Quenching for semidiscretizations of a semilinear heat equation with Dirichlet and Neumann boundary conditions

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Abstract. This paper concerns the study of the numerical approximation for the following boundary value problem:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = -u^{-p}(x,t), & 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, & u(1,t) = 1,\ t > 0, \\ u(x,0) = u_0(x) > 0, & 0 \le x \le 1, \end{cases}$$

where $p > 0$. We obtain some conditions under which the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time. Finally, we give some numerical experiments to illustrate our analysis.

Keywords: semidiscretizations, discretizations, heat equations, quenching, semidiscrete quenching time, convergence

Classification: 35K55, 35B40, 65M06

1. Introduction

Consider the following boundary value problem:

$$u_t(x,t) - u_{xx}(x,t) = -u^{-p}(x,t),\ 0 < x < 1,\ t > 0, \tag{1}$$

$$u_x(0,t) = 0, \quad u(1,t) = 1,\ t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0,\ 0 \le x \le 1, \tag{3}$$

where $p > 0$, $u_0'(0) = 0$, $u_0(1) = 1$, $u_0(x) < 1$ for $x \in [0,1)$.

Definition 1.1. We say that a solution u of (1)–(3) *quenches* in a finite time if there exists a finite time T_q such that $\|u(x,t)\|_{\inf} > 0$ for $t \in [0, T_q)$, but

$$\lim_{t \to T_q} \|u(x,t)\|_{\inf} = 0,$$

where $\|u(x,t)\|_{\inf} = \min_{0 \le x \le 1} u(x,t)$. The time T_q is called the *quenching time* of the solution u.

The theoretical study of solutions for semilinear heat equations which quench in a finite time has been the subject of investigations of many authors (see [2], [4]–[8] and the references cited therein). Under some conditions, the authors have proved that the solution u of (1)–(3) quenches in a finite time and have given some estimates of the quenching time.

In this paper, we are interested in the numerical study of the phenomenon of quenching using a semidiscrete form of (1)–(3). We give some conditions under which the solution of the semidiscrete form quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. A similar study has been undertaken by some authors concerning the phenomenon of blow-up (we say that a solution blows up in a finite time if it takes an infinite value in a finite time)(see [1]). In [3], some schemes have been used to study the phenomenon of extinction.

This paper is organised as follows. In the next section, we construct a semidiscrete scheme and give some lemmas which will be used later. In Section 3, under some conditions, we prove that the solution of a semidiscrete form of (1)–(3) quenches in a finite time and estimate its semidiscrete quenching time. In Section 4, we study the convergence of the semidiscrete quenching time. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. A semidiscrete problem

In this section, we give some lemmas which will be used later. We start by the construction of a semidiscrete scheme as follows. Let I be a positive integer, and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. Approximate the solution u of the problem (1)–(3) by the solution $U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - (U_i(t))^{-p},\ 0 \le i \le I-1,\ t \in (0, T_q^h), \tag{4}$$

$$U_I(t) = 1,\ t \in (0, T_q^h),\ U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{5}$$

where $\varphi_i < 1$ for $0 \le i \le I-1$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2},\ 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}.$$

Here $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, where $\|U_h(t)\|_{\inf} = \min_{0 \le i \le I} U_i(t)$. When T_q^h is finite, then we say that the solution $U_h(t)$ of (4)–(5) quenches in a finite time, and the time T_q^h is called the semidiscrete quenching time of the solution $U_h(t)$.

The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let $\alpha_h \in C^0([0,T),\mathbb{R}^{I+1})$ and let $V_h \in C^1([0,T),\mathbb{R}^{I+1})$ be such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + \alpha_i(t)V_i(t) \geq 0,\ 0 \leq i \leq I-1,\ t \in (0,T), \tag{6}$$

$$V_I(t) \geq 0,\ t \in (0,T), \tag{7}$$

$$V_i(0) \geq 0,\ 0 \leq i \leq I. \tag{8}$$

Then $V_i(t) \geq 0$ for $0 \leq i \leq I$, $t \in (0,T)$.

