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# On approximation of functions by certain operators preserving $x^2$

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*Abstract.* In this paper we extend the Duman-King idea of approximation of functions by positive linear operators preserving $e_k(x) = x^k$, $k = 0, 2$. Using a modification of certain operators $L_n$ preserving $e_0$ and $e_1$, we introduce operators $L_n^*$ which preserve $e_0$ and $e_2$ and next we define operators $L_{n;r}^*$ for $r$-times differentiable functions. We show that $L_n^*$ and $L_{n;r}^*$ have better approximation properties than $L_n$ and $L_{n;r}$.

*Keywords:* positive linear operators, polynomial weighted space, degree of approximation

*Classification:* 41A25, 41A36

## 1. Introduction

**1.1.** It is well known ([3-5]) that many of classical approximation operators $L_n$ satisfy the following conditions for the functions $e_k(x) = x^k$, $k = 0, 1, 2$:

$$L_n(e_0; x) = 1, \qquad L_n(e_1; x) = x, \tag{1}$$

and

$$L_n(e_2; x) = x^2 + \frac{ax^2 + bx}{\lambda_n}, \tag{2}$$

for $x \in X$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, where $a, b$ are given non-negative numbers, $a^2 + b^2 > 0$, and $(\lambda_n)_1^\infty$, $\lambda_1 \geq 1$, is a fixed increasing and unbounded sequence of numbers.

We say that the operators $L_n$ preserve the functions $e_0$ and $e_1$ if the conditions (1) are satisfied.

The conditions (1) and (2) hold, in particular, for the Szász-Mirakyan, Baskakov, Post-Widder and Stancu operators ([1]–[5], [7], [11]–[14]).

In the papers [6]–[8], there were introduced certain modified Bernstein, Szász-Mirakyan and Meyer-König and Zeller operators, which preserve the functions $e_0$ and $e_2$ and have better approximation properties than classical operators.

In the paper [13] we have extended the Duman-King idea, [6]–[8], to the Post-Widder and Stancu operators considered in polynomial weighted spaces.

**1.2.** G. Kirov [9] and other authors (e.g. [10], [11]) have examined approximation properties of linear operators

$$L_{n;r}(f;x) := L_n\left(F_r(t,x);x\right), \quad n \in \mathbb{N}, \tag{3}$$

with

$$F_r(t,x) := \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j, \tag{4}$$

for $r$-times differentiable functions $f$, using operators $L_n$ with conditions (1). These authors have shown that the order of approximation of an $r$-times differentiable function $f$ by $L_{n;r}(f)$ is dependent on $r$ and it improves if $r$ grows.

**1.3.** Let $\mathbb{N}_0$ and $\mathbb{R}$ be sets of non-negative integers and real numbers, correspondingly, and let $I$ be the interval $(0,\infty)$ (or $[0,\infty)$).

Analogously to [2] let $p \in \mathbb{N}_0$,

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \ \text{ if } \ p \geq 1, \tag{5}$$

for $x \in I$, and let $B_p \equiv B_p(I)$ be the set of all functions $f : I \to \mathbb{R}$ for which $fw_p$ is bounded on $I$ and the norm is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in I} w_p(x)|f(x)|. \tag{6}$$

Next let $C_p \equiv C_p(I)$, $p \in \mathbb{N}_0$, be the set of all $f \in B_p$ for which $fw_p$ is uniformly continuous on $I$ and the norm is given by (6). $C_p$ is called the polynomial weighted space.

Moreover, let $C^r \equiv C^r(I)$, with a fixed $r \in \mathbb{N}$, be the set of all $r$-times differentiable functions $f \in C_r$ with derivatives $f^{(k)} \in C_{r-k}$ for $k = 0, 1, \ldots, r$ and the norm in $C^r$ is given by (6).

It is obvious that if $p, q \in \mathbb{N}_0$ and $p < q$, then $B_p \subset B_q$, $C_p \subset C_q$ and $\|f\|_q \leq \|f\|_p$ for $f \in B_p$. Obviously, for every $p \in \mathbb{N}_0$ we have $w_p \in C_0$ and $\frac{1}{w_p} \in C^p$ (here $C^0 \equiv C_0$).

**1.4.** The purpose of this paper is to extend the Duman-King and Kirov methods to the classes of operators $L_n$ and $L_{n;r}$ satisfying the conditions (1)–(4), defined in polynomial weighted spaces $C_p$ and $C^r$.

In Section 2 we shall introduce the operators $L_n$, $L_n^*$, $L_{n;r}$ and $L_{n;r}^*$ for functions $f \in C_p$ and $f \in C^r$, correspondingly, and we shall give some of their basic properties.

In Section 3 we shall give the main approximation theorems.

