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How sensitive is $C_p(X,Y)$ to changes in X and/or Y?

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Abstract. We investigate how the Lindelöf property of the function space $C_p(X,Y)$ is influenced by slight changes in X and/or Y.

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1. Introduction

Consider the statement: *"If X has property $\mathcal{P}$, then $C_p(X)$ has property $\mathcal{Q}$."* This statement always suggests the question: *"Does $C_p(X)^\omega$ have $\mathcal{Q}$?"* It is a classical fact [ARH] that $C_p(X)^\omega$ is homeomorphic to $C_p(\oplus_n X)$. If $\mathcal{P}$ is preserved by countable free sums, then the answer is obviously affirmative. However many nice properties are not so. For example, compactness is not preserved by countable free sums. To explain motivation of the proposed study, assume this statement: *"If X is compact, then $C_p(X)$ has property $\mathcal{Q}$."* Let $Y = (\oplus_n X) \cup \{\infty\}$ be a one-point compactification of $\oplus_n X$. Since Y is compact, $C_p(Y) \in \mathcal{Q}$. Thus the question whether $C_p(X)^\omega \in \mathcal{Q}$ leads to the question whether $C_p(Y \setminus \{\infty\}) \in \mathcal{Q}$. This example shows a natural relation between countable productivity of a property in function spaces and a single point alteration of the domain space. Motivated by this example we suggest to study how adding or/and removing points to/from X affect the function space. In this paper we will direct our efforts at the following problem.

Problem. *Suppose $C_p(X,Z)$ is Lindelöf and Y is a subspace or a superspace of X. Under what conditions is $C_p(Y,Z)$ Lindelöf?*

In notation and terminology we will follow [ENG] and [ARH]. All spaces are assumed to be Tychonoff. The study of the suggested topic will begin in Section 3. In Section 2, we will prove modifications of classical facts to be used in our study.

2. $C_p(X)$ if $\mathrm{Ind}(X) = 0$

In this section we will repeat (with some changes) the argument in [ARH] to show that $C_p(X,\omega^\omega)$ admits a continuous map onto $C_p(X)$ if $\mathrm{Ind}(X) = 0$.

Lemma 2.1 (A version of Mardešić Factorization Theorem [MAR])**.** *Let X be locally compact and Lindelöf and* $\operatorname{ind}(X) = 0$. *Let $f : X \to R$ be continuous. Then there exist continuous $g : X \to \omega^\omega$ and continuous $h : \omega^\omega \to R$ such that $f = h \circ g$.*

Proof: Let $\mathcal{B}$ be a countable base for the topology of R. For each $B \in \mathcal{B}$, fix a countable family $\mathcal{O}_B$ of clopen sets in X such that $f^{-1}(B) = \bigcup \mathcal{O}_B$. Fix a countable family $\mathcal{O}_\infty$ of clopen compact sets whose union is X. Let $\mathcal{O} = \{O : O \in \mathcal{O}_\infty \text{ or } O \in \mathcal{O}_B, B \in \mathcal{B}\}$.

For each $O \in \mathcal{O}$, define $g_O : X \to \omega$ by letting $g_O(O) = \{0\}$ and $g_O(X \setminus O) = \{1\}$. The function $G_1 = \Delta\{g_O : O \in \mathcal{O}\}$ is a continuous function from X to ω^ω. Due to the presence of elements of $\mathcal{O}_\infty$ in $\mathcal{O}$, G_1 is a perfect map. Hence, $G_1(X)$ is locally compact. Therefore, there exists a homeomorphism G_2 of $G_1(X)$ onto a closed subset of ω^ω. Put $g = G_2 \circ G_1$.

Define $H : g(X) \to R$ as follows. For each $y \in g(X)$, put $H(y) = f(x)$, where $g(x) = y$. Let us show that g is well defined. Let $x_1 \neq x_2$ and $y = g(x_1) = g(x_2)$. We need to show that $f(x_1) = f(x_2)$. Since $g(x_1) = g(x_2)$, the family $\mathcal{O}$ does not separate x_1 from x_2. Therefore, f does not separate them either. To show continuity of g, fix $B \in \mathcal{B}$. By the definition of H, $H^{-1}(B) = g(f^{-1}(B))$. By the definition of g, $g(O)$ is open for all $O \in \mathcal{O}$. Since $f^{-1}(B) = \bigcup \mathcal{O}_B$, where $\mathcal{O}_B \subset \mathcal{O}$, the set $g(f^{-1}(B))$ is open. Since $g(X)$ is closed in ω^ω, there exists continuous $h : \omega^\omega \to R$ that coincides with H on $g(X)$. Clearly, $f = h \circ g$. □

Next two lemmas are analogous to Lemma IV.3.6 and Lemma IV.3.7 in [ARH]. We will follow both the proofs and notations.

