Comment.Math.Univ.Carolin. 50,1 (2009)113–124

Addition theorems, D-spaces and dually discrete spaces

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Abstract. A *neighbourhood assignment* in a space X is a family $\mathcal{O} = \{O_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for any $x \in X$. A set $Y \subseteq X$ is *a kernel of* $\mathcal{O}$ if $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\} = X$. If every neighbourhood assignment in X has a closed and discrete (respectively, discrete) kernel, then X is said to be a D-space (respectively a dually discrete space). In this paper we show among other things that every GO-space is dually discrete, every subparacompact scattered space and every continuous image of a Lindelöf P-space is a D-space and we prove an addition theorem for metalindelöf spaces which answers a question of Arhangel'skii and Buzyakova.

Keywords: neighbourhood assignment, D-space, dually discrete space, discrete kernel, scattered space, paracompactness, GO-space

Classification: Primary 54D20; Secondary 54G99

1. Introduction

A *neighbourhood assignment* in a space X is a family $\mathcal{O} = \{O_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for any $x \in X$. A set $Y \subseteq X$ is *a kernel of* $\mathcal{O}$ if $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\} = X$.

For any class (or property) $\mathcal{P}$ we define a dual class $\mathcal{P}^d$ which consists of spaces X such that, for any neighbourhood assignment $\mathcal{O}$ in the space X there exists a subspace $Y \subseteq X$ such that $\mathcal{O}(Y) = X$ and $Y \in \mathcal{P}$; the spaces from $\mathcal{P}^d$ are called *dually* $\mathcal{P}$. Thus a space is *dually discrete* if every neighbourhood assignment in X has a discrete kernel and is a *D-space* if it has a closed and discrete kernel. It is an immediate consequence of the definition, that if X is dually discrete, then $L(X) \le s(X)$ (where $L(X)$ is the *Lindelöf number* of X and $s(X)$ is the *spread* of X; definitions can be found in [12]).

The concept of a D-space was introduced in [9] and has attracted a great deal of attention recently (see for example [4], [5] and [11]). Possibly the first mention of dually discrete spaces can be found in [16] and their study was continued in [3] and [7] and most recently [1]. On consulting these papers it is immediately obvious that the class of dually discrete spaces is "very large" — in some sense it is difficult to construct spaces which are not dually discrete. However, in [7],

Research supported by Programa Integral de Fortalecimiento Institucional (PIFI), grant no. 34536-55 (México) and Fundação de Amparo a Pesquisa do Estado de São Paulo (Brasil).

examples of (Hausdorff, some even Tychonoff) spaces which are not dually discrete were constructed in ZFC but all the known examples depend on the existence of spaces X in which $hd(X) < hL(X)$ (where $hd(X)$ denotes the *hereditary density* of X and $hL(X)$ the *hereditary Lindelöf number* of X).

All spaces are assumed to be T_1 and all undefined notation and terminology is taken from [12].

2. Addition theorems

In this section we consider the conditions under which the properties of being a D-space, being dually discrete and being metalindelöf are preserved under finite unions. The main result of this section (Theorem 2.11) answers a question posed in [5].

Theorem 2.1. *If (X,τ) is a T_1-space and $F \subseteq X$ is the union of a σ-locally finite family of closed (in X) D-subspaces (respectively, dually discrete subspaces), then $(F, \tau|F)$ is a D-space (respectively, a dually discrete space).*

Proof: We prove the theorem for D-subspaces, the proof for dually discrete subspaces is virtually identical. So, assume that $F = \bigcup\{\bigcup \mathcal{F}_n : n \in \omega\}$, where each $\mathcal{F}_n$ is a locally finite family of closed (in X), D-subspaces (in the relative topology) and $\mathcal{O} = \{O_x : x \in F\}$ is a neighbourhood assignment in F. Note first that for each $n \in \omega$, $C_n = \bigcup \mathcal{F}_n$ is a D-space since for each $C \in \mathcal{F}_n$ we can choose a closed and discrete set $D_C \subseteq C$ such that $\mathcal{O}(D_C) \supseteq C$. It is immediate that $\bigcup\{D_C : C \in \mathcal{F}_n\}$ is a closed discrete kernel of $\mathcal{O}$.