In this paper we shall denote by $M_k(\alpha,\beta)$, $k \in \mathbb{N}$, suitable positive constants depending only on the indicated parameters $\alpha$ and $\beta$.

## 2. Definitions and auxiliary results

**2.1.** Let $(L_n)_{n=1}^{\infty}$ (or $n \geq n_0$) be a sequence of positive linear operators with the following properties:

(i) $L_n : C_p \to B_p$ for every $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

(ii) $L_n$ satisfies the conditions (1) and (2) for $x \in I$ and $n \in \mathbb{N}$, with fixed $a$, $b$ and $(\lambda_n)$,

(iii) there exists $M_1 \equiv M_1(a, b, p) =$ const. $> 0$ such that for the functions

$$T_{n;p}(x) := L_n\left(\varphi_x^p(t); x\right), \qquad x \in I, \quad n \in \mathbb{N}, \quad 2 \leq p \in \mathbb{N}, \tag{7}$$

with

$$\varphi_x(t) := t - x, \qquad t \in I, \tag{8}$$

there holds

$$\|T_{n;2p}\|_{2p} \leq M_1 \lambda_n^{-p} \quad \text{for} \quad n \in \mathbb{N}. \tag{9}$$

Using the above operators $L_n$, we define for $f \in C_p$, $p \in \mathbb{N}_0$, the following operators:

$$L_n^*(f; x) := L_n(f; u_n(x)) \quad \text{for} \quad x \in I, \quad n \in \mathbb{N}, \tag{10}$$

where

$$u_n(x) := \frac{-b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2}}{2(a + \lambda_n)}. \tag{11}$$

Next, for the functions $f \in C^r$, $r \in \mathbb{N}$, $x \in I$ and $n \in \mathbb{N}$, we introduce the operators $L_{n;r}$ by formulas (3) and (4) and the operators $L_{n;r}^*$:

$$L_{n;r}^*(f; x) := L_n^*\left(F_r(t, x); x\right), \qquad x \in I, \quad n \in \mathbb{N}, \tag{12}$$

where $F_r$ is defined by (4).

From the properties of the above operators $L_n$ and formulas (10) and (11), it follows that $L_n^*$, $n \in \mathbb{N}$, is a positive linear operator acting from the space $C_p$ to $B_p$ for every $p \in \mathbb{N}_0$ and by (1), (2) and (8) we have

$$L_n^*(e_0; x) = 1, \quad L_n^*(e_1; x) = u_n(x), \quad L_n^*(e_2; x) = x^2, \tag{13}$$

$$L_n\left(\varphi_x^2(t); x\right) = \frac{ax^2 + bx}{\lambda_n} \tag{14}$$

and

$$L_n^*\left(\varphi_x^2(t); x\right) = 2x\left(x - u_n(x)\right), \tag{15}$$

for $x \in I$ and $n \in \mathbb{N}$. Moreover, from (3), (4) and (10)–(12) we deduce that $L_{n;r}$ and $L_{n;r}^*$ for $n, r \in \mathbb{N}$, are well defined on the space $C^r$ and

$$L_{n;r}^*(f; x) = L_{n;r}\left(f; u_n(x)\right), \quad x \in I, \quad n \in \mathbb{N}, \tag{16}$$

for every $f \in C^r$.

**2.2.** Here we shall give some lemmas on basic properties of the introduced operators.

By (i)–(iii) and (10) and (11) we easily obtain the following two lemmas.

**Lemma 1.** *Let $u_n$ be defined by (11) for $x \in I$ and $n \in \mathbb{N}$, with fixed numbers $a, b \geq 0$, $a^2 + b^2 > 0$ and $(\lambda_n)_1^\infty$ given by (2). Then we have*

$$0 \leq u_n(x) \leq x, \quad 0 \leq x - u_n(x) \leq \frac{ax+b}{\lambda_n}, \tag{17}$$

$$\sqrt{\frac{ax^2+bx}{\lambda_n}} - \sqrt{2x\,(x-u_n(x))} \geq \frac{2ax+b}{4\,(2ax+b+2\lambda_n x)}\sqrt{\frac{ax^2+bx}{\lambda_n}}, \tag{18}$$

*for $x \in I$ and $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} \lambda_n\,(x-u_n(x)) = \frac{ax+b}{2} \quad \text{at every } x \in I. \tag{19}$$

**Lemma 2.** *For every $f, g \in C_p$, $p \in \mathbb{N}$, there holds*

$$|L_n\,(f(t)g(t);x)| \leq \Big(L_n\left(f^2(t);x\right)\Big)^{\frac{1}{2}} \Big(L_n\left(g^2(t);x\right)\Big)^{\frac{1}{2}}, \quad x \in I,\ n \in \mathbb{N}.$$

*The identical inequality holds for the operators $L_n^*$.*

By (5) and (17) we easily derive the following inequalities

$$w_p^2(x) \leq w_{2p}(x),\ 1/w_p^2(x) \leq 2/w_{2p}(x),\ 0 < w_p(x)/w_p(u_n(x)) \leq 1, \tag{20}$$

for $x \in I$ and $p \in \mathbb{N}_0$.