Proof: Let $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t}V_h(t)$, where λ is such that $\alpha_i(t) - \lambda > 0$, $0 \leq i \leq I$, $t \in [0,T_0]$. Let

$$m = \min_{0\leq i\leq I, 0\leq t\leq T_0} Z_i(t).$$

For $i = 0,\ldots,I$, $Z_i(t)$ is a continuous function on the compact $[0,T_0]$. Then, there exist $i_0 \in \{0,1,\ldots,I\}$ and $t_0 \in [0,T_0]$ such that $m = Z_{i_0}(t_0)$. If $i_0 \in \{0,1,\ldots,I-1\}$, then we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \leq 0, \tag{9}$$

$$\delta^2 Z_{i_0}(t_0) = \delta^2 Z_0(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0, \tag{10}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1. \tag{11}$$

Using (6), a straightforward computation yields

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0 \quad \text{if} \quad 0 \leq i_0 \leq I-1. \tag{12}$$

From the inequalities (9)–(12), it is not hard to see that $(\alpha_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$, $0 \leq i_0 \leq I-1$. Due to (7) and the fact that $\alpha_{i_0}(t_0) - \lambda > 0$, we see that $Z_h(t_0) \geq 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0,T_0]$ which leads us to the desired result. □

The lemma below shows a property of the semidiscrete solution.

Lemma 2.2. *Let U_h be the solution of (4)–(5). Then*

$$U_i(t) < 1,\ 0 \leq i \leq I-1,\ t \in (0,T_q^h). \tag{13}$$

Proof: Let t_0 be the first $t \in (0, T_q^h)$ such that $U_i(t) < 1$ for $t \in [0, t_0)$, $0 \leq i \leq I-1$, but $U_{i_0}(t_0) = 1$ for a certain $i_0 \in \{0, \ldots, I-1\}$. We observe that

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \geq 0, \tag{14}$$

$$\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} \leq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1, \tag{15}$$

$$\delta^2 U_{i_0}(t_0) = \delta^2 U_0(t_0) = \frac{2U_1(t_0) - 2U_0(t_0)}{h^2} \leq 0 \quad \text{if} \quad i_0 = 0, \tag{16}$$

which implies that

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) + (U_{i_0}(t_0))^{-p} > 0.$$

But, this contradicts (4) and the proof is complete. □

Another version of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.3. *Let $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. If V_h, $W_h \in C^1([0,T), \mathbb{R}^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t), t) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + f(W_i(t), t), \tag{17}$$
$$0 \leq i \leq I-1, \; t \in (0,T),$$

$$V_I(t) < W_I(t), \; t \in (0,T), \tag{18}$$

$$V_i(0) < W_i(0), \; 0 \leq i \leq I, \tag{19}$$

then $V_i(t) < W_i(t)$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof: Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \leq i \leq I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \ldots, I\}$.
We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0.$$

Therefore if $i_0 \in \{0, \ldots, I-1\}$, then we have

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) < 0,$$

which contradicts (17). If $i_0 = I$, then we have a contradiction because of (18). This ends the proof. □

The lemma below reveals a property of the operator δ^2.

Lemma 2.4. *Let V_h and $U_h \in \mathbb{R}^{I+1}$. If $\delta^+(U_0)\delta^+(V_0) \geq 0$ and*

$$\delta^+(U_i)\delta^+(V_i) \geq 0, \quad \delta^-(U_i)\delta^-(V_i) \geq 0, \quad 1 \leq i \leq I-1,$$

then

$$\delta^2(U_iV_i) \geq U_i\delta^2 V_i + V_i\delta^2 U_i, \quad 0 \leq i \leq I-1,$$

where $\delta^+(U_i) = \frac{U_{i+1}-U_i}{h}$, $\delta^-(U_i) = \frac{U_{i-1}-U_i}{h}$.

