10], there are compact Hausdorff spaces $\tilde{Y_i}$, in which Y_i are dense such that $C(\tilde{Y_i})_{|Y_i} = B_i$. There is a one-to-one, onto, linear, order-preserving, sup-norm preserving mapping from $C(\tilde{Y_i})$ to B_i. Thus we get measures $\tilde{\mu_i} : C(\tilde{Y_i}) \to E_i$. By Theorem 7, we get the product measure $\mu = \tilde{\mu_1} \otimes \tilde{\mu_2} : C(\tilde{Y_1} \times \tilde{Y_2}) \to E$. This can be considered as a regular Baire measure. Take a compact G_δ subset $C \subset \tilde{Y_1} \times \tilde{Y_2} \setminus Y_1 \times Y_2$. There is a sequence $\{f_n\} \subset C(\tilde{Y_1} \times \tilde{Y_2})$ such that $f_n \downarrow \chi_C$. Because of the norm-denseness of $C(\tilde{Y_1}) \otimes C(\tilde{Y_2})$ in $C(\tilde{Y_1} \times \tilde{Y_2})$, there is a norm-bounded sequence $\{h_n\} \subset C(\tilde{Y_1}) \otimes C(\tilde{Y_2})$ such that $h_n \to \chi_C$, pointwise on $\tilde{Y_1} \times \tilde{Y_2}$.

For $i = (1,2)$, let S_i be $\sigma(E_i', E_i)$-closed, convex and equicontinuous subsets of E_i'. E can be considered as a subspace of $C(S_1 \times S_2)$. Since μ is a weakly compact mapping, $\{\mu(h_n) : n \in \mathbb{N}\}$ is relatively weakly compact in E and so its weak convergence is the same as pointwise convergence on $S_1 \times S_2$. For $g_i \in S_i$,

$$\langle (g_1, g_2), \mu(C) \rangle = \lim_{n \to \infty} \int h_n \, d((g_1 \circ \tilde{\mu_1}) \otimes (g_2 \circ \tilde{\mu_2})).$$

Now $(g_1 \circ \mu_1) \otimes (g_2 \circ \mu_2)$ is the product measure, $(h_n)_{|(Y_1 \times Y_2)} \in B_1 \otimes B_2$ and $h_n \to 0$ on $Y_1 \times Y_2$. This gives $\langle (g_1, g_2), \mu(C) \rangle = 0$ and so $\mu(C) = 0$. By regularity, $\mu(Q) = 0$, for every Baire set $Q \subset \tilde{Y_1} \times \tilde{Y_2} \setminus Y_1 \times Y_2$. Now $(\mathcal{B}_0(\tilde{Y_1} \times \tilde{Y_2})) \cap (Y_1 \times Y_2) \supset \mathcal{A}_1 \times \mathcal{A}_2$ and so for a $P \in \mathcal{A}_1 \times \mathcal{A}_2$, there is a Baire set P_0 in $\tilde{Y_1} \times \tilde{Y_2}$ such that $P_0 \cap (Y_1 \times Y_2) = P$; now we can define $(\mu_1 \otimes \mu_2)(P) = \mu(P_0)$. The required properties are easily verified. $\square$

4. Product of vector-valued τ-smooth measures on completely regular Hausdorff spaces

For a completely regular Hausdorff space X and a quasi-complete locally convex space F, a countably additive $\mu : \mathcal{B}(X) \to F$ is called τ-*smooth* if for an increasing net $\{V_\alpha\}$ of open subsets of X, $\mu(\bigcup_\alpha V_\alpha) = \lim \mu(V_\alpha)$. This μ gives rise to a weakly compact linear map $\mu : C_b(X) \to F$ with the property that for every $f \in F'$, $f \circ \mu \in M_\tau(X)$. Conversely if a weakly compact linear map $\mu : C_b(X) \to F$ has the property for every $f \in F'$, $f \circ \mu \in M_\tau(X)$, then it is easy to prove that such a μ gives a unique τ-smooth Borel measure. To prove this, we get a linear, continuous, weakly compact $\tilde{\mu} : C(\tilde{X}) \to F$ and so $\tilde{\mu}$ can be considered as a regular Borel measure on $\tilde{X}$. Also we have $\mathcal{B}(\tilde{X}) \cap X = \mathcal{B}(X)$. Take a closed set

$C \subset \tilde{X} \setminus X$; there exists a net $\{f_\alpha\} \subset C(\tilde{X})$ such that $f_\alpha \downarrow \chi_C$. This means, in $(C_b(X), \beta_\tau)$, that $(f_\alpha)_{|X} \to 0$ ([17]). Thus for every closed set $C \subset \tilde{X} \setminus X$, $\tilde{\mu}(C) = 0$, and so, by regularity, for every $p \in P$, $\bar{\tilde{\mu}}_p(B) = 0$, for all Borel sets $B \subset \tilde{X} \setminus X$. For any Borel set $A \subset X$, define $\nu(A) = \tilde{\mu}(B)$, B being any Borel subset of $\tilde{X}$ with $B \cap X = A$. It is a routine verification that ν is well-defined, countably additive and for any $f \in C_b(X)$, $\int f\,d\nu = \int f\,d\mu$. Also by the regularity of $\tilde{\mu}$, it can be easily verified that ν is τ-smooth. Other things need routine verification.