**Lemma 3.** *Let $p \in \mathbb{N}_0$ and let $a$, $b$ and $\lambda_n$ be fixed numbers connected with operators $L_n$ given by the formula (2). Then there exists $M_2 = M_2(a,b,p) =$ const. $> 0$ such that*

$$\|L_n^*(1/w_p)\|_p \leq \|L_n(1/w_p)\|_p \leq M_2 \quad \text{for } n \in \mathbb{N}. \tag{21}$$

*Moreover, for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$\|L_n^*(f)\|_p \leq \|L_n(f)\|_p \leq M_2\|f\|_p. \tag{22}$$

*The formulas (10) and (11) and the inequality (22) show that $L_n^*$, $n \in \mathbb{N}$, is a positive linear operator acting from the space $C_p$ into $B_p$ for every $p \in \mathbb{N}_0$.*

Proof: If $p = 0$, then by (5), (6), (1) and (13) it follows that $\|L_n^*(1/w_0)\|_0 = \|L_n(1/w_0)\|_0 = 1$ for $n \in \mathbb{N}$.

If $p \in \mathbb{N}$, then by the linearity of $L_n$ and (5), (1) and (8) we have

$$L_n\left(1/w_p(t);x\right) = 1 + L_n(e_p;x) \le 1 + 2^p\left(x^p + L_n\left(|\varphi_x(t)|^p;x\right)\right),$$

which by (5)–(9), (20) and Lemma 2 implies that

$$\begin{aligned} w_p(x)L_n\left(1/w_p;x\right) &\le 2^p + 2^p\left(w_{2p}(x)L_n\left(\varphi_x^{2p}(t);x\right)\right)^{\frac{1}{2}} \\ &\le 2^p\left(1+\sqrt{M_1/\lambda_n^p}\right) \le 2^p\left(1+\sqrt{M_1}\right), \end{aligned}$$

for $x \in I$ and $n \in \mathbb{N}$. Hence the inequality (21) is proved for $L_n$.

By (10), (20) and (6) we can write

$$w_p(x)L_n^*\left(1/w_p;x\right) \le w_p\left(u_n(x)\right)L_n\left(1/w_p;u_n(x)\right) \le \|L_n(1/w_p)\|_p$$

for $x \in I$ and $n \in \mathbb{N}$, which by (6) yields (21) for $L_n^*$.

The inequality (22) for $f \in C_p$, $n \in \mathbb{N}_0$, follows by (10), (20), (6), (21) and the following estimate

$$\begin{aligned} w_p(x)\left|L_n^*(f;x)\right| &\le w_p(u_n(x))\left|L_n(f;u_n(x)\right| \le \|L_n(f)\|_p \\ &\le \|f\|_p\|L_n(1/w_p)\|_p \le M_2\|f\|_p, \quad x \in I, \quad n \in \mathbb{N}. \end{aligned}$$

□

**Lemma 4.** *Let $r \in \mathbb{N}$ and let $L_{n;r}$ and $L_{n;r}^*$ be operators defined by* (3), (4) *and* (10)–(12) *with fixed parameters $a, b$ and $\lambda_n$ connected with $L_n$. Then there exists $M_3 = M_3(a,b,r) =$ const. $> 0$ such that for every $f \in C^r$ and $n \in \mathbb{N}$ there holds*

$$\|L_{n;r}^*(f)\|_r \le \|L_{n;r}(f)\|_r \le \|f\|_r + M_3\|f^{(r)}\|_0. \tag{23}$$

*The formulas* (3), (4) *and* (12) *and the inequalities* (23) *show that $L_{n;r}$ and $L_{n;r}^*$, $n \in \mathbb{N}$, are linear operators acting from the space $C^r$ to $B_r$.*

Proof: Choose $f \in C^r$ with a fixed $r \in \mathbb{N}$ and $t \in I$. Then, by the modified Taylor formula we have

$$f(x) = \sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(x-t)^j + I_r(t,x), \quad x \in I, \tag{24}$$

where

$$I_r(t,x) := \frac{(x-t)^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\left[f^{(r)}\left(t+u(x-t)\right) - f^{(r)}(t)\right]\,du. \tag{25}$$