Proof: A straightforward computation yields

$$\delta^2(U_0V_0) = 2\delta^+(U_0)\delta^+(V_0) + U_0\delta^2 V_0 + V_0\delta^2 U_0,$$

$$\delta^2(U_iV_i) = \delta^+(U_i)\delta^+(V_i) + \delta^-(U_i)\delta^-(V_i) + U_i\delta^2 V_i + V_i\delta^2 U_i, \quad 1 \leq i \leq I-1.$$

Using the assumptions of the lemma, we obtain the desired result. □

The following result shows another property of the semidiscrete solution.

Lemma 2.5. *Let U_h be the solution of (4)–(5) such that the initial data at (5) satisfy*

$$\varphi_{i+1} > \varphi_i, \; 0 \leq i \leq I-1. \tag{20}$$

Then, we have

$$U_{i+1}(t) > U_i(t), \; 0 \leq i \leq I-1, \quad t \in (0, T_q^h). \tag{21}$$

Proof: Let $t_0 \in (0, T_q^h)$ be the first $t > 0$ such that $U_{i+1}(t) > U_i(t)$ for $t \in (0, t_0)$, $0 \leq i \leq I-1$, but

$$U_{k+1}(t_0) = U_k(t_0) \quad \text{for a certain} \quad k \in \{0, \ldots, I-1\}.$$

Without loss of generality, we may suppose that k is the smallest integer which satisfies the above equality.
If $k = I-1$ then $U_I(t_0) = U_{I-1}(t_0) = 1$. But, this contradicts Lemma 2.2. If $k \in \{0, \ldots, I-2\}$, then letting $Z_k(t) = U_{k+1}(t) - U_k(t)$, we observe that

$$\frac{dZ_k(t_0)}{dt} = \lim_{k \to 0} \frac{Z_k(t_0) - Z_k(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_k(t_0) = \delta^2 Z_0(t_0) = \frac{Z_1(t_0) - 3Z_0(t_0)}{h^2} > 0 \quad \text{if} \quad k = 0,$$

$$\delta^2 Z_k(t_0) = \frac{Z_{k+1}(t_0) - 2Z_k(t_0) + Z_{k-1}(t_0)}{h^2} > 0 \quad \text{if} \quad 1 \leq k \leq I-2.$$

Therefore, if $0 \leq k \leq I-2$, we get

$$\frac{dZ_k(t_0)}{dt} - \delta^2 Z_k(t_0) + (U_{k+1}(t_0))^{-p} - (U_k(t_0))^{-p} < 0,$$

which contradicts (4). This ends the proof. □

Remark 2.1. The above result reveals that if the initial data of the semidiscrete solution are increasing in space, then the semidiscrete solution is also increasing in space. This property will be used later to show that the semidiscrete solution attains its minimum at the first node.

3. Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (4)–(5) quenches in a finite time and estimate its semidiscrete quenching time.

Let us give another property of the operator δ^2 useful in this section.

Lemma 3.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $U_h > 0$. Then, we have*

$$\delta^2 U_i^{-p} \geq -pU_i^{-p-1}\delta^2 U_i \quad \textit{for} \quad 0 \leq i \leq I-1.$$

Proof: Apply Taylor's expansion to obtain

$$\begin{aligned}
\delta^2 U_0^{-p} &= -pU_0^{-p-1}\delta^2 U_0 + (U_1 - U_0)^2 \frac{p(p+1)}{h^2}\theta_0^{-p-2},\\
\delta^2 U_i^{-p} &= -pU_i^{-p-1}\delta^2 U_i + (U_{i+1} - U_i)^2 \frac{p(p+1)}{2h^2}\theta_i^{-p-2}\\
&\quad + (U_{i-1} - U_i)^2 \frac{p(p+1)}{2h^2}\eta_i^{-p-2} \quad \text{if} \quad 1 \leq i \leq I-1,
\end{aligned}$$

where θ_i is an intermediate value between U_{i+1} and U_i and η_i the one between U_{i-1} and U_i. Use the fact that $U_h > 0$ to complete the rest of the proof. □

Our result about the quenching time is the following.