The set of all F-valued τ-smooth measures on X will be denoted by $M_\tau(X, F)$.

The following result is well-known ([1]); we give a different proof.

Lemma 9. (a) *For $i = (1,2)$, let $\mu_i \in M_\tau(X_i)$. Then there is a unique $\mu \in M_\tau(X_1 \times X_2)$ such that $\mu(f_1 f_2) = \mu_1(f_1) \otimes \mu_2(f_2)$ for any $f_1 \in C_b(X_1)$ and any $f_2 \in C_b(X_2)$. Also for any $f \in C_b(X_1 \times X_2)$,*

$$\mu(f) = \int \Big(\int f(x,y)\,d\mu_2(y) \Big)\,d\mu_1(x) = \int \Big(\int f(x,y)\,d\mu_1(x) \Big)\,d\mu_2(y).$$

(b) *For any μ-integrable $f : X_1 \times X_2 \to K$, for μ_1-almost all x_1, $f(x_1, \cdot)$ is μ_2-integrable and for μ_2-almost all x_2, $f(\cdot, x_2)$ is μ_1-integrable, and*

$$\mu(f) = \int \Big(\int f(x_1,x_2)\,d\mu_2(x_2) \Big)\,d\mu_1(x_1) = \int \Big(\int f(x_1,x_2)\,d\mu_1(x_1) \Big)\,d\mu_2(x_2).$$

Proof: (a) We break up the proof into several steps:

I. For any $f \in C_b(X_1 \times X_2)$, the function $h(x) = \int f(x,y)\,d\mu_2(y)$ is in $C_b(X_1)$. Using the τ-additivity of μ_2, it is easy to verify this.

II. First assume that for $i = (1,2)$, $\mu_i \in M_\tau^+(X_i)$. If $f_\alpha \downarrow 0$ in $C_b(X_1 \times X_2)$ and $h_\alpha(x) = \int f_\alpha(x,y)\,d\mu_2(y)$ then $h_\alpha \downarrow 0$ in $C_b(X_1)$. This means, for $f \in C_b(X_1 \times X_2)$, that the measures $\nu_1(f) = \int(\int f(x,y)\,d\mu_2(y))\,d\mu_1(x)$ and $\nu_2(f) = \int(\int f(x,y)\,d\mu_1(x))\,d\mu_2(y)$ are in $M_\tau^+(X_1 \times X_2)$. Also $\nu_1 = \nu_2$ on $C_b(X_1) \otimes C_b(X_2)$.
In the general case, the real and the imaginary parts of μ_i can be written as the difference of positive elements of $M_\tau^+(X_i)$ and so the above result holds without the positivity of μ_1 and μ_2.

III. For any $\mu \in M_\tau^+(X_1 \times X_2)$, consider $C_b(X_1 \times X_2)$ with the norm induced by $L^1(\mu)$. Then $C_b(X_1) \otimes C_b(X_2)$ is dense in $C_b(X_1 \times X_2)$. Suppose this is not true. Then there is a $g \in L^\infty(\mu)$ such that $\int hg\,d\mu = 0$ for every $h \in C_b(X_1) \otimes C_b(X_2)$, but $\int fg\,d\mu \neq 0$ for some $f \in C_b(X_1 \times X_2)$. Since $\mu_0 = g\mu \in M_\tau(X_1 \times X_2)$, $\mu_0 \equiv 0$ on $C_b(X_1) \otimes C_b(X_2)$. This means that for an open set $V_1 \times V_2 \subset X_1 \times X_2$, $\mu_0(V_1 \times V_2) = 0$. Thus $\mu_0(V) = 0$ for every open set $V \subset X_1 \times X_2$ and so $\mu_0 \equiv 0$ on $C_b(X_1 \times X_2)$. From this it follows that $g = 0$ *a.e.* $[\nu]$ and so we have a contradiction.

IV. Since $\nu_1 = \nu_2$ on $C_b(X_1) \otimes C_b(X_2)$, by II, III $\nu_1 = \nu_2$ on $C_b(X_1 \times X_2)$. This is the product measure and we denote it by $(\mu_1 \otimes \mu_2)$.

(b) The problem can be easily reduced to positive μ_1, μ_2. Suppose first that f is a real-valued, bounded and lower semi-continuous. Take a bounded net $\{f_\alpha\} \subset C_b(X_1 \times X_2)$, $f_\alpha \uparrow f$. It is easily verified that, for all x_1, $f(x_1, \cdot)$ is μ_2-integrable and for all x_2, $f(\cdot, x_2)$ is μ_1-integrable, and

$$\mu(f) = \int \Big(\int f(x_1, x_2)\, d\mu_2(x_2) \Big)\, d\mu_1(x_1) = \int \Big(\int f(x_1, x_2)\, d\mu_1(x_1) \Big)\, d\mu_2(x_2).$$