To complete the proof it is clearly sufficient to prove that a countable union of closed D-subspaces is a D-space. To this end, suppose that $F = \bigcup\{C_n : n \in \omega\}$, where each set C_n is a closed D-subspace of X and $\{O_x : x \in F\}$ is a neighbourhood assignment in F; then since C_0 is a D-space, it follows that there is some closed and discrete set $D_0 \subseteq C_0$ such that $\bigcup\{O_x : x \in D_0\} \supseteq C_0$.

Having chosen closed discrete sets $\{D_0, D_1, \ldots, D_{n-1}\}$ so that

$$D_k \subseteq C_k \setminus \bigcup\{O_x : x \in \bigcup\{D_j : 0 \leq j < k\}\} \subseteq \bigcup\{O_x : x \in D_k\}$$

for each $k \leq n-1$, it follows that $C_n \setminus \bigcup\{O_x : x \in \bigcup\{D_j : 0 \leq j \leq n-1\}\}$ is a closed subset of C_n and hence is a D-space. Thus we can choose a closed discrete subset $D_n \subseteq X$ such that

$$D_n \subseteq C_n \setminus \bigcup\{O_x : x \in \bigcup\{D_j : 0 \leq j < n\}\} \subseteq \bigcup\{O_x : x \in D_n\}.$$

Let $D = \bigcup\{D_k : k \in \omega\}$; it is clear that $\bigcup\{O_x : x \in D\} \supseteq F$ and we claim that D is closed and discrete in F. To see this, suppose that $z \in F$ and let $m \in \omega$ be the minimal integer such that $z \in \mathcal{O}(D_m)$. Clearly $z \notin \mathrm{cl}(\bigcup\{D_k : 1 \leq k \leq m-1\})$, and since $z \in \mathcal{O}(D_m)$ and $\mathcal{O}(D_m) \cap D_k = \emptyset$ for each $k > m$, it follows from the fact that D_m is closed and discrete that z is not an accumulation point of D. $\square$

Corollary 2.2. *If F is an F_σ-set in a D-space (respectively, a dually discrete space) (X,τ), then $(F,\tau|F)$ is a D-space (respectively, a dually discrete space).*

Corollary 2.3. *The product of a σ-compact space and a dually discrete space is dually discrete.*

PROOF: It is an immediate consequence of Theorem 2.7 of [7] that the product of a compact T_1-space and a dually discrete T_1-space is dually discrete. The result now follows from Theorem 2.1. □

Theorem 2.4. *If a space X is the union of two dually discrete subspaces Y and Z where Z is closed in X, then X is dually discrete.*

PROOF: Let $\mathcal{O} = \{O_x : x \in X\}$ be a neighbourhood assignment in X. Then $\mathcal{O}_Z = \{O_x \cap Z : x \in Z\}$ is a neighbourhood assignment in Z and hence has a discrete kernel, D_Z. Now $W = Y \setminus \bigcup\{O_x : x \in D_Z\}$ is a closed subspace of the dually discrete space Y and hence is dually discrete. Thus the neighbourhood assignment in W, $\mathcal{O}_W = \{O_x \cap W : x \in W\}$ has a discrete kernel D_Y, say and it is straightforward to check that $D_Y \cup D_Z$ is a discrete kernel of $\mathcal{O}$. □

Corollary 2.5. *If a space X is the finite union of dually discrete spaces $\{Z_1, \dots, Z_n\}$ where, for each $1 \le j \le n-1$, the subspace Z_j is closed, then X is dually discrete.*

We say that a topological space is *adequate* if every closed subspace with countable extent is Lindelöf. It is easy to see that a D-space is adequate.

Theorem 2.6. *Let $X = Y \cup Z$ be a space of countable extent. If Y is adequate and Z is a D-space, then X is linearly Lindelöf.*

PROOF: Suppose to the contrary that X is not linearly Lindelöf; then there is some strictly increasing open cover $\{U_\alpha : \alpha \in \kappa\}$ of uncountable regular cardinality which has no countable subcover. Define $f : X \to \kappa$ by $f(x) = \min\{\alpha \in \kappa : x \in U_\alpha\}$ and a neighbourhood assignment $\mathcal{O}$ by $O_x = U_{f(x)}$.