Lemma 2.2. *Let X be a Tychonoff space and* $\operatorname{Ind}(X) = 0$. *Then there exists a continuous map $\phi : \omega^\omega \to R$ such that for every continuous $f \in C_p(X)$ there exists continuous $g_f \in C_p(X, \omega^\omega)$ such that $f = \phi \circ g_f$.*

Proof: For every $f \in C_p(X)$ let $X_f = X \times \{f\}$ and let e_f be the homeomorphism of X with X_f defined by $e_f(x) = (x, f)$. Let $Z = \oplus\{X_f : f \in C_p(X)\}$. It is clear that $\operatorname{Ind}(Z) = 0$ and $\operatorname{Ind}(\beta Z) = 0$.

Consider $h : Z \to R$, where $h(x, f) = f(x)$ for all $x \in X$ and $f \in C_p(X)$. The map h is continuous. Let $\tilde{h} : \beta Z \to \beta R$ be the continuous extension of h to the Čech-Stone compactifications. Let $Z' = \tilde{h}^{-1}(R)$ and let h' be the restriction of $\tilde{h}$ to Z'. Clearly, Z' is locally compact, Lindelöf, and $\operatorname{ind}(Z') = 0$.

By Lemma 2.1, there exist continuous $s : Z' \to \omega^\omega$ and continuous $\phi : \omega^\omega \to R$ such that $h' = \phi \circ s$.

Let us show that ϕ is as desired. Take any $f \in C_p(X)$. Put $g_f = s \circ e_f : X \to \omega^\omega$. Then $\phi \circ g_f = f$. □

Lemma 2.3. *Let X be a Tychonoff space and* $\operatorname{Ind}(X) = 0$. *Then $C_p(X)$ is a continuous image of $C_p(X, \omega^\omega)$.*

Proof: Fix ϕ that satisfies the conclusion of Lemma 2.2. Define $p : C_p(X, \omega^\omega) \to C_p(X)$ by letting $p(g) = \phi \circ g$. By Lemma 2.2, p is "onto". Continuity of p is clear. □

Theorem 2.4. *Let X be a Tychonoff space and* $\mathrm{Ind}(X) = 0$. *If $C_p(X, \omega^\omega)$ is Lindelöf then so is $C_p(X)^\omega$.*

Proof: The space $C_p(X, \omega^\omega)$ is homeomorphic to $C_p(X, (\omega^\omega)^\omega)$. The latter is homeomorphic to $C_p(X, \omega^\omega)^\omega$, which, by Lemma 2.3, admits a continuous map onto $C_p(X)^\omega$. □

According to Theorem 2.4, if the answer to the following question is "Yes" then the Lindelöf property is preserved by countable powers in the class of function spaces over spaces X with $\mathrm{Ind}(X) = 0$.

Question 2.5. *Let $C_p(X)$ be Lindelöf and* $\mathrm{Ind}(X) = 0$. *Is $C_p(X, \omega^\omega)$ Lindelöf?*

3. Study

In [D&S], A. Dow and P. Simon constructed a consistent example of a countably tight compactum $Y = X \cup \{p\}$ such that X is first-countable, p has countable tightness in Y, $C_p(X)$ is Lindelöf, and $C_p(Y)$ is not Lindelöf. That is, adding a countably tight point to the domain space destroys Lindelöf property in the resulting function space. This example prompts the following question.

Question 3.1. *Let $C_p(X)$ be Lindelöf. Suppose p has countable character in $Y = X \cup \{p\}$. Is $C_p(Y)$ Lindelöf?*

We do not know an answer to this question. However, if we replace $C_p(X)$ with $C_p(X, Z)$, where Z is a discrete countable space, then the answer is "Yes".