Let $\mathcal{F} = \{f : X_1 \times X_2 \to K : f$ Borel measurable, $\|f\| \le 1\}$ and $\mathcal{F}_0 = \{f \in \mathcal{F} : f$ satisfies the conditions of (b)$\}$. It is a simple verification that $\mathcal{F}_0$ is sequentially closed in $\mathcal{F}$. Combining this with the fact that the lower semi-continuous f with $\|f\| \le 1$ are in $\mathcal{F}_0$, we easily see that $\mathcal{F} = \mathcal{F}_0$. Combining these results, we see that Fubini theorem holds for any bounded, Borel measurable, μ-integrable $f : X_1 \times X_2 \to K$. Suppose a bounded, non-negative $f : X_1 \times X_2 \to K$ is such that $f = 0$, μ-almost everywhere. We get a Borel measurable, bounded, non-negative function $f_0 : X_1 \times X_2 \to K$ such that $f \le f_0 = 0$ μ-almost everywhere and so Fubini theorem holds for f; this means that Fubini theorem holds for any bounded, μ-integrable function $f : X_1 \times X_2 \to K$. Now let $h : X_1 \times X_2 \to K$ be μ-integrable and $h \ge 0$. For an $n \in \mathbb{N}$, put $h_n = \inf(h, n)$. This means that

$$\mu(h) = \lim_n \mu(h_n) = \lim_n \int \Big(\int h_n\, d\mu_1 \Big)\, d\mu_2 = \lim_n \int \Big(\int h_n\, d\mu_2 \Big)\, d\mu_1$$

and so

$$\mu(h) = \int \Big(\int h\, d\mu_1 \Big)\, d\mu_2 = \int \Big(\int h\, d\mu_2 \Big)\, d\mu_1.$$

So $\int h\, d\mu_1$ is finite almost everywhere and integrable relative to μ_2 and also $\int h\, d\mu_2$ is finite almost everywhere and integrable relative to μ_1. Hence, Fubini theorem holds for all μ-integrable functions $f : X_1 \times X_2 \to K$. □

For proving the next theorem, we need the following result:

Lemma 10. (a) *Let $\nu \in M_\tau^+(X_1 \times X_2)$. Then, in $L_1(\nu)$, the closed unit ball of $C_b(X_1) \otimes C_b(X_2)$ is dense in the closed unit ball of $C_b(X_1 \times X_2)$.*

(b) *For any $f \in C_b(X_1 \times X_2)$, $\|f\| \le 1$, there is a net $\{f_\alpha\}$ in the closed unit ball of $C_b(X_1) \otimes C_b(X_2)$ such that $f_\alpha \to f$, pointwise on $M_\tau(X_1 \times X_2)$.*

Proof: (a) We will make use of the following well-known result which follows easily from the regularity of measure:

Let μ be a finite, positive, regular Borel measure on a compact Hausdorff space Y. Then, in $L_1(\mu)$, the closed unit ball of $C(Y)$ is dense in the set of all scalar-valued, Borel measurable functions, bounded by 1, on Y.

We assume $\nu(1) = 1$. Fix an $f \in C_b(X_1 \times X_2)$ with $|f| \leq 1$. By Lemma 9, III, there is a sequence $\{f_n\} \subset C_b(X_1) \otimes C_b(X_2)$ such that $\nu(|f_n - f|) \to 0$. By taking a subsequence, if necessary, we assume that $f_n \to f$ *a.e.* $[\nu]$.

Denote the Borel set $B = \{x \in (X_1 \times X_2) : \lim f_n(x)$ exists and is finite$\}$ and define $g : (X_1 \times X_2) \to K$ as $g(x) = \lim f_n(x)$ if it exists and is finite, and 0 otherwise. Then g is Borel measurable, $\nu(B) = 1$, $|g\chi_B| \leq 1$ *a.e.* $[\nu]$ and $f = g\chi_B$ *a.e.* $[\nu]$.

Define the linear, continuous, and positive mapping $\tilde{\mu} : C(\tilde{X_1}) \otimes C(\tilde{X_2}) \to K$, $\tilde{\mu}(\sum \tilde{f_i^1} \otimes \tilde{f_i^2}) = \nu(\sum f_i^1 \otimes f_i^2)$. Since $C(\tilde{X_1}) \otimes C(\tilde{X_2})$ is norm-dense in $C(\tilde{X_1} \times \tilde{X_2})$, this uniquely extends to a linear, continuous, and positive mapping $\tilde{\mu} : C(\tilde{X_1} \times \tilde{X_2}) \to K$ which may be considered as a regular Borel measure on $\tilde{X_1} \times \tilde{X_2}$. Since ν is τ-smooth, for any bounded Borel measurable function $h : \tilde{X_1} \times \tilde{X_2} \to K$, $\tilde{\mu}(h) = \nu(h_{|(X_1 \times X_2)})$. From $\tilde{\mu}(|\tilde{f_n} - \tilde{f_m}|) \to 0$, by taking a subsequence if necessary, we get that $\tilde{f_n}$ is convergent *a.e.* $[\tilde{\mu}]$ on $\tilde{X_1} \times \tilde{X_2}$. Let B_0 be the Borel subset of $\tilde{X_1} \times \tilde{X_2}$ on which $\tilde{f_n}$ is convergent and is finite and define $g_0 : (\tilde{X_1} \times \tilde{X_2}) \to K$ as $g_0(x) = \lim \tilde{f_n}(x)$ if it exists and is finite, and 0 otherwise. g_0 is Borel measurable. We also have $\tilde{\mu}(B_0) = 1 = \nu(B)$, $B_0 \cap (X_1 \times X_2) \supset B$, and $g_0\chi_B = g\chi_B$. Thus there is a sequence $\{h_n\}$ in the closed unit ball of $C(\tilde{X_1}) \otimes C(\tilde{X_2})$ such that $\tilde{\mu}(|h_n - g_0\chi_{B_0}|) \to 0$. Translating to ν, there is a sequence $w_n = (h_n)_{|(X_1 \times X_2)}$ in the closed unit ball of $C_b(X_1) \otimes C_b(X_2)$ such that $\nu(|w_n - g\chi_B|) \to 0$ and so $\nu(|w_n - f|) \to 0$. This completes the proof.