Since Z is a D-space, there is some closed (in Z) discrete set $D \subseteq Z$ such that

$$\bigcup\{O_x : x \in D\} \supseteq Z.$$

Now $F = \mathrm{cl}_X(D)\setminus D$ is a (possibly empty) closed subset of X which is contained in Y. It follows that F has countable extent and since X is adequate, F is Lindelöf. Thus there is a countable set $S \subseteq X$ such that $F \subseteq \bigcup\{O_x : x \in S\}$; now $D \setminus \bigcup\{O_x : x \in S\}$ is closed and discrete in X, hence is countable, and so there is a countable set $T \subseteq X$ such that $\mathrm{cl}_X(D) \subseteq \bigcup\{O_t : t \in T\}$. Let $\gamma = \sup\{f(t) : t \in T\} < \kappa$ and $z \in Z$; then there is $d \in D$ such that $z \in O_d$ and $t \in T$ such that $d \in O_t$. Hence $f(d) \le f(t) \le \gamma$ and $z \in U_{f(d)}$.

The set $X \setminus U_\gamma$ is closed in X, is contained in Y and has countable extent, so again, since Y is adequate, $X\setminus U_\gamma$ is Lindelöf; thus there is a countable $Q \subseteq X$ such that $X \setminus U_\gamma \subseteq \bigcup\{O_q : q \in Q\}$. Let $\delta = \sup\{f(q) : q \in Q\}$ and $\eta = \max\{\gamma, \delta\} + 1$. Since κ has uncountable cofinality, we have $\eta < \kappa$, but $X = \bigcup\{U_\alpha : \alpha < \eta\} \subseteq U_\eta$, a contradiction. □

Recall that a space X is *metalindelöf* if every open cover of X has a point-countable open refinement.

The following lemma and its corollaries, each having easy proofs, are part of the folklore.

Lemma 2.7. *For each open cover $\mathcal{U}$ of a topological space X, there is a closed discrete set $D \subseteq X$ such that $\bigcup\{\mathrm{St}(d,\mathcal{U}) : d \in D\} = X$.*

Corollary 2.8. *If X is a metalindelöf space then $L(X) = e(X)$.*

Corollary 2.9. *A metalindelöf space of countable extent is Lindelöf, hence linearly Lindelöf.*

Recall that a cover $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ is a *shrinking* of a cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ if $V_\alpha \subseteq U_\alpha$ for all $\alpha \in I$ ($V_\alpha = \emptyset$ is not excluded).

In [14], Gruenhage proved that if a space X has countable extent and is a finite union of D-spaces, then it is linearly Lindelöf. Below we prove a analogous theorem, involving a finite union of metalindelöf subspaces, which allows us to answer a question of Arhangel'skii and Buzyakova. First we need a simple lemma.

Lemma 2.10. *If an open cover of a space X has a point-countable open refinement, then it has a point-countable open shrinking.*

Proof: Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of X and $\mathcal{C}$ a point-countable open refinement of $\mathcal{U}$. For each $C \in \mathcal{C}$, choose $\alpha(C) \in I$ so that $C \subseteq U_{\alpha(C)}$ and define

$$W_\alpha = \bigcup\{C \in \mathcal{C} : \alpha(C) = \alpha\}.$$

Clearly $W_\alpha \subseteq U_\alpha$ for each $\alpha \in I$ and $\bigcup\{W_\alpha : \alpha \in I\} = X$; hence to complete the proof we must show that $\mathcal{W} = \{W_\alpha : \alpha \in I\}$ is a point-countable family. To this end, we fix $x \in X$ and enumerate the countable set $\{C \in \mathcal{C} : x \in C\}$ as $\{C_n : n \in \omega\}$. It is then clear that $x \in W_\beta$ if and only if $\beta \in \{\alpha(C_n) : n \in \omega\}$, which completes the proof. □

Theorem 2.11. *If a space X of countable extent is the finite union of metalindelöf spaces, then it is linearly Lindelöf.*

Proof: Suppose that X is a space of countable extent which is a finite union of metalindelöf subspaces. The proof is by induction on the number n of such subspaces. It follows from Corollary 2.9 that the theorem is true if $n = 1$. So suppose that the theorem is true for any union of n metalindelöf subspaces and assume that $X = \bigcup\{X_k : 1 \leq k \leq n+1\}$ where each subspace X_k is metalindelöf.