From (24), (25) and (4) it results that

$$F_r(t,x) = f(x) - I_r(t,x) \quad \text{for} \;\; t,x \in I, \tag{26}$$

which next by (3) and (1) implies that

$$L_{n;r}\left(f(t);x\right) = f(x) - L_n\left(I_r(t,x);x\right) \tag{27}$$

and consequently

$$w_r(x)\left|L_{n;r}(f(t);x)\right| \le \|f\|_r + w_r(x)L_n\left(\left|I_r(t,x)\right|;x\right), \tag{28}$$

for $x \in I$ and $n \in \mathbb{N}$. But if $f \in C^r$, then $f^{(r)} \in C_0$ and by (25) and (8) we have

$$|I_r(t,x)| \le (2/r!)\|f^{(r)}\|_0 \, |\varphi_x(t)|^r$$

and next by Lemma 2, (20) and (7)–(9) we get

$$\begin{aligned} w_r(x)L_n\left(\left|I_r(t,x)\right|;x\right) &\le \frac{2}{r!}\|f^{(r)}\|_0\left(w_{2r}(x)L_n\left(\varphi_x^{2r}(t);x\right)\right)^{\frac{1}{2}} \\ &\le \frac{2}{r!}\|f^{(r)}\|_0\,(M_1/\lambda_n^r)^{\frac{1}{2}} \le \left(2\sqrt{M_1}/r!\right)\|f^{(r)}\|_0, \end{aligned} \tag{29}$$

for $x \in I$ and $n \in \mathbb{N}$. Now, using (29) to (28), we obtain the inequality (23) for $L_{n;r}$.

The formula (16) and the inequality (20) imply that for $f \in C^r$ we can write

$$\begin{aligned} w_r(x)\left|L^*_{n;r}(f;x)\right| &\le w_r(u_n(x))\left|L_{n;r}\left(f;u_n(x)\right)\right| \\ &\le \|L_{n;r}(f)\|_r \;\; \text{for} \;\; x \in I, \quad n \in \mathbb{N}, \end{aligned}$$

which by (6) completes the proof of (23). □

## 3. Theorems

**3.1.** In this section we shall estimate the orders of approximation of a function $f \in C_p$ by $L_n(f)$ and $L^*_n(f)$, and also $f \in C^r$ by $L_{n;r}(f)$ and $L^*_{n;r}(f)$. We shall use the modulus of continuity of a function $f \in C_p$, i.e.

$$\omega(f;t)_p := \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p \quad \text{for} \;\; t \ge 0, \tag{30}$$

where $\Delta_h f(x) = f(x+h) - f(x)$.

**Theorem 1.** *Assume that $p \in \mathbb{N}_0$ is a fixed number and $L_n$ and $L_n^*$ are operators defined in Section 2. Then there exists $M_4 = M_4(a,b,p) =$ const. $> 0$ such that for every $f \in C_p$ having the first derivative $f'$ belonging to $C_p$ there holds*

$$w_p(x)\,|L_n(f;x) - f(x)| \le M_4 \|f'\|_p \sqrt{(ax^2+bx)/\lambda_n} \tag{31}$$

*and*

$$w_p(x)\,|L_n^*(f;x) - f(x)| \le M_4 \|f'\|_p \sqrt{2x\,(x-u_n(x))}, \tag{32}$$

*for $x \in I$ and $n \in \mathbb{N}$.*

Proof: Choose $p \in \mathbb{N}$ and $f \in C_p$ for which $f' \in C_p$. Then for a fixed $x \in I$ we can write

$$f(t) - f(x) = \int_x^t f'(u)\,du, \quad t \in I.$$

Using now $L_n$ and (1), (8), Lemma 2 and (20), we get

$$\begin{aligned} |L_n(f(x);x) - f(x)| &\le L_n\left(\left|\int_x^t f'(u)\,du\right|;x\right) \\ &\le \|f'\|_p L_n\left(\left|\int_x^t \frac{du}{w_p(u)}\right|;x\right) \\ &\le \|f'\|_p \left(L_n\left(|\varphi_x(t)|\,/w_p(t);x\right) + L_n\left(|\varphi_x(t)|\,;x\right)\right) \\ &\le \|f'\|_p \left(L_n\left(\varphi_x^2(t);x\right)\right)^{\frac{1}{2}} \left(\left(2L_n\left(1/w_{2p}(t);x\right)\right)^{\frac{1}{2}} + 1\right), \end{aligned}$$

for $n \in \mathbb{N}$. From this and by (14), (20) and (21) we immediately obtain the desired inequality (31).