Theorem 3.2. *Let $C_p(X, Z)$ be Lindelöf, where Z is a discrete countable space. If p has countable character in $Y = X \cup \{p\}$, then $C_p(Y, Z)$ is Lindelöf.*

Proof: Enumerate elements of Z as $\{z_n : n \in \omega\}$. Fix a countable base $\{B_n\}_n$ at p in Y. Define $S_{n,m} \subset C_p(X, Z)$ as follows: $f \in S_{n,m}$ iff $f(B_n \setminus \{p\}) = \{z_m\}$. It is clear that $S_{n,m}$ is closed in $C_p(X, Z)$ for any $n, m \in \omega$, and therefore, is Lindelöf.

Define $\phi_{n,m} : S_{n,m} \to C_p(Y, Z)$ as follows: $\phi(f) = f_{n,m}$, where $f_{n,m}$ coincides with f on X and $f_{n,m}(p) = z_m$. Clearly, $f_{n,m}$ is a continuous function of Y to Z. Let us show that $\phi_{n,m}$ is continuous. Fix $f \in S_{n,m}$ and open $V \subset C_p(Y, Z)$ that contains $f_{n,m}$. We may assume that $V = V_1 \cap V_2$, where $V_1 = \{g \in C_p(Y, Z) : g(p) = f_{n,m}(p)\}$ and $V_2 = \{g \in C_p(Y, Z) : g(a) = f_{n,m}(a)\}$, where a is a fixed element of X. Put $U = \{h \in S_{n,m} : h(a) = f(a)\}$. Since $a \in X$, $h_{n,m}(a) = h(a) = f_{n,m}(a)$. Thus, $h_{n,m} \in V_2$. Since $h \in S_{n,m}$, $h_{n,m}(p) = z_m = f_{n,m}(p)$. Thus, $h_{n,m} \in V_1$.

Observe that $C_p(Y,Z)$ is equal to $\bigcup\{\phi_{n,m}(S_{n,m}) : n,m \in \omega\}$. Since each term in the union is a continuous image of a Lindelöf space, the union is Lindelöf as well. $\square$

Remark. Observe that the argument of the proof of Theorem 3.2 proves more than promised in the statement. Namely, the Lindelöf property in the statement of the theorem can be replaced by any property that is preserved by countable unions and continuous maps.

In connection with this result, it is interesting to know what happens to $C_p(X)$ if we remove one point from X. It is known that removing a point of countable tightness from a zero-dimensional compactum may destroy the Lindelöf property of a function space. For example, let $X = D \cup \{\infty\}$ be the one-point compactification of an uncountable discrete space D. It is known that X is an Eberlein compactum. Therefore, $C_p(X)$ is Lindelöf [TAL]. The space $C_p(D)$ is not Lindelöf because it is homeomorphic to R^D. It turns out, however, that removing a point of countable character from a zero-dimensional compactum does not affect the Lindelöf property of the function space. To prove this result we will make a repeated use of the fact that $C_p(\oplus_n X, Y)$, $C_p(X, Y^\omega)$, and $C_p(X,Y)^\omega$ are homeomorphic (see Propositions 0.3.3 and 0.3.4 in [ARH]).

Theorem 3.3. *Let X be a zero-dimensional compactum and let $p \in X$ have countable character in X. If $C_p(X)$ is Lindelöf, then so is $C_p(X \setminus \{p\})$.*

PROOF: If p is isolated, then $C_p(X)$ is homeomorphic to $C_p(X\setminus\{p\})\times R$. Assume p is a limit point for X. Since X is zero-dimensional and $\chi(p,X) = \omega$ we can write $X\setminus\{p\}$ as $\oplus_{i\in\omega}X_i$, where X_i is compact for all i. Clearly, $\mathrm{Ind}(X\setminus\{p\}) = 0$.

Since $C_p(X)$ is Lindelöf and X is a zero-dimensional compactum we have $C_p(X)^\omega$ is Lindelöf (see [POL1] or [ARH]). Since $C_p(X)^\omega$ is homeomorphic to $C_p(X, R^\omega)$ the latter is Lindelöf. Since ω embeds in R as a closed subspace, we have $C_p(X, \omega^\omega)$ is Lindelöf. Therefore, $C_p(X, (\omega^\omega)^\omega)$ is Lindelöf. Therefore, $C_p(\oplus_{i\in\omega}X, \omega^\omega)$ is Lindelöf.