We suppose to the contrary that X is not linearly Lindelöf; then there is some uncountable regular cardinal κ and a strictly increasing open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ which has no countable subcover. Without loss of generality we may assume that the open cover $\mathcal{V} = \{U_\alpha \cap X_{n+1} : \alpha \in \kappa\}$ of X_{n+1} has no countable subcover. Since X_{n+1} is metalindelöf, it follows from Lemma 2.10 that the open cover $\mathcal{V}$ of X_{n+1} has a point-countable open (in X_{n+1}) shrinking $\{W_\alpha : \alpha < \kappa\}$. For each $\alpha \in \kappa$ we may then find open sets Y_α in X such that $Y_\alpha \cap X_{n+1} = W_\alpha$ and $Y_\alpha \subseteq U_\alpha$; let $Y = \bigcup\{Y_\alpha : \alpha \in \kappa\}$. Then Y is an open subset of X which contains X_{n+1} and so $X \setminus Y = \bigcup\{X_k \setminus Y : 1 \leq k \leq n\}$ is a closed subspace of a space of countable extent which is the union of at most n metalindelöf subspaces and hence by the induction hypothesis it is linearly Lindelöf. Now $\{U_\alpha \cap (X \setminus Y) : \alpha \in \kappa\}$ is a strictly increasing open cover of $X \setminus Y$ and since κ is regular and uncountable, for some $\lambda < \kappa$, $U_\lambda \supseteq X \setminus Y$.

We now consider the open cover $\mathcal{F} = \{U_\lambda\} \cup \{Y_\alpha : \alpha \in \kappa\}$ of X. Fix $x_0 \in X_{n+1}$; since each point of X_{n+1} is contained in at most countably many sets Y_α, $\mathcal{V}$ has no countable subcover and $Y_\alpha \subseteq U_\alpha$ for each $\alpha \in \kappa$, it follows that $\mathrm{St}(x_0, \mathcal{F}) \not\supseteq X_{n+1}$ and we may find $x_1 \in X_{n+1} \setminus \mathrm{St}(x_0, \mathcal{F})$. Now suppose for some $\alpha < \omega_1 \leq \kappa$ and for each $\beta < \alpha$ we have chosen $x_\beta \in X_{n+1} \setminus \bigcup\{\mathrm{St}(x_\gamma, \mathcal{F}) : \gamma < \beta\}$, then since $\{F \in \mathcal{F} : x_\gamma \in F \text{ for some } \gamma < \alpha\}$ is countable, it follows that $X_{n+1} \setminus \bigcup\{\mathrm{St}(x_\gamma, \mathcal{F}) : \gamma < \alpha\} \neq \emptyset$ and we may choose $x_\alpha \in X_{n+1} \setminus \bigcup\{\mathrm{St}(x_\gamma, \mathcal{F}) : \gamma < \alpha\}$. Thus we construct a closed (in X_{n+1}) discrete subset $D = \{x_\alpha : \alpha \in \omega_1\}$ of X_{n+1} with the property that no countable subcollection of $\mathcal{F}$ covers D. Since X has countable extent, D cannot be closed in X and so the set $\mathrm{cl}_X(D) \setminus D$ is a closed non-empty subspace of $\bigcup\{X_k : 1 \leq k \leq n\}$ which by the induction hypothesis must be linearly Lindelöf. Thus there is a countable subset $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathrm{cl}_X(D) \setminus D \subseteq \bigcup \mathcal{G} = U$. Now $D \setminus U$ is a closed and discrete subset of X and hence is countable. But then, $D \subseteq \mathrm{cl}_X(D)$ is contained in a countable subcollection of $\mathcal{F}$, which is a contradiction; thus X is linearly Lindelöf. □

The next result gives a positive answer to Question 21 of [5].