The proof of (32) is similar. □

**Theorem 2.** *Let $p$, $L_n$ and $L_n^*$ satisfy the assumptions of Theorem 1. Then there exists $M_5 = M_5(a,b,p) =$ const. $> 0$ such that*

$$w_p(x)\,|L_n(f;x) - f(x)| \le M_5 \omega\left(f; \sqrt{(ax^2+bx)/\lambda_n}\right)_p \tag{33}$$

*and*

$$w_p(x)\,|L_n^*(f;x) - f(x)| \le M_5 \omega\left(f; \sqrt{2x(x-u_n(x))}\right)_p, \tag{34}$$

*for $x \in I$ and $n \in \mathbb{N}$.*

Proof: Similarly to [2] and [13] we use the Steklov function $f_h$ of $f \in C_p$, i.e.

$$f_h(x) := \frac{1}{h} \int_0^h f(x+t)\,dt, \quad x \in I, \quad h > 0.$$

It is easily verified that $f_h$ and the derivative $f_h'$ belong to $C_p$ as well, and by (30) we have:

$$\|f_h - f\|_p \le \omega(f;h)_p, \tag{35}$$

and

$$\|f_h'\|_p \le h^{-1}\omega(f;h)_p \quad \text{for} \quad h > 0. \tag{36}$$

Applying (13), (22), Theorem 1, (30), (35) and (36), we get

$$\begin{aligned}
&w_r(x)\,|L_n^*(f;x) - f(x)| \\
&\le w_r(x)\left(|L_n^*\left(f(t) - f_h(t);x\right)| + |L_n\left(f_h(t);x\right) - f_h(x)| + |f_h(x) - f(x)|\right) \\
&\le M_2\|f - f_h\|_p + M_4\|f_h'\|_p\sqrt{2x(x - u_n(x))} + \|f_h - f\|_p \\
&\le \omega(f;h)_p\left(M_1 + 1 + M_4 h^{-1}\sqrt{2x(x - u_n(x))}\right)
\end{aligned}$$

for $x \in I$, $n \in \mathbb{N}$ and $h > 0$. Now setting $h = \sqrt{2x(x - u_n(x))}$, we obtain the desired estimate (34).

The proof of (33) is identical. □

From Theorem 2 and Lemma 1 we can derive the following two corollaries.

**Corollary 1.** *For every $f \in C_p$, $p \in \mathbb{N}_0$, there holds*

$$\lim_{n\to\infty} L_n(f;x) = f(x) = \lim_{n\to\infty} L_n^*(f;x) \quad \text{at} \quad x \in I.$$

*This convergence is uniform on every interval $[x_1, x_2]$, $x_1 > 0$.*

**Corollary 2.** *The inequalities* (17), (18), (33) *and* (34) *show that the operators $L_n^*$, $n \in \mathbb{N}$, have better approximation properties than $L_n$ for functions $f \in C_p$, $p \in \mathbb{N}_0$.*

**Theorem 3.** *Let $r \in \mathbb{N}$ and let $L_{n;r}$ and $L_{n;r}^*$ be operators defined in Section* 2. *Then for every $f \in C^r$ we have:*

$$w_r(x)\left|L_{n;r}(f;x) - f(x)\right| \le \frac{2}{r!}\left(M_1/\lambda_n^r\right)^{\frac{1}{2}}\omega\left(f^{(r)};\sqrt{(ax^2+bx)/\lambda_n}\right)_0 \tag{37}$$

*and*

$$w_r(x)\left|L^*_{n;r}(f;x)-f(x)\right| \le \frac{2}{r!}\left(M_1/\lambda_n^r\right)^{\frac{1}{2}}\omega\left(f^{(r)};\sqrt{2x\left(x-u_n(x)\right)}\right)_0 \tag{38}$$

*for* $x \in I$ *and* $n \in \mathbb{N}$, *where* $M_1 =$ *const.* $> 0$ *is given by* (9).

Proof: First we prove the inequality (37). Fix $r \in \mathbb{N}$, $f \in C^r$ and $x \in I$. Then by (27) it follows that

$$\left|L_{n;r}(f;x)-f(x)\right| \le L_n\left(\left|I_r(t,x)\right|;x\right) \quad \text{for} \quad n \in \mathbb{N}.$$