Theorem 3.1. *Let U_h be the solution of* (4)–(5). *Assume that there exists a constant $A > 0$ such that the initial data at* (5) *satisfy*

$$\delta^2 \varphi_i - \varphi_i^{-p} \leq -A\cos(ih\frac{\pi}{2})\varphi_i^{-p}, \quad 0 \leq i \leq I-1, \tag{22}$$

$$1 - \frac{\pi^2}{2A(p+1)}\|\varphi_h\|_{\inf}^{p+1} > 0. \tag{23}$$

If (20) *holds, then U_h quenches in a finite time T_q^h which satisfies the following estimate*

$$T_q^h < -\frac{8}{\pi^2}\ln\left(1 - \frac{\pi^2}{2A(p+1)}\|\varphi_h\|_{\inf}^{p+1}\right).$$

Proof: Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ such that

$$J_i(t) = \frac{dU_i(t)}{dt} + C_i(t)U_i^{-p}(t), \quad 0 \leq i \leq I, \quad t \in [0, T_q^h),$$

where $C_i(t) = Ae^{-\lambda_h t}\cos(ih\frac{\pi}{2})$ with $\lambda_h = \frac{2-2\cos(h\frac{\pi}{2})}{h^2}$. It is not hard to see that

$$\frac{dC_i(t)}{dt} - \delta^2 C_i(t) = 0, \quad C_{i+1}(t) < C_i(t),\ 0 \le i \le I-1. \tag{24}$$

Using Lemma 2.5, we observe that

$$\delta^+(U_0^{-p})\delta^+(C_0) \ge 0 \quad \text{and} \quad \delta^+(U_i^{-p})\delta^+(C_i) \ge 0, \quad \delta^-(U_i^{-p})\delta^-(C_i) \ge 0 \tag{25}$$

for $1 \le i \le I-1$. A straightforward computation gives

$$\begin{aligned}\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) &= \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) + U_i^{-p}\frac{dC_i(t)}{dt} - pC_i(t)U_i^{-p-1}\frac{dU_i(t)}{dt}\\ &\quad - \delta^2(C_i(t)U_i^{-p}(t)), \quad 0 \le i \le I-1.\end{aligned}$$

It follows from (25), Lemmas 2.4 and 3.1 that

$$\delta^2(C_i(t)U_i^{-p}(t)) \ge U_i^{-p}(t)\delta^2 C_i(t) - pC_i(t)U_i^{-p-1}(t)\delta^2 U_i(t), \quad 0 \le i \le I-1.$$

We deduce that

$$\begin{aligned}\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) &\le \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) - pC_i(t)U_i^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t))\\ &\quad + U_i^{-p}(t)(\frac{dC_i(t)}{dt} - \delta^2 C_i(t)), \quad 0 \le i \le I-1.\end{aligned}$$

In virtue of (4) and (24), we arrive at

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \le pU_i^{-p-1}(t)J_i(t),\ 0 \le i \le I-1,\ t \in (0, T_q^h).$$

Obviously, $J_I(t) = 0$. From the assumption (22), we get $J_h(0) \le 0$. It follows from Lemma 2.1 that $J_h(t) \le 0$ for $t \in (0, T_q^h)$. This estimate may be rewritten as follows

$$\frac{dU_i(t)}{dt} \le -Ae^{-\lambda_h t}\cos(ih\frac{\pi}{2})U_i^{-p}(t),\ 0 \le i \le I,\ t \in (0, T_q^h).$$