Corollary 2.12. *If X has countable extent and is the union of finitely many paracompact subspaces, then X is linearly Lindelöf.*

Proof: A paracompact space is metalindelöf. □

3. Scattered spaces

Recall that a T_1-space is *scattered* if every non-empty subspace has an isolated point. Given a scattered T_1-space X, for each ordinal number γ, the *γ-th derived set of X*, X_γ, is defined recursively as follows: $X_0 = X$, $X_{\gamma+1}$ is the derived set of X_γ, and if γ is limit then $X_\gamma = \bigcap\{X_\beta : \beta < \gamma\}$. The minimal ordinal μ such that $X_\mu = \emptyset$ is called the *Cantor-Bendixson height of X* (or more simply in the sequel, *the height of* X) and will be denoted by $\mathrm{ht}(X)$. The family of subspaces $\{X_\gamma : \gamma < \mathrm{ht}(X)\}$ is called the *Cantor-Bendixson decomposition of X*.

It is known from [9] that every left-separated T_1-space is a D-space. Since every scattered space of finite height is left-separated, the following result is immediate (and a direct proof is an easy exercise).

Theorem 3.1. *Each scattered space of finite height is a D-space.*

Corollary 3.2. *The product of a dually discrete space and a scattered space of finite height is dually discrete.*

Proof: Suppose that Y is dually discrete and X is a scattered space of height $m \in \omega$. If $m = 1$, then $X \times Y$ is the topological union of dually discrete spaces and hence is dually discrete. The proof proceeds by induction on the height m of X. If the result is true for each scattered space X of height $m - 1$, then we write $X = (X \setminus X_1) \cup X_1$. The set $X \setminus X_1$ is discrete and X_1 is a scattered space of height $m - 1$. Thus $X \times Y$ is the union of two dually discrete subspaces, one of which, X_1, is closed, and the result follows from Theorem 2.4. □

As is well-known, the space ω_1 with its order topology is not a D-space and so not every scattered T_1-space is a D-space. Our next result gives a large class of scattered spaces which are D-spaces.

Recall that a space is *subparacompact* if every open cover has a closed σ-discrete refinement (we do not assume any separation axiom stronger than T_1). It is well known that every paracompact Hausdorff space is subparacompact.

Theorem 3.3. *A subparacompact scattered space is a D-space.*

Proof: Assume that X is a non-empty subparacompact scattered space; if $\mathrm{ht}(X) = 1$, then X being discrete, is a D-space. Proceeding inductively assume that α is an ordinal and that any subparacompact space Y with $\mathrm{ht}(Y) < \alpha$ is a D-space. Now suppose that a space X has height α and let $\{X_\beta : \beta < \alpha\}$ be the Cantor-Bendixson decomposition of X. Take an arbitrary neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X.

If α is a successor then $\alpha = \beta + 1$ and X_β is a closed discrete subspace of X; let $U = \mathcal{O}(X_\beta)$. The set $F = X \setminus U$ is closed in X and it follows from $F \cap X_\beta = \emptyset$ that $\mathrm{ht}(F) < \alpha$ and hence F is a D-space by the induction hypothesis. Choose a closed discrete set $D \subseteq F$ such that $\mathcal{O}(D) \supseteq F$. It is evident that $D \cup X_\beta$ is a closed discrete kernel of $\mathcal{O}$ so X is a D-space.