(b) Putting $P = M_\tau^+(X_1 \times X_2)$, we see that P is filtering upwards with natural order. Take a $\lambda \in P$ and an $n \in \mathbb{N}$. By (a), there is function $f_{(\lambda,n)}$ in the closed unit ball of $C_b(X_1) \otimes C_b(X_2)$ such that $\lambda(|f - f_{(\lambda,n)}|) \leq \frac{1}{n}$. Taking $\alpha = (\lambda, n)$, the result follows. $\square$

Now we come to the product of vector-valued τ-smooth measures:

Theorem 11. *For $i = (1, 2)$, let $\mu_i \in M_\tau(X_i, E_i)$. Then*

(a) *there exists a unique $\mu \in M_\tau(X_1 \times X_2, E_1 \breve{\otimes} E_2)$ such that $\mu(f_1 f_2) = \mu_1(f_1) \otimes \mu_2(f_2)$ for any $f_1 \in C_b(X_1)$ and any $f_2 \in C_b(X_2)$; also for Borel sets $B_i \subset X_i$ $(i = (1, 2))$, $\mu(B_1 \times B_2) = \mu_1(B_1) \otimes \mu_2(B_2)$. This measure μ is denoted by $\mu_1 \otimes \mu_2$.*

(b) *(Fubini-type result) Take an $f(x_1, x_2) \in L^1(\mu)$ and suppose, for $i = (1, 2)$, that there are $\phi_i(x_i) \in L^1(\mu_i)$ such that $|f(x_1, x_2) \leq |\phi_1(x_1)||\phi_2(x_2)|$ on $X_1 \times X_2$. Then*

   (i) *for every $x_1 \in X_1$, $h_2(x_1) = \int f(x_1, \cdot)\, d\mu_2$ is in $L^1(\mu_1, E_2)$ and for every $x_2 \in X_2$, $h_1(x_2) = \int f(\cdot, x_2)\, d\mu_1$ is in $L^1(\mu_2, E_1)$;*

   (ii) *$\int h_2\, d\mu_1 = \int h_1\, d\mu_2 = \int f\, d\mu$.*

Proof: (a) By Theorem 7, μ is defined on $C_b(X_1)\otimes C_b(X_2)$ and the closed unit ball B, of $C_b(X_1)\otimes C_b(X_2)$, is mapped into a relatively weakly compact subset of E. Thus the closure of $(\mu_1\otimes\mu_2)(B)$ in E, denoted by Q, is convex and weakly compact. For $i=(1,2)$, let S_i be equicontinuous, convex, $\sigma(E_i', E_i)$-compact subsets of E_i'. Considering $E\subset C(S_1\times S_2)$, the pointwise and weak topologies on Q are identical. For an $h\in C_b(X_1\times X_2)$, define

$$\mu(h): S_1\times S_2\to K,\ \langle (g_1,g_2),\mu(h)\rangle = \int h\, d((g_1\circ\mu_1)\otimes(g_2\circ\mu_2)).$$

Now assume that $\|h\|\le 1$. Using Lemma 10, take a net $\{h_\alpha\}\subset B$ such that $h_\alpha\to h$, pointwise on $M_\tau(X_1\times X_2)$. Since $((g_1\circ\mu_1)\otimes(g_2\circ\mu_2))\in M_\tau(X_1\times X_2)$ (Lemma 9), $\mu(h)\in Q\subset C(S_1\times S_2)$. Thus the mapping $\mu=\mu_1\otimes\mu_2: C_b(X_1\times X_2)\to E$ is weakly compact. Now $Q\subset C(S_1\times S_2)$ and is weakly compact, so weak and pointwise topologies, on $C(S_1\times S_2)$, coincide on Q. Since for any $(g_1,g_2)\in E_1'\times E_2'$, $(g_1,g_2)\circ\mu = ((g_1\circ\mu_1)\otimes(g_2\circ\mu_2))\in M_\tau(X_1\times X_2)$, we get that for every $\phi\in E'$, $\phi\circ\mu\in M_\tau(X_1\times X_2)$. This proves that $\mu_1\otimes\mu_2\in M_\tau(X_1\times X_2, E)$.