Since $X \setminus \{p\} = \oplus_{i\in\omega}X_i$ is a clopen subspace of $\oplus_{i\in\omega}X$, any continuous function from $\oplus_{i\in\omega}X_i$ to ω^ω can be continuously extended to a continuous function of $\oplus_{i\in\omega}X$ to ω^ω. Therefore, $C_p(X \setminus \{p\}, \omega^\omega)$ is a continuous image of $C_p(\oplus_{i\in\omega}X, \omega^\omega)$. Therefore, the former is Lindelöf as well. By Theorem 2.4, $C_p(X \setminus \{p\})^\omega$ is Lindelöf. $\square$

Question 3.4. *Let X be a compactum and $\chi(p,X) = \omega$. Suppose $C_p(X)$ is Lindelöf. Is $C_p(X \setminus \{p\})$ Lindelöf? What if the requirement on compactness of X is replaced by some other compact-type property or simply dropped?*

It is interesting that Theorem 3.3 does not hold if $C_p(X)$ is replaced with $C_p(X,2)$. To describe an example, we need one technical statement about the following classical structure.

Definition of Structure $F(\oplus_i X_i)$**:** Let X_i be a topological space for all $i \in \omega$ and $\infty \notin \bigcup_i X_i$. The space $F(\oplus_i X_i)$ consists of points of $\{\infty\} \cup (\oplus_i X_i)$. Each X_i is clopen in $F(\oplus_i X_i)$ and keeps its original topology. The base neighborhoods at ∞ are in form $\{\infty\} \cup (\oplus_{i>n} X_i)$.

The proof of the next technical statement is similar to that of Theorem 3.2.

Proposition 3.5. *Let Z be a countable discrete space. Suppose $C_p(\oplus_{i<n} X_i, Z)$ is Lindelöf for all n. Then $C_p(F(\oplus_i X_i), Z)$ is Lindelöf.*

Proof: Enumerate elements of Z as $\{z_m : m \in \omega\}$. Define a function $\phi_{n,m}$ from $C_p(\oplus_{i<n} X_i, Z)$ to $C_p(F(\oplus_i X_i), Z)$ as follows: $\phi_{n,m}(f) = f_{n,m}$, where $f_{n,m}$ coincides with f on $\oplus_{i<n} X_i$ and $f_{n,m}(\{\infty\} \cup (\oplus_{i \geq n} X_i)) = \{z_m\}$. Since $\oplus_{i<n} X_i$ is clopen in $F(\oplus_i X_i)$, $f_{n,m}$ is continuous.

Let us show that $\phi_{n,m}$ is continuous. Fix $f \in C_p(\oplus_{i<n} X_i, Z)$ and open $V \subset C_p(F(\oplus_i X_i), Z)$ that contains $f_{n,m} = \phi_{n,m}(f)$. We may assume that $V = V_1 \cap V_2$, where $V_1 = \{g : g(p) = f_{n,m}(p)\}$ with $p \in \oplus_{i<n} X_i$ fixed, and $V_2 = \{g : g(q) = f_{n,m}(q)\}$, where q is a fixed element of $\{\infty\} \cup (\oplus_{i \geq n} X_i)$. Put $U = \{h \in C_p(\oplus_{i<n} X_i, Z) : h(p) = f(p)\}$. Fix $h \in U$. We need to show that $h_{n,m} \in V$. Since $p \in \oplus_{i<n} X_i$, $h_{n,m}(p) = h(p) = f_{n,m}(p)$. Thus, $h_{n,m} \in V_1$. By definition of $\phi_{n,m}$, $h_{n,m}(q) = z_m = f_{n,m}(q)$. Thus, $h_{n,m} \in V_2$.

Since the domain of $\phi_{n,m}$ is Lindelöf, its image is Lindelöf as well. According to the definition of $F(\oplus_i X_i)$, the sequence of sets (X_i) converges to ∞ in $F(\oplus_i X_i)$. Therefore, for any $f \in C_p(F(\oplus_i X_i), Z)$ there exists n such that f is constant on $\{\infty\} \cup (\oplus_{i>n} X_i)$. Therefore, $C_p(F(\oplus_i X_i), Z)$ is equal to $\bigcup\{\phi_{n,m}(C_p(\oplus_{i<n} X_i, Z)) : n, m \in \omega\}$. Thus, $C_p(F(\oplus_i X_i), Z)$ is Lindelöf as the union of countably many Lindelöf subspaces. $\square$

Example 3.6. *Assume Continuum Hypothesis. Then there exists a compactum X and $p \in X$ of countable character such that $C_p(X, 2)$ is Lindelöf while $C_p(X \setminus \{p\}, 2)$ is not.*

Proof: In [POL2], R. Pol constructed (under CH) a compactum Y with the following properties:

1. $C_p(Y, 2)^n$ is Lindelöf for all n; and
2. $C_p(Y, 2)^\omega$ is not Lindelöf.