Next assume that α is a limit ordinal and hence $\bigcap\{X_\beta : \beta < \alpha\} = \emptyset$. For any point $x \in X$ there exists $\beta < \alpha$ such that $x \notin X_\beta$; we can find an open neighbourhood U_x of the point x such that $U_x \cap X_\beta = \emptyset$ and hence the height of the space U_x is strictly less than α. Since X is subparacompact, there exists a σ-discrete closed refinement of the cover $\{U_x : x \in X\}$ which we denote by $\mathcal{K} = \bigcup\{\mathcal{K}_n : n \in \omega\}$, where for each $n \in \omega$, $\mathcal{K}_n$ is a discrete family of closed sets. It is clear that for each $n \in \omega$ and each $K \in \mathcal{K}_n$, the height of the subspace K is strictly less than α so the induction hypothesis implies that K is a D-space. It remains only to apply Theorem 2.1 to conclude that X is a D-space. □

Corollary 3.4. *Each regular Lindelöf scattered space is a D-space.*

Recall that F. Galvin [14] and R. Telgársky [17] introduced *the point-open game* $\mathcal{PO}$ in which at the n-th move the first player I picks a point $x_n \in X$ while the second player II replies by choosing an open set $U_n \subseteq X$ with $x_n \in U_n$. The game is finished after ω moves and I is deemed to be the winner if $\bigcup\{U_n : n \in \omega\} = X$; otherwise player II wins the game $\{(x_n, U_n) : n \in \omega\}$. A space X is called *I-favorable* (*II-favorable*) for the point-open game if the first (second) player has a winning strategy on X.

It is easy to see that any space which fails to be Lindelöf, is II-favorable for the point-open game. Therefore every space which is not II-favorable (in particular each I-favorable space) is Lindelöf.

The class of (regular) spaces which are I-favorable or II-favorable for the point-open game has received a lot of attention recently. Telgársky proved in [17] that a regular Lindelöf scattered space is I-favorable for the point-open game and it is easy to see that not every I-favorable space is scattered. Therefore the following result is stronger than Corollary 3.4.

Theorem 3.5. *If a regular space X is not II-favorable for the point-open game then X is a D-space. In particular, any I-favorable space is a D-space.*

Proof: Given a neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X define a strategy σ of the second player as follows: if x_0 is the first move of I then let $U_0 = \sigma(x_0) = O_{x_0}$. Assume that $n \in \omega$ and moves $x_0, U_0, \dots, x_n, U_n$ have been made in the point-open game on X. If I selects x_{n+1} for his move $(n+1)$ then let $\sigma(x_0, \dots, x_n, x_{n+1}) = U_0 \cup \ldots \cup U_n$ if $x_{n+1} \in U_0 \cup \ldots \cup U_n$; if not, then let $\sigma(x_0, \dots, x_n, x_{n+1}) = O_{x_{n+1}}$.

By our assumption the strategy σ is not winning for the second player so there is a play $\{x_i, U_i : i \in \omega\}$ on the space X in which II applies the strategy σ and loses, that is, $\bigcup_{n\in\omega} U_n = X$. Let $A = \{n \in \omega : x_{n+1} \in U_0 \cup \ldots \cup U_n\}$ and enumerate the set $\omega \setminus A$ as $\{n_i : i < \alpha\}$ for some ordinal $\alpha \leq \omega$ in such a way that $i < j$ implies $n_i < n_j$. It takes a trivial induction to see that $U_{n_i} = O_{x_{n_i}}$ and $x_{n_{i+1}} \notin O_{x_{n_0}} \cup \ldots \cup O_{x_{n_i}}$ for any $i < \alpha$ while $\bigcup_{n\in\omega} U_n = \bigcup_{i\in\omega} O_{x_{n_i}} = X$. It is immediate that $D = \{x_{n_i} : i < \alpha\}$ is a closed discrete kernel of $\mathcal{O}$ so X is a D-space as promised. □

Corollary 3.6. *Every continuous image of a regular Lindelöf P-space is a D-space.*

Proof: It is well-known (and easy to prove) that the property of not being II-favorable for the first player in the point-open game is preserved by continuous images. Since each Lindelöf P-space is not II-favorable for the point-open game (see Theorem 6.10 of [18]), Theorem 3.5 applies. □

Corollary 3.7. *Every continuous image of a regular Lindelöf scattered space is a D-space.*

Proof: If X is a Lindelöf scattered space then let Y be the set X with the topology generated by all G_δ-subsets of X. It is clear that X is a continuous image of Y and Y is a P-space. By Proposition 1 of [19], the space Y is also Lindelöf[1], and so every continuous image of X is a continuous image of a Lindelöf P-space; Corollary 3.6 now completes the proof. □