(b) First we will prove that $h_2(x_1)$ exists for every $x_1\in X_1$. As for every $x_1\in X_1$, $|f(x_1,\cdot)|\le|\phi_1(x_1)||\phi_2(\cdot)|$ by [12, Theorem 1, p. 27], $f(x_1,\cdot)$ is μ_2-integrable and so for each $x_1\in X_1$, $h_2: X_1\to E_2$, $h_2(x_1)=\int f(x_1,\cdot)\,d\mu_2$ is well-defined and for any $g_2\in E_2'$, $g_2\circ h_2(x_1)=\int f(x_1,\cdot)\,d(g_2\circ\mu_2)$. Now we want to prove that $h_2\in L^1(\mu_1,E_2)$.

Take an $A\in\mathcal{A}_1$. For any $(g_1,g_2)\in E_1'\times E_2'$,

$$(g_1,g_2)\circ\mu = (g_1\circ\mu_1)\otimes(g_2\circ\mu_2)$$

on $C_b(X_1)\otimes C_b(X_2)$ and, since both are τ-smooth,

$$(g_1,g_2)\circ\mu = (g_1\circ\mu_1)\otimes(g_2\circ\mu_2)$$

on $C_b(X_1\times X_2)$; and so, as τ-smooth measures, they are equal.

Now $\chi_A f\in L^1(\mu)$ and so $\chi_A f$ is integrable relative to $(g_1\circ\mu_1)\otimes(g_2\circ\mu_2)$. Let $\int\chi_A f\,d\mu = z$.

$$\langle (g_1,g_2),z\rangle = \int\Big(\int f(x_1,\cdot)\,d(g_2\circ\mu_2)\Big)\chi_A d(g_1\circ\mu_1) = \int \chi_A(g_2\circ h_2(x_1))\,d(g_1\circ\mu_1).$$

So $h_2\in L^1(\mu_1,E_2)$ and $\int f\,d\mu=\int h_2\,d\mu_1$. The case of h_1 can be dealt with in a similar way. □

5. Product of vector-valued tight measures on completely regular Hausdorff spaces

For $i=(1,2)$, let $\mu_i\in M_t(X_i)$ ([17], [8]). Then $\mu_i\in M_\tau(X_i)$. By Lemma 9, $\mu=\mu_1\otimes\mu_2\in M_\tau(X_1\times X_2)$. It is easy to see that $\mu\in M_t(X_1\times X_2)$. To prove this, we see that $|\mu| \le |\mu_1| \otimes |\mu_2|$ and, for any compact subsets $C_i \subset X_i$ $(i = 1, 2)$, $X_1 \times X_2 \setminus C_1 \times C_2 \subset ((X_1 \setminus C_1) \times X_2) \cup (X_1 \times (X_2 \setminus C_2))$. This means that $|\mu|(X_1 \times X_2 \setminus C_1 \times C_2) \le |\mu_1|(X_1 \setminus C_1)|\mu_2|(X_2) + |\mu_1|(X_1)|\mu_2|(X_2 \setminus C_2)$ and from this it follows that $\mu \in M_t(X_1 \times X_2)$.

For a completely regular Haurdorff space X, and a locally convex space F, a measure $\mu : \mathcal{B}(X) \to F$ is called *tight* if for every $f \in F'$, $f \circ \mu \in M_t(X)$; this does imply that, in the original topology of F, it is inner regular by the compact subsets of X ([13]). The set of all F-valued tight measures on X will be denoted by $M_t(X, F)$.

Now we prove the main theorem of this section.

Theorem 12. (a) *For $i = (1, 2)$, let $\mu_i \in M_t(X_i, E_i)$. Then there exists a unique $\mu \in M_t(X_1 \times X_2, E)$ such that $\mu(f_1 f_2) = \mu_1(f_1) \otimes \mu_2(f_2)$ for any $f_1 \in C_b(X_1)$ and any $f_2 \in C_b(X_2)$; also for Borel sets $B_i \subset X_i$ $(i = (1, 2))$, $\mu(B_1 \times B_2) = \mu_1(B_1) \otimes \mu_2(B_2)$. This measure μ is denoted by $\mu_1 \otimes \mu_2$.*

(b) *(Fubini-type result) Take an $f(x_1, x_2) \in L^1(\mu)$ and suppose, for $i = (1, 2)$, there are $\phi_i(x_i) \in L^1(\mu_i)$ such that $|f(x_1, x_2)| \le |\phi_1(x_1)||\phi_2(x_2)|$ on $X_1 \times X_2$. Then*

(i) *for every $x_1 \in X_1$, $h_2(x_1) = \int f(x_1, \cdot)\, d\mu_2$ is in $L^1(\mu_1, E_2)$ and for every $x_2 \in X_2$, $h_1(x_2) = \int f(\cdot, x_2)\, d\mu_1$ is in $L^1(\mu_2, E_1)$;*