Using (8), (30) and properties of the modulus of continuity of $f^{(r)} \in C_0$, we deduce from (25):

$$\begin{aligned}|I_r(t,x)| &\le \frac{|\varphi_x(t)|^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\omega\left(f^{(r)};u|\varphi_x(t)|\right)_0 du\\ &\le \omega\left(f^{(r)};|\varphi_x(t)|\right)_0 \frac{|\varphi_x(t)|^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\,du\\ &\le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)_0\left(|\varphi_x(t)|^r+\delta^{-1}|\varphi_x(t)|^{r+1}\right),\end{aligned}$$

for $t \in I$ and every fixed $\delta > 0$. Consequently,

$$\left|L_{n;r}(f;x)-f(x)\right| \le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)_0\left(L_n\left(|\varphi_x(t)|^r;x\right)+\delta^{-1}L_n\left(|\varphi_x(t)|^{r+1};x\right)\right),$$

which by Lemma 2, (1), (20), (7)–(9) and (14) implies that

$$\begin{aligned}w_r(x)\left|L_{n;r}(f;x)-f(x)\right| &\le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)_0\|T_{n;2r}\|_{2r}\\ &\quad\times\left(1+\delta^{-1}\left(L_n\left(\varphi_x^2(t);x\right)\right)^{\frac{1}{2}}\right)\\ &\le (1/r!)\omega\left(f^{(r)};\delta\right)_0\left(M_1\lambda_n^{-r}\right)^{\frac{1}{2}}\left(1+\delta^{-1}\sqrt{(ax^2+bx)/\lambda_n}\right)\end{aligned} \tag{39}$$

for $n \in \mathbb{N}$. Setting $\delta = \sqrt{(ax^2+bx)/\lambda_n}$ to (39), we obtain (37) for chosen $x \in I$ and $n \in \mathbb{N}$.

Applying (12), (26), (25) and (13), and arguing as above, we can write the following analogues of (27) and (39) for $f \in C^r$ and $L^*_{n;r}(f)$, i.e.

$$L^*_{n;r}(f;x)-f(x) = -L^*_n\left(I_r(t,x);x\right)$$

and

$$(40)\qquad \begin{aligned} w_r(x)\left|L^*_{n;r}(f;x)-f(x)\right| &\le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)_0 \\ &\times\left(w_{2r}(x)L^*_n\left(\varphi_x^{2r}(t);x\right)\right)^{\frac{1}{2}}\left\{1+\delta^{-1}\left(L^*_n\left(\varphi_x^2(t);x\right)\right)^{\frac{1}{2}}\right\}, \end{aligned}$$

for $x\in I$, $n\in\mathbb{N}$ and every fixed $\delta>0$. But by (23) and (6)–(9) it follows that

$$(41)\qquad w_{2r}(x)L^*_n\left(\varphi_x^{2r}(t);x\right)\le \|T_{n;2r}\|_{2r}\le M_1\lambda_n^{-r}$$

for $x\in I$ and $n\in\mathbb{N}$. Using (41) and (15) to (40) and next putting $\delta=\sqrt{2x\,(x-u_n(x))}$, we obtain the estimate (38). □

From Theorem 3 and (17) and (18) we can derive:

**Corollary 3.** *Let $r\in\mathbb{N}$ and $f\in C^r$. Then*

$$\lim_{n\to\infty}\lambda_n^{r/2}\left(L_{n;r}(f;x)-f(x)\right)=0=\lim_{n\to\infty}\lambda_n^{r/2}\left(L^*_{n;r}(f;x)-f(x)\right)$$

*at every $x\in I$. This convergence is uniform on every interval $[x_1,x_2]$, $x_1>0$.*

**Corollary 4.** *The inequalities (33) and (37) show that the order of approximation of an $r$-times differentiable function $f\in C^r$ by $L_{n;r}(f)$ is better than by $L_n(f)$. This order of approximation of $f\in C^r$ by $L_{n;r}(f)$ improves if $r\in\mathbb{N}$ grows.*

*The identical properties have operators $L^*_n$ and $L^*_{n;r}$ in spaces $C^r$, $r\in\mathbb{N}$. Moreover, the inequalities (37), (38), (17) and (18) show that operators $L^*_{n;r}$ have better approximation properties than $L_{n;r}$ for functions $f\in C^r$, $r\in\mathbb{N}$.*

**3.2.** Here we present the Voronovskaya type theorems for the operators considered.

**Theorem 4.** *Suppose that $p\in\mathbb{N}_0$ and a function $f\in C_p$ has derivatives $f',f''\in C_p$. Then*

$$(42)\qquad \lim_{n\to\infty}\lambda_n\left(L_n(f;x)-f(x)\right)=\frac{ax^2+bx}{2}f''(x)$$

*and*

$$(43)\qquad \lim_{n\to\infty}\lambda_n\left(L^*_n(f;x)-f(x)\right)=-\frac{ax+b}{2}f'(x)+\frac{ax^2+bx}{2}f''(x),$$

*at every $x\in I$.*

PROOF: We show only (43) because the proof of (42) is analogous.