Put $X_i = Y$ for all $i \in \omega$. Let $X = F(\oplus_i X_i)$. Clearly X is compact and $p = \infty$ has countable character in X. By property 1, $C_p(\oplus_{i<n} X_i, 2)$ is Lindelöf for all $n \in \omega$. By Proposition 3.5, $C_p(X, 2)$ is Lindelöf. Recall that $X \setminus \{p\} = \oplus_{i \in \omega} Y$. By property 2, $C_p(X \setminus \{p\}, 2)$ is not Lindelöf. $\square$

In the construction of Example 3.6, if we replace Y from [POL2] by any space that meets properties 1 and 2, then we get a space Z with $C_p(Z, 2)$ Lindelöf and $C_p(Z \setminus \{\infty\}, 2)$ not Lindelöf. It is shown in [HST] that there exists a consistent example of a maximal almost disjoint family $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}), 2)^n$ is Lindelöf for all n. The fact that $C_p(\Psi(\mathcal{A}), 2)^\omega$ is never Lindelöf, where $\Psi(\mathcal{A})$ is the Mrowka space determined by a maximal almost disjoint family $\mathcal{A}$, is implicitly proved in [POL1] but not stated. A stronger result is obtained in [D&S], namely, that $C_p(\Psi(\mathcal{A})$, convergent sequence) is not Lindelöf for any maximal almost disjoint family $\mathcal{A}$. Thus, if in Example 3.6 we trade compactness for pseudocompactness, we obtain a first-countable pseudocompact Z with the desired properties. But can we have both compactness and first-countability?

Question 3.7. *Is there a first-countable* (*or weakly first-countable*) *compactum Z such that $C_p(Z, 2)$ is Lindelöf while $C_p(Z \setminus \{p\}, 2)$ is not for some $p \in Z$?*

Above discussion suggests relaxing "countable character" to some property closer to "countable tightness". It is natural to add points of countable tightness from the Čech-Stone compactification.

Question 3.8. *Let $C_p(X)$ be Lindelöf and $p \in \beta X$ have countable tightness in βX. Is $C_p(X \cup \{p\})$ Lindelöf?*

Although we do not know the answer to the question in general case, we have a solution if X is initially ω_1-compact. Recall that X is *initially ω_1-compact* if every infinite set of cardinality at most ω_1 has a complete accumulation point in X. This is equivalent to the condition that every open cover of X of cardinality at most ω_1 contains a finite subcover. A point $p \in X$ is a *complete accumulation point* for a set $A \subset X$ if every open neighborhood of p meets A by a subset of cardinality of A.

Theorem 3.9. *Let X be initially ω_1-compact. If $C_p(X)$ is Lindelöf then so is $C_p(\beta X)$.*

Proof: By Baturov's theorem [BAT] it suffices to show that $C_p(\beta X)$ has countable extent (="every closed discrete subset is countable"). Fix an ω_1-sized subset F of $C_p(\beta X)$. If $g \in C_p(X)$ we denote by $\tilde{g}$ the continuous extension of g over βX. Put $G = \{g \in C_p(X) : \tilde{g} \in F\}$. Clearly, the cardinality of G is ω_1. By hypothesis, there exists a complete accumulation point $g^* \in C_p(X)$ for G. Put $f^* = \tilde{g}^*$. Let us show that f^* is a complete accumulation point for F. First let us prove the following claim.

Claim. *Suppose $x \in \beta X \setminus X$. Then the set $G_x = \{g \in G : \tilde{g}(x) \in (f^*(x) - 1/5, f^*(x) + 1/5)\}$ is uncountable and g^* is a complete accumulation point for G_x.*

Assume that at least one of the two conclusions does not hold. Then $G' = G \setminus G_x$ is uncountable and g^* is a complete accumulation point for G'. Assume $f^*(x) = 0$. For every $g \in G'$ put $S_g = g^{-1}(R \setminus (-1/5, 1/5))$. Put $S_{g^*} = g^{*-1}(0)$. Observe that the closure of every S_g in βX is a closed G_δ-set containing x. Since X is pseudocompact, the intersection of any countable collection of S_g's is non-empty. Since X is initially ω_1-compact, there exists

$z \in S_{g^*} \cap (\bigcap_{g \in G'} S_g)$. Then $g(z) \notin (-1/5, 1/5)$ for every $g \in G$ while $g^*(z) \in (-1/5, 1/5)$. This contradicts the fact that g^* is a complete accumulation point for G'. The claim is proved.