(ii) *$\int h_2\, d\mu_1 = \int h_1\, d\mu_2 = \int f\, d\mu$.*

Proof: (a) By Theorem 11, there is a unique measure $\mu_1 \otimes \mu_2 \in M_\tau(X_1 \times X_2, E)$. The only thing to be verified is that $\mu_1 \otimes \mu_2 \in M_t(X_1 \times X_2, E)$. For $i = (1, 2)$, fix $p_i \in P_i$ and let

$$S_i = \left\{ g \in E_i' : |g(p_i^{-1}([0, 1]))| \le 1 \right\}.$$

E can be considered as a subspace of $C(S_1 \times S_2)$. Since $\mu = \mu_1 \otimes \mu_2$ has relatively weakly compact range in $E_1 \breve{\otimes} E_2$, the weak topology on the range is identical with the pointwise topology on $S_1 \times S_2$. Since for any $(g_1, g_2) \in S_1 \times S_2$, $(g_1 \circ \mu_1) \otimes (g_2 \circ \mu_2) \in M_t(X_1 \times X_2)$, μ is tight in the weak topology and so it is tight ([13]).

(b) This follows from Theorem 11(b). □

6. Product of vector-valued measures when both are not τ-smooth

It is shown in [1] for $i = (1, 2)$ and $\mu_i \in M_\sigma(X_i)$, unless both μ_1 and μ_2 are in $M_\tau(X_1)$ and $M_\tau(X_2)$, the product measure may not exist in $M_\sigma(X_1 \times X_2)$ for which the Fubini theorem works for functions in $C_b(X_1 \times X_2)$. In this section we consider some special cases and prove the existence of product Baire measures satisfying some form of Fubini's theorem.

In this section we suppose that X_2 is compact and the measures we consider on X_1 are in $M_\infty(X_1)$ ([17], [8]); in [17] M_∞ is denoted by M_s and these measures are called separable measures. First we make some comments on separable measures on a completely regular Hausdorff space X:

Let $\{f_\alpha\}$ be an e.b. set (that is, uniformly bounded equicontinuous subset of $C_b(X)$) such that $f_\alpha \to 0$, pointwise on X. If a $\mu \in M_\sigma(X)$ has the property that $\mu(f_\alpha) \to 0$ for all such e.b. sets, then $\mu \in M_\infty(X)$. For a quasi-complete locally convex space F, $M_\infty(X, F)$ denotes those linear weakly compact $\mu : C_b(X) \to F$ which have the property that $f \circ \mu \in M_\infty(X)$ for all $f \in F'$. There is a locally convex topology, called β_∞, on $C_b(X)$ such that $\mu : C_b(X) \to K$ is in $M_\infty(X)$ iff μ is continuous ([17]); this topology is Mackey. So if a linear, weakly compact $\mu : C_b(X) \to F$ has the property that $f \circ \mu \in M_\infty(X, F)$ for all $f \in F'$, then $\mu : (C_b(X), \beta_\infty) \to F$ is continuous with weak topology on F and, since β_∞ is Mackey, it is also continuous in the original topology on F.

We start with a lemma.

Lemma 13. *Let $f \in C_b(X_1 \times X_2)$, with $\|f\| \le 1$, and $\varepsilon > 0$. Then there is a partition of unity $\{g_\alpha\}$ in X_1 and $\{h_\alpha\} \subset C(X_2)$ with $\|h_\alpha\| \le 1$ for all α, such that $\|f - \sum_\alpha g_\alpha h_\alpha\| \le \varepsilon$.*

PROOF: As in [8, p. 201], define a continuous semimetric d on X_1, $d(x, y) = \sup_{x_2} |f(x, x_2) - f(y, x_2)|$. Proceeding as in [8, p. 201], we get the result. □

Lemma 14. *Let $f \in C_b(X_1 \times X_2)$ with $\|f\| \le 1$, $\mu_1 \in M_\infty(X_1)$ and $\mu_2 \in M(X_2) = M_\infty(X_2)$. Then the functions $\int f\, d\mu_1$ and $\int f\, d\mu_2$ are Baire measurable and*

$$\int \Big(\int f\, d\mu_1 \Big)\, d\mu_2 = \int \Big(\int f\, d\mu_2 \Big)\, d\mu_1.$$

PROOF: In Lemma 13, take $\varepsilon = \frac{1}{n}$. There is a partition of unity $\{g^n_\alpha\}$ in X_1 and $\{h^n_\alpha\} \subset C(X_2)$ with $\|h^n_\alpha\| \le 1$ for all α such that $\|f - f_n\| \le \frac{1}{n}$ where $f_n = \sum_\alpha g^n_\alpha h^n_\alpha$. Now $\int f_n\, d\mu_1 = \sum_\alpha c^n_\alpha h^n_\alpha$, where $c^n_\alpha = \int g^n_\alpha\, d\mu_1$, is continuous on X_2 and so $\int f\, d\mu_1$ is Baire measurable; in a similar way, it is easily seen that $\int f\, d\mu_2$ is Baire measurable. Now it is easily verified that $\int (\int f\, d\mu_1)\, d\mu_2 = \int (\int f\, d\mu_2)\, d\mu_1$. □