Choose a function $f$ satisfying the above assumptions and $x \in I$. Then by the Taylor formula we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \alpha(t,x)(t-x)^2, \quad t \in I,$$

where $\alpha(t) \equiv \alpha(t,x)$ is a function belonging to $C_p$ and $\lim_{t\to x} \alpha(t) = \alpha(x) = 0$. Using the operator $L_n^*$ and next (8), (13) and (15), we get

$$\begin{aligned} L_n^*(f(t);x) &= f(x) + (u_n(x) - x)\, f'(x) + x\,(x - u_n(x))\, f''(x) \\ &\quad + L_n^*\left(\alpha(t)\varphi_x^2(t);x\right), \quad n \in \mathbb{N}. \end{aligned} \tag{44}$$

Applying Lemma 2, we get

$$\left|L_n^*\left(\alpha(t)\varphi_x^2(t);x\right)\right| \le \left(L_n^*\left(\alpha^2(t);x\right)\right)^{\frac{1}{2}} \left(L_n^*\left(\varphi_x^4(t);x\right)\right)^{\frac{1}{2}} \quad \text{for} \quad n \in \mathbb{N},$$

and moreover, by the properties of $\alpha(\cdot)$, Corollary 1 and (41) we have

$$\begin{aligned} &\lim_{n\to\infty} L_n^*\left(\alpha^2(t);x\right) = \alpha^2(x) = 0, \\ &\lambda_n^2 L_n^*\left(\varphi_x^4(t);x\right) \le M_1/w_4(x), \quad n \in \mathbb{N}. \end{aligned}$$

From the above it follows that

$$\lim_{n\to\infty} \lambda_n L_n^*\left(\alpha(t)\varphi_x^2(t);x\right) = 0. \tag{45}$$

Applying (19) and (45), we immediately derive (43) from (44). □

**Theorem 5.** *Let $r \in \mathbb{N}$ and let $f \in C^r$ be a function whose derivatives $f^{(r+1)}$ and $f^{(r+2)}$ belong to $C_0$. Then, for the operators $L_{n;r}^*$, the following asymptotic formula holds:*

$$\begin{aligned} L_{n;r}^*(f;x) - f(x) &= \frac{(-1)^r f^{(r+1)}(x) L_n^*\left(\varphi_x^{r+1}(t);x\right)}{(r+1)!} \\ &\quad + \frac{(-1)^r (r+1) f^{(r+2)}(x) L_n^*\left(\varphi_x^{r+2};x\right)}{(r+2)!} \\ &\quad + o_x\left(\lambda_n^{-(r+2)/2}\right) \quad \text{as} \quad n \to \infty, \end{aligned} \tag{46}$$

*at every $x \in I$.*

*The analogous asymptotic formula holds for the operators $L_{n;r}$.*

Proof: Choose $r \in \mathbb{N}$, $x \in I$ and $f \in C^r$ satisfying the above assumptions. Then the derivative $f^{(j)}$, $0 \le j \le r+2$, is an $(r+2-j)$-times differentiable function on $I$. Hence for every $f^{(j)}$, $0 \le j \le r$, we can write the Taylor formula at given $x$:

$$f^{(j)}(t) = \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \alpha_j(t,x)(t-x)^{r+2-j}, \quad t \in I,$$

where $\alpha_j(t) \equiv \alpha_j(t,x)$ is a function belonging to $C_0$ and $\lim_{t\to x} \alpha_j(t) = \alpha_j(x) = 0$. Using this formula to $F_r$ given by (4), we get

$$\begin{aligned} F_r(t,x) &= \sum_{j=0}^{r} \frac{(-1)^j}{j!} \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^{j+i} \\ &\quad + (t-x)^{r+2} \sum_{j=0}^{r} \frac{(-1)^j}{j!}\alpha_j(t) \\ &= \sum_{j=0}^{r}(-1)^j \sum_{s=j}^{r} \binom{s}{j} \frac{f^{(s)}(x)}{s!}(t-x)^s \\ &\quad + \frac{f^{(r+1)}(x)(t-x)^{r+1}}{(r+1)!} \sum_{j=0}^{r}(-1)^j \binom{r+1}{j} \\ &\quad + \frac{f^{(r+2)}(x)(t-x)^{r+2}}{(r+2)!} \sum_{j=0}^{r}(-1)^j \binom{r+2}{j} \\ &\quad + (t-x)^{r+2} A_r(t) \quad \text{for } t \in I, \end{aligned} \tag{47}$$

with

$$A_r(t) := \sum_{j=0}^{r} \frac{(-1)^j}{j!}\alpha_j(t). \tag{48}$$

The following identities for $m \in \mathbb{N}_0$

$$\sum_{j=0}^{m} \binom{m}{j}(-1)^j = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \in \mathbb{N} \end{cases},$$

$$\sum_{j=0}^{m} \binom{m+1}{j}(-1)^j = (-1)^m, \qquad \sum_{j=0}^{m} \binom{m+2}{j}(-1)^j = (m+1)(-1)^m,$$