To show that f^* is a complete accumulation point for F fix an open neighborhood $U = U_1 \cap \dots \cap U_n$ of f^*, where $U_i = \{f \in C_p(\beta X) : f(x_i) \in (f^*(x_i) - \epsilon_i, f^*(x_i) + \epsilon_i)\}$ with $x_i \in \beta X$ and $\epsilon_i > 0$ fixed. Put $G_0 = G$. For $0 < i \leq n$, put $G_i = \{g \in G : \tilde{g}(x_j) \in (f^*(x_j) - \epsilon_j, f^*(x_j) + \epsilon_j), j \leq i\}$. That is, G_i consists of all elements of G whose extensions over βX are elements of $U_1 \cap \dots \cap U_i$. Suppose that for every $i < k \leq n$, we have proved that G_i is uncountable and g^* is a complete accumulation point for G_i. Let us prove the same for G_k. Since $G_0 = G$, we may assume that $k > 0$.

If $x_k \in X$ then $V_k = \{g \in C_p(X) : \tilde{g}(x_k) \in (f^*(x_k) - \epsilon_k, f^*(x_k) + \epsilon_k)\}$ is an open neighborhood of g^* in $C_p(X)$. Therefore, $G_k = G_{k-1} \cap V_k$ is uncountable and g^* is a complete accumulation point for G_k.

If $x_k \in \beta X \setminus X$ then by substituting G_{k-1} for G in the hypothesis of Claim we get $G_k = G_{x_k}$ is uncountable and g^* is a complete accumulation point for G_k.

Thus, G_n is uncountable and g^* is a complete accumulation point for G_n. Since $G_n = \{g \in G : \tilde{g} \in U\}$ and $\tilde{g} \in F$ for all $g \in G_n$ we conclude that $F \cap U$ is uncountable. Therefore, f^* is a limit point for F. □

To strengthen the conclusion of this theorem, we need the following folklore statement.

Lemma 3.10. *If $C_p(\beta X)$ is Lindelöf and X is dense in Y then $C_p(Y)$ is Lindelöf.*

Proof: The space βX admits a continuous map onto βY. Since these spaces are compact, the map in question is quotient. Therefore, $C_p(\beta Y)$ is homeomorphic to a closed subspace of $C_p(\beta X)$ [ARH, Corollary 0.4.8], and therefore, is Lindelöf. It is clear that $C_p(\beta Y)$ admits a continuous surjection onto $C_p(Y)$. □

Corollary 3.11. *Let X be initially ω_1-compact with $C_p(X)$ Lindelöf. If X is dense in Y then $C_p(Y)$ is Lindelöf.*

Theorem 3.9 and Lemma 3.10 imply that if X is an initially ω_1-compact space and $C_p(X)$ is Lindelöf then $C_p(bX)$ is Lindelöf for some compactification bX of X. To reverse this statement we have to answer the following question.

Question 3.12. *Let X be a compactum with $C_p(X)$ Lindelöf. Let Y be an initially ω_1-compact subspace of X. Is $C_p(Y)$ Lindelöf? What if Y is countably compact?*

Next is a simple observation related to this question.

Theorem 3.13. *Assume Martin Axiom and Negation of Continuum Hypothesis. Let X be a compactum with $C_p(X)$ Lindelöf. Let Y be a countably compact subspace of X. Then $C_p(Y)$ is Lindelöf. Moreover, Y is compact.*

Proof: It suffices to show that Y is compact. By Asanov's theorem [ASA], X has countable tightness. Assume Y is not compact. Due to countable tightness, there exists a countable set $A \subset Y$ whose closure in X meets $X \setminus Y$. By Reznichenko's theorem (see [ARH]), Martin Axiom and negation of CH imply that every separable compactum Z with Lindelöf $C_p(Z)$ is metrizable. Therefore, $\mathrm{Cl}_X(A)$ is metrizable. This means that there exists a sequence of elements of Y that converges to a point in $X \setminus Y$. This contradicts countable compactness of X. $\square$

We would like to finish our study with several more questions related to our results.