Lemma 15. *Let $\{f_\alpha\} \subset C_b(X_1 \times X_2)$ be an e.b. set and $\varepsilon > 0$. Then there is a partition of unity $\{g_\beta\}$ in X_1 and $\{h^\alpha_\beta\} \subset C(X_2)$ with $\|h^\alpha_\beta\| \le 1$ for all α, β and such that $\|f_\alpha - \sum_\beta g_\beta h^\alpha_\beta\| \le \varepsilon$ for all α.*

PROOF: As in Lemma 14, define a continuous metric d on X_1, $d(x, y) = \sup_{(x_2, \alpha)} |f_\alpha(x, x_2) - f_\alpha(y, x_2)|$. As in Lemma 13, we get the result. □

Theorem 16. *Given $\mu_1 \in M_\infty(X_1)$ and $\mu_2 \in M(X_2)$, there is a unique Baire measure $\mu = \mu_1 \otimes \mu_2 \in M_\infty(X_1 \times X_2)$ such that*

(a) *for any $f \in C_b(X_1 \times X_2)$, $\int(\int f\, d\mu_2)\, d\mu_1 = \int(\int f\, d\mu_1)\, d\mu_2$; in particular $\int(f_1 f_2)\, d(\mu_1 \otimes \mu_2) = (\int f_1\, d\mu_1)(\int f_2\, d\mu_2)$, for $f_1 \in C_b(X_1)$ and $f_2 \in C_b(X_2)$;*

(b) *for Baire sets $B_i \subset X_i$ $(i = (1,2))$, $(\mu_1 \otimes \mu_2)(B_1 \times B_2) = \mu_1(B_1) \otimes \mu_2(B_2)$;*

(c) *for any μ-integrable $f : X_1 \times X_2 \to K$, for μ_1-almost all x_1, $f(x_1, \cdot)$ is μ_2-integrable and for μ_2-almost all x_2, $f(\cdot, x_2)$ is μ_1-integrable, and*

$$\mu(f) = \int \Big(\int f(x_1, x_2)\, d\mu_2(x_2) \Big)\, d\mu_1(x_1) = \int \Big(\int f(x_1, x_2)\, d\mu_1(x_1) \Big)\, d\mu_2(x_2).$$

PROOF: (a) Define $\int f\, d(\mu) = \int f\, d(\mu_1 \otimes \mu_2) = \int(\int f\, d\mu_1)\, d\mu_2$. By Lemma 14, it is also equal to $\int(\int f\, d\mu_2)\, d\mu_1$. To prove that $\mu \in M_\infty(X_1 \times X_2)$, take an e.b. set $\{f_\alpha\} \subset C_b(X_1 \times X_2)$ such that $|f_\alpha| \le 1$ for all α and $f_\alpha \to 0$, pointwise. Fix $n \in \mathbb{N}$. By Lemma 15, there is partition of unity $\{g_{\beta,n}\}$ in X_1 and $\{h^\alpha_{\beta,n}\} \subset C(X_2)$ with $\|h^\alpha_{\beta,n}\| \le 1$ for all α and β such that $\|f_\alpha - \sum_\beta g_{\beta,n} h^\alpha_{\beta,n}\| \le \frac{1}{n}$. Now the set $\phi_\alpha = \sum_\beta g_{\beta,n} h^\alpha_{\beta,n}$ is an e.b. set and is pointwise convergent to, say ϕ (note that n is fixed). It is easy to see that $\int(\int \phi_\alpha\, d\mu_1)\, d\mu_2 \to \int(\int \phi\, d\mu_1)\, d\mu_2$. Also $|f_\alpha - \phi_\alpha| \le \frac{1}{n}$ and so $|\phi| \le \frac{1}{n}$. This proves that $\int\int f_\alpha\, d\mu_1\, d\mu_2 \to 0$.

(b) This follows form the regularity properties of measures and (a).

(c) The proof is very similar to Lemma 9(b). □

To extend the above theorem to the vector case, we start with a lemma:

Lemma 17. (a) *Fix a $\mu \in M^+_\infty(X_1 \times X_2)$ and consider on $C_b(X_1 \times X_2)$ the topology induced by $L_1(\mu)$. Then the closed unit ball of $C_b(X_1) \otimes C_b(X_2)$ is dense in the closed unit ball of $C_b(X_1 \times X_2)$.*

(b) *For any $f \in C_b(X_1 \times X_2)$, $\|f\| \le 1$, there is a net $\{f_\alpha\}$ in the closed unit ball of $C_b(X_1) \otimes C_b(X_2)$, such that $f_\alpha \to f$, pontwise on $M_\infty(X_1 \times X_2)$.*

PROOF: (a) We assume $\mu(1) = 1$. Fix an f in the unit ball of $C_b(X_1 \times X_2)$ and an $\varepsilon > 0$. By Lemma 13, there is partition of unity $\{g_\alpha\}$ in X_1 and $\{h_\alpha\} \subset C(X_2)$ with $\|h_\alpha\| \le 1$ for all α such that $\|f - \sum_\alpha g_\alpha h_\alpha\| \le \varepsilon$. Since $\mu \in M_\infty(X_1 \times X_2)$, there is a finite subset $J \subset I$ such that $\mu(\sum_{\alpha \in I \setminus J}) < \varepsilon$. Let $h = \sum_{\alpha \in J} g_\alpha h_\alpha$. We have

$$\mu|f - h| \le \varepsilon + \mu\Big(\Big|\sum_{\alpha \in I\setminus J} g_\alpha h_\alpha\Big|\Big) \le \varepsilon + \mu\Big(\sum_{\alpha \in I\setminus J} g_\alpha\Big) \le 2\varepsilon.$$

This proves the result.