imply that

$$\sum_{j=0}^{r}(-1)^j \sum_{s=j}^{r} \binom{s}{j} \frac{f^{(s)}(x)}{s!}(t-x)^s = \sum_{s=0}^{r} \frac{f^{(s)}(x)(t-x)^s}{s!} \sum_{j=0}^{s} \binom{s}{j} (-1)^j = f(x),$$

which applied to (47) yields

$$F_r(t,x) = f(x) + \frac{(-1)^r f^{(r+1)}(x)(t-x)^{r+1}}{(r+1)!} + \frac{(-1)^r (r+1) f^{(r+2)}(x)(t-x)^{r+2}}{(r+2)!} + (t-x)^{r+2} A_r(t),$$

for $t \in I$. From this and (12), (13) and (8) we deduce that

$$\begin{aligned} L^*_{n;r}(f(t);x) = f(x) &+ \frac{(-1)^r f^{(r+1)}(x) L^*_n\left(\varphi_x^{r+1}(t);x\right)}{(r+1)!} \\ &+ \frac{(-1)^r (r+1) f^{(r+2)}(x) L^*_n\left(\varphi_x^{r+2}(t);x\right)}{(r+2)!} \\ &+ L^*_n\left(A_r(t)\varphi_x^{r+2}(t);x\right) \quad \text{for} \quad n \in \mathbb{N}. \end{aligned} \tag{49}$$

We observe that, by the properties of the functions $\alpha_j$, (48) and Corollary 1,

$$\lim_{n\to\infty} L^*_n\left(A_r^2(t);x\right) = A_r^2(x) = 0. \tag{50}$$

Arguing as in the proof of Theorem 4 and applying (50) and (41), we obtain

$$L^*_n\left(A_r(t)\varphi_x^{r+2}(t);x\right) = o_x\left(\lambda_n^{-(r+2)/2}\right) \quad \text{as} \quad n \to \infty,$$

which, applied to (49), yields the desired asymptotic formula (46). □

## 4. Examples

Finally we present four examples of well-known positive linear operators $L_n$ which satisfy conditions (i)–(iii) given in Section 2.

**1.** The Szász-Mirakyan operators ([2]–[5])

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \geq 0, \quad n \in \mathbb{N}, \tag{51}$$

satisfy the conditions (1) and (2) with $a = 0$, $b = 1$ and $\lambda_n = n$ for $n \in \mathbb{N}$.

**2.** The Baskakov operators ([2], [5])

$$V_n(f;x) := (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right), \tag{52}$$
$$x \ge 0, \quad n \in \mathbb{N},$$

satisfy also the conditions (1) and (2) with $a = b = 1$ and $\lambda_n = n$ for $n \in \mathbb{N}$.

**3.** The Post-Widder operators ([5], [13]) are defined for $f \in C_p$, $p \in \mathbb{N}_0$, by the following integral formula:

$$(53) \quad \begin{aligned} P_n(f;x) &:= \int_0^\infty f(t)p_n(x,t)\,dt, \quad x > 0, \quad n \in \mathbb{N}, \\ p_n(x,t) &:= \frac{(n/x)^n t^{n-1}}{(n-1)!} \exp(-nt/x). \end{aligned}$$

These operators satisfy (1) and (2) with $a = 1$, $b = 0$ and $\lambda_n = n$ for $n \in \mathbb{N}$.

**4.** The beta Stancu operators ([14], [13]) are defined for $f \in C_p$, $p \in \mathbb{N}_0$, by the formula:

$$\widetilde{L}_n(f;x) := \int_0^\infty f(t)s_n(x,t)\,dt, \quad x > 0, \quad n \ge p+2, \tag{54}$$

where

$$s_n(x,t) := \frac{t^{nx-1}}{B(nx, n+1)(1+t)^{nx+n-1}}$$

and $B$ is the Euler beta function. Now the conditions (1) and (2) hold with $a = b = 1$ and $\lambda_n = n - 1$ for $2 \le n \in \mathbb{N}$.

Using the formulas (3), (4), (10)–(12) and (51)–(54), we can define the modified Szász-Mirakyan, Baskakov, Post-Widder and Stancu operators: $S_n^*$, $V_n^*$, $P_n^*$ and $\widetilde{L}_n^*$ in the space $C_p$, $p \in \mathbb{N}_0$, and the corresponding operators $L_{n;r}$ and $L_{n;r}^*$.

Hence, from Theorems 1–5 and Corollaries 1–4 we can deduce approximation properties of operators $S_n$, $V_n$, $P_n$ and $\widetilde{L}_n$ and their modifications for functions $f \in C_p$ and $f \in C^r$, correspondingly.

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