We observe that $\lambda_h \le \frac{\pi^2}{2}$ for h small enough. Hence, we get

$$U_0^p(t)dU_0(t) \le -Ae^{-\frac{\pi^2}{2}t}dt \quad \text{for} \quad t \in (0, T_q^h). \tag{26}$$

From Lemma 2.5, $U_0(t) = \|U_h(t)\|_{\inf}$. Therefore, integrating (26) over $(0, T_q^h)$, we obtain

$$T_q^h \le -\frac{8}{\pi^2}\ln(1 - \frac{\pi^2}{2A(p+1)}\|U_h(0)\|_{\inf}^{p+1}).$$

Use the fact that $U_h(0) = \varphi_h$ and (23) to complete the rest of the proof. $\square$

Remark 3.1. Assume that there exists a time $t_0 \in (0, T_q^h)$ such that

$$1 - \frac{\pi^2}{2A(p+1)} e^{-\frac{\pi^2}{2} t_0} \|U_h(t_0)\|_{\inf}^{p+1} > 0.$$

Integrating the inequality (26) over (t_0, T_q^h), and using the fact that $U_0(t_0) = \|U_h(t_0)\|_{\inf}$, we arrive at

$$T_q^h - t_0 \leq -\frac{8}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A(p+1)} e^{-\frac{\pi^2}{2} t_0} \|U_h(t_0)\|_{\inf}^{p+1}\right).$$

Remark 3.2. It is easy to find a vector φ_h and a positive constant A such that (22), (23) hold. In fact, one may find a vector ψ_h and a constant $A \in (0,1)$ such that

$$\delta^2 \psi_i - \psi_i^{-p} \leq -A \psi_i^{-p}, \quad 0 \leq i \leq I-1,$$

which implies that

$$\delta^2 \psi_i - \psi_i^{-p} \leq -A \cos(ih\frac{\pi}{2}) \psi_i^{-p}, \quad 0 \leq i \leq I-1.$$

Let $\varphi_h = \varepsilon \psi_h$ where $0 < \varepsilon < 1$. It is not hard to see that

$$\delta^2 \varphi_i - \varphi_i^{-p} \leq -A \cos(ih\frac{\pi}{2}) \varphi_i^{-p}, \quad 0 \leq i \leq I-1,$$

and the inequality (22) follows. To obtain (23), it suffices to take ε small enough.

4. Convergence of the semidiscrete quenching time

In this section, under some assumptions, we prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero.

We denote

$$u_h(t) = (u(x_0, t), \ldots, u(x_I, t))^T.$$

In order to obtain the convergence of the semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

Theorem 4.1. *Assume that the problem* (1)–(3) *has a solution* $u \in C^{4,1}([0,1] \times [0,T])$ *such that* $\min_{0 \leq t \leq T} \|u(x,t)\|_{\inf} = \rho > 0$ *and the initial data at (5) satisfy*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as} \quad h \to 0. \tag{27}$$

Then, for h sufficiently small, the problem (4)–(5) *has a unique solution* $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ *such that*

$$\max_{0 \leq t \leq T} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad \text{as} \quad h \to 0. \tag{28}$$

PROOF: Problem (4)–(5) has for each h a unique solution $U_h \in C^1([0,T_q^h),\mathbb{R}^{I+1})$. Let $t(h)$ be the greatest value of $t>0$ such that

$$\|U_h(t)-u_h(t)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad t\in(0,t(h)). \tag{29}$$

Relation (27) implies that $t(h)>0$ for h sufficiently small. Let $t^*(h) = \min\{t(h),T\}$. From the triangle inequality, we get

$$\|U_h(t)\|_{\inf} \geq \|u_h(t)\|_{\inf} - \|U_h(t)-u_h(t)\|_\infty \quad \text{for} \quad t\in(0,t^*(h)),$$

which implies that

$$\|U_h(t)\|_{\inf} \geq \rho - \frac{\rho}{2} = \frac{\rho}{2} \quad \text{for} \quad t\in(0,t^*(h)). \tag{30}$$