(b) The proof is very similar to Lemma 10(b). □

Now we prove the vector form of Theorem 16.

Theorem 18. *Suppose $\mu_1 \in M_\infty(X_1, E_1)$ and $\mu_2 \in M(X_2, E_2)$ (note that X_2 is compact). Then*

(a) *there exists a unique $\mu \in M_\infty(X_1 \times X_2, E)$ such that $\mu(f_1 f_2) = \mu_1(f_1) \otimes \mu_2(f_2)$ for any $f_1 \in C_b(X_1)$ and any $f_2 \in C_b(X_2)$; also for Baire sets $B_i \subset X_i$ $(i = (1,2))$, $\mu(B_1 \times B_2) = \mu_1(B_1) \otimes \mu_2(B_2)$. This measure μ is denoted by $\mu_1 \otimes \mu_2$.*

(b) *(Fubini-type result) Take an $f(x_1, x_2) \in L^1(\mu)$ and suppose, for $i = (1,2)$, there are $\phi_i(x_i) \in L^1(\mu_i)$ such that $|f(x_1, x_2)| \leq |\phi_1(x_1)||\phi_2(x_2)|$ on $X_1 \times X_2$. Then*

   (i) *for every $x_1 \in X_1$, $h_2(x_1) = \int f(x_1, \cdot)\, d\mu_2$ is in $L^1(\mu_1, E_2)$ and for every $x_2 \in X_2$, $h_1(x_2) = \int f(\cdot, x_2)\, d\mu_1$ is in $L^1(\mu_2, E_1)$;*

   (ii) *$\int h_2\, d\mu_1 = \int h_1\, d\mu_2 = \int f\, d\mu$.*

Proof: Using Theorem 16 and Lemma 17, the proof is similar to that of Theorem 11.

□

Acknowledgment. We are very thankful to the referee for pointing out typographical errors and making some other suggestions which has improved the paper.

References

[1] Babiker A.G., Knowles J.D., *Functions and measures on product spaces*, Mathematika **32** (1985), 60–67.

[2] Diestel J., Uhl J.J., *Vector Measures*, Mathematical Surveys, no. 15, American Mathematical Society, Providence, R.I., 1977.

[3] Duchoň M., Kluvánek I., *Inductive tensor product of vector-valued measures*, Mat. Časopis Sloven. Akad. Vied **17** (1967), 108–112; **20** (1972), 269–286.

[4] Fremlin D., Garling D., Haydon R., *Bounded measures on topological spaces*, Proc. Lon. Math. Soc. **25** (1972), 115–136.

[5] Freniche F.J., García-Vázquez J.C., *The Bartle bilinear integration and Carleman operators*, J. Math. Anal. Appl. **240** (1999), 324–339.

[6] Grothendieck A., *Sur les applicationes linéaires faiblement compactes d'espaces du type $C(K)$*, Canadian J. Math. **5** (1953), 129–173.

[7] Jarchow H., *Locally Convex Spaces*, B.G. Teubner, Stuttgart, 1981.

[8] Khurana S.S., *Topologies on spaces of continuous vector-valued functions*, Trans Amer. Math. Soc. **241** (1978), 195–211.

[9] Khurana S.S., *Topologies on spaces of continuous vector-valued functions II*, Math. Ann. **234** (1978), 159–166.

[10] Khurana S.S., *A topology associated with vector measures*, J. Indian Math. Soc. **45** (1981), 167–179.

[11] Khurana S.S., *Vector measures on topological spaces*, Georgian Math. J., to appear.

[12] Kluvánek I, Knowles G., *Vector Measures and Control Systems*, North-Holland Publishing Co., Amsterdam; American Elsevier Publishing Co., New York, 1976.

[13] Lewis D.R., *Integration with respect to vector measures*, Pacific J. Math. **33** (1970), 157–165.

[14] Phelps R.R., *Lectures on Choquet Theorem*, Van Nostrand, Princeton, 1966.
[15] Schaefer H.H., *Topological Vector Spaces*, Springer, New York-Berlin, 1971.
[16] Sentilles F.D., *Bounded continuous functions on completely regular spaces*, Trans. Amer. Math. Soc. **168** (1972), 311–336.
[17] Wheeler R.F., *Survey of Baire measures and strict topologies*, Exposition. Math. **1** (1983), no. 2, 97–190.
[18] Varadarajan V.S., *Measures on topological spaces*, Amer. Math. Soc. Transl. (2) **48** (1965), 161–220.

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242, U.S.A.
E-mail: khurana@math.uiowa.edu

(*Received* March 13, 2007, *revised* December 18, 2007)