Question 3.14. *Is there an initially ω_1-compact space X which is not compact and has $C_p(X)$ Lindelöf?*

Question 3.15. *Let X be a compactum with $C_p(X)$ Lindelöf. Let $X \cup X'$ be the Alexandroff double of X. Is $C_p(X \cup X')$ Lindelöf? What if X is a Corson compactum. What if X is countably compact and not compact?*

Observe that the answer to this question is in affirmative if X is an Eberlein compactum. Indeed, the Alexandroff double of an Eberlein compactum is an Eberlein compactum and the function space of every Eberlein compactum is Lindelöf [TAL].

Question 3.16. *Let X be the Alexandroff double of the Σ-product of ω_1-many copies of R. Is $C_p(X)$ Lindelöf?*

Question 3.17. *Let $C_p(X,2)$ be Lindelöf and let p have a countable weak base in $Y = \{p\} \cup X$. Is $C_p(Y,2)$ Lindelöf? What if $C_p(\cdot,2)$ is replaced with $C_p(\cdot)$?*

Question 3.18. *Let $Y = X \cup A$ be a first countable compactum, where all elements of A are isolated in Y. Suppose that $C_p(X)$ is Lindelöf and A is countable. Is $C_p(Y)$ Lindelöf?*

Question 3.19. *Let X_n be a first countable compactum such that $C_p(X_n)$ is Lindelöf. Suppose that $X = \bigcup_n X_n$ and $X_n \subset X_{n+1}$. Is $C_p(X)$ Lindelöf? What if X is compact?*

In connection with the last two questions we would like to mention that there exists ([POL1]) a compact space $X \cup A$ such that X is compact with Lindelöf $C_p(X)$, A is a countable dense subset of isolated points in $X \cup A$, and $C_p(X \cup A)$ is not Lindelöf. That is, adding countably many isolated points to a compactum may destroy Lindelöf property in a function space. The existence of such a space follows from the theorem of R. Pol [POL1] that states the following: *If Z is a compact separable space with the ω_1th derived set empty and $C_p(Z)$ Lindelöf then Z is countable.* Therefore, for the promised example one can take the one-point compactification of the Cantor Tree.

Another fact that should be mentioned is that $C_p(\mathrm{FR})$ is not Lindelöf, where FR is the Franklin-Rajagopalan space [F&R] with the derived set ω_1. This follows from the argument of the above mentioned theorem of Pol [POL1] (can also be found in [ARH, Proposition IV.7.4]). Thus, compactness is important in Questions 3.18 and 3.19. In fact, the argument of [D&S, Proposition 1] implies that even C_p(FR, convergent sequence) is not Lindelöf. It is not clear, however, whether $C_p(\mathrm{FR}, 2)$ is Lindelöf or not.

References

[ARH] Arhangel'skii A.V., *Topological Function Spaces*, Math. Appl., vol. 78, Kluwer Academic Publisher, Dordrecht, 1992.

[ASA] Asanov M.O., *On cardinal invariants of function spaces*, Modern Topology and Set Theory, Igevsk, (2) 1979, 8–12.

[BAT] Baturov D., *On subspaces of function spaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., 1987, no. 4, 66–69.

[D&S] Dow A., Simon P., *Spaces of continuous functions over a Ψ-space*, Topology Appl. **153** (2006), no. 13, 2260–2271.

[ENG] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann, Berlin, revised ed., 1989.

[F&R] Franklin S.P., Rajagopalan M., *Some examples in topology*, Trans. Amer. Math. Soc. **155** (1971), 305–314.

[HST] Hrušák M., Szeptycki P.J., Tamariz-Mascarúa A., *Spaces of continuous functions defined on Mrowka spaces*, Topology Appl. **148** (2005), no. 1–3, 239–252.

[MAR] Mardešić S., *On covering dimension and inverse limits of compact spaces*, Illinois J. Math. **4** (1960), 278–291.

[POL1] Pol R., *Concerning function spaces on separable compact spaces*, Bull. Acad. Polon. Sci. **25** (1977), no. 10, 993–997.

[POL2] Pol R., *The Lindelöf property and its analogue in function spaces with weak topology*, Topology 4-th Colloq., Budapest, 1978, Vol. 2; Amsterdam, 1980, pp. 965–969.

[TAL] Talagrand M., *Sur une conjecture de H.H. Corson*, Bull. Sci. Math. (2) **99** (1975), no. 4, 211–212.

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