Consider the error

$$e_h(t) = U_h(t) - u_h(t).$$

By a direct calculation, we find that for $t\in(0,t^*(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = p(\Theta_i(t))^{-p-1}e_i(t) + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t), \quad 0\leq i\leq I-1, \tag{31}$$

where Θ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Let $M>0$ be such that

$$\frac{\|u_{xxxx}(x,t)\|_\infty}{12} \leq M \quad \text{for} \quad t\in[0,T], \quad p(\frac{\rho}{2})^{-p-1} \leq M. \tag{32}$$

Using (30)–(31), it is not hard to see that

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \leq M|e_i(t)| + Mh^2,\ 0\leq i\leq I-1,\ t\in(0,t^*(h)).$$

Introduce the vector $z_h(t)$ such that

$$z_i(t) = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Mh^2),\ 0\leq i\leq I,\ t\in[0,T].$$

A straightforward computation yields

$$\frac{dz_i(t)}{dt} - \delta^2 z_i(t) > M|z_i(t)| + Mh^2,\ 0\leq i\leq I-1,\ t\in(0,t^*(h)), \tag{33}$$

$$z_I(t) > e_I(t),\ t\in(0,t^*(h)), \tag{34}$$

$$z_i(0) > e_i(0),\ 0\leq i\leq I. \tag{35}$$

It follows from Comparison Lemma 2.3 that

$$z_i(t) > e_i(t) \quad \text{for} \quad t \in (0, t^*(h)),\ 0 \le i \le I.$$

In the same way, we also show that

$$z_i(t) > -e_i(t) \quad \text{for} \quad t \in (0, t^*(h)),\ 0 \le i \le I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Mh^2),\ t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (29), we obtain

$$\frac{\rho}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Mh^2).$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $\frac{\rho}{2} \le 0$, which is impossible. Consequently $t^*(h) = T$ and the proof is complete. □

Now, we are in a position to prove the main result of this section.

Theorem 4.2. *Suppose that the solution u of* (1)–(3) *quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$ and the initial data at* (5) *satisfy condition* (27). *Under the assumptions of Theorem* 3.1, *problem* (4)–(5) *admits a unique solution $U_h(t)$ which quenches in a finite time T_q^h with* $\lim_{h\to 0} T_q^h = T_q$.

Proof: Let $0 < \varepsilon < T_q/2$. There exists a constant $R > 0$ such that

$$-\frac{8}{\pi^2}\ln\left(1 - \frac{\pi^2}{2A(p+1)}x^{p+1}\right) < \frac{\varepsilon}{2} \quad \text{for} \quad x \in [0, R]. \tag{36}$$

Since u quenches in a finite time T_q, then there exists $T_1 \in (T_q - \frac{\varepsilon}{2}, T_q)$ such that

$$0 < \|u(x,t)\|_{\inf} < \frac{R}{2} \quad \text{for} \quad t \in (T_1, T_q).$$

Let $T_2 = \frac{T_1+T_q}{2}$. Obviously, we have $0 < \|u(x,t)\|_{\inf} < \frac{R}{2}$ for $t \in [0, T_2]$. It follows from Theorem 4.1 that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{R}{2} \quad \text{for} \quad t \in [0, T_2],$$

which implies that

$$\|U_h(T_2) - u_h(T_2)\|_\infty < \frac{R}{2}\,.$$

Applying the triangle inequality, we obtain

$$\|U_h(T_2)\|_{\inf} \leq \|U_h(T_2) - u_h(T_2)\|_\infty + \|u_h(T_2)\|_{\inf} \leq \frac{R}{2} + \frac{R}{2} = R.$$

We deduce from Remark 3.1 and (36) that

$$|T_q^h - T_2| \leq -\frac{8}{\pi^2} \ln\left(1 - \frac{\pi^2}{2A(p+1)} e^{-\frac{\pi^2}{2}T_2} \|U_h(T_2)\|_{\inf}^{p+1}\right) < \frac{\varepsilon}{2}.$$

Consequently, we find that

$$|T_q^h - T_q| \leq |T_q^h - T_2| + |T_2 - T_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the proof is complete. □

5. Numerical experiments

In this section, we consider the problem (1)–(3) in the case where $p = 1$, $u_0(x) = 0.05 + 0.95\sin(\frac{\pi}{2}x)$. We give some computational results concerning some approximations of the real quenching time. We start by proposing some schemes which will be used later for our numerical experiments.

At first, we approximate the solution $u(x,t)$ of the problem (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} - (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \ 0 \leq i \leq I-1,$$

$$U_I^{(n)} = 1, \quad U_i^{(0)} = \varphi_i, \ 0 \leq i \leq I,$$

where $n \geq 0$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T_q, we need to adapt the size of the time step so that we take $\Delta t_n = \min\{\frac{h^2}{2}, \tau\|U_h^{(n)}\|_{\inf}^{p+1}\}$ with $\tau = \text{const} \in (0,1)$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution.

At second, we approximate the solution $u(x,t)$ of the problem (1)–(3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} - (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \ 0 \leq i \leq I-1,$$

$$U_I^{(n)} = 1, \ U_i^{(0)} = \varphi_i, \ 0 \leq i \leq I,$$

where $n \geq 0$. As in the case of the explicit scheme, here, we choose $\Delta t_n = \tau\|U_h^{(n)}\|_{\inf}^{p+1}$ with $\tau = \text{const} \in (0,1)$. For the implicit scheme, the existence and positivity of the discrete solution is also guaranteed using standard methods (see, for instance, [3]).

In both schemes, we take $\varphi_i = 0.05 + 0.95\sin(\frac{\pi}{2}ih)$, $\tau = h^2$.

We need the following definition.

Definition 5.1. We say that the solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to+\infty} \|U_h^{(n)}\|_{\inf} = 0$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In Tables 1 and 2, in rows, we present the numerical quenching times, the number of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T^{n+1} - T^n| \leq 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Table 1:
Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method:

I	T^n	n	CPU_t	s
16	0.5619	4632	1	-
32	0.5661	18026	4	-
64	0.5671	69898	27	2.07
128	0.5672	270200	687	3.03

Table 2:
Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method:

I	T^n	n	CPU_t	s
16	0.5634	4633	1	-
32	0.5664	18030	10	-
64	0.5672	69899	430	1.91
128	0.5674	270249	7200	2.00

References

[1] Abia L.M., López-Marcos J.C., Martinez J., *On the blow-up time convergence of semidiscretizations of reaction-diffusion equations*, Appl. Numer. Math. **26** (1998), 399–414.

[2] Acker A., Kawohl B., *Remarks on quenching*, Nonlinear Anal. **13** (1989), 53–61.

[3] Boni T.K., *Extinction for discretizations of some semilinear parabolic equations*, C.R. Acad. Sci. Paris Sér. I Math. **333** (2001), 795–800.

[4] Boni T.K., *On quenching of solutions for some semilinear parabolic equations of second order*, Bull. Belg. Math. Soc. Simon Stevin **7** (2000), 73–95.

[5] Fila M., Kawohl B., Levine H.A., *Quenching for quasilinear equations*, Comm. Partial Differential Equations **17** (1992), 593–614.

[6] Guo J.S., Hu B., *The profile near quenching time for the solution of a singular semilinear heat equation*, Proc. Edinburgh Math. Soc. **40** (1997), 437–456.

[7] Guo J., *On a quenching problem with Robin boundary condition*, Nonlinear Anal. **17** (1991), 803–809.

[8] Levine H.A., *Quenching, nonquenching and beyond quenching for solutions of some parabolic equations*, Annali Mat. Pura Appl. **155** (1990), 243–260.

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