Question 3.8. *Is every metacompact scattered Hausdorff space dually discrete (or even a D-space)?*

Recall that a *submaximal space* (respectively, *nodec space*) is a dense-in-itself space in which every dense set is open (respectively, every nowhere dense set is closed); again we assume no separation axiom beyond T_1. Clearly a submaximal space is nodec. From Corollary 3.4 of [2], under $V = L$, every submaximal Hausdorff space is strongly σ-discrete and hence from Theorem 2.1 every Hausdorff submaximal space is dually discrete. In fact an even stronger result is true in ZFC.

Theorem 3.9. *Every nodec space is a D-space.*

Proof: Suppose that X is a nodec space and $\mathcal{O} = \{O_x : x \in X\}$ is a neighbourhood assignment in X. It was proved in Proposition 2.1 of [7] that every space is dually scattered so we can find a scattered kernel $F \subseteq X$ for the assignment $\mathcal{O}$. However, every scattered subspace of a dense-in-itself space is nowhere dense. Since X is nodec, F is a closed and discrete kernel of $\mathcal{O}$. □

The space Γ of [10] is a locally compact, scattered Hausdorff space of height ω, which is not a D-space and so we are led to ask:

Question 3.10. *Is Γ dually discrete? More generally, is every scattered Hausdorff space (or even T_1-space) of countable height, dually discrete?*

A related result is the following:

Theorem 3.11. *A countably compact, scattered T_1-space of countable height is compact.*

We omit the simple proof which is by induction on the scattering height.

4. Dual discreteness of generalized ordered spaces

Let $(X, \tau, <)$ be a GO-space and C its Dedekind compactification, that is to say, the minimal ordered compactification of X. By the term *left pseudogap* of X,

[1]The referee has pointed out to us that this result was known to Paul R. Meyer in 1966, but was apparently never published.

we mean a pair (A, B) of open subsets of X such that $a < b$ for all $a \in A$ and $b \in B$, $A \cup B = X$ and A has no maximum element. A *right pseudogap* is defined analogously. The pair (A, B) is called a gap of X if it is both a right and a left pseudogap. If $(\emptyset, X)$ (respectively, $(X, \emptyset)$) is a gap then it is called the *left end gap* (respectively, *right end gap*) of X.

Recall that a left pseudogap (A, B) of X is a *left Q-pseudogap* if for some regular cardinal κ there is a strictly increasing transfinite sequence $\{d_\alpha : \alpha < \kappa\}$ in A which is closed and discrete as a subspace of X and cofinal in A, that is to say, $\sup_C(A) = \sup_C(D)$. *Right Q-pseudogaps* are defined analogously. For simplicity, we say that a left (respectively, right) pseudogap which is not a left Q-pseudogap (respectively, not a right Q-pseudogap) is a *left* (respectively, *right*) *N-pseudogap*.

We define an ordered compactification K of X as follows: For each non-end gap (A, B) of X, add two points a^*, b^* such that $a < a^* < b^* < b$ for all $a \in A$ and $b \in B$ and for each left pseudogap (A, B) which is not a gap (respectively, right pseudogap (C, D) which is not a gap) add a point p_A (respectively, p_D) such that $a < p_A < b$ for all $a \in A$ and $b \in B$ (such that $c < p_D < d$ for all $c \in C$ and $d \in D$). Also add a minimal point m if X has a left end gap and a maximal point M if X has a right end gap. In the sequel, we identify the points $m, M, a^*, b^*, p_A, p_D \in K$ with the left and/or right pseudogaps of X. In [15], Lutzer showed that a GO-space is paracompact if and only if each of its pseudogaps is a Q-pseudogap.

We denote the set of left (respectively, right) Q-pseudogaps of X (considered as subsets of K) by L_Q (respectively R_Q) and the set of left (respectively, right) N-pseudogaps by L_N (respectively R_N).

It was shown in [8] that a GO-space is a D-space if and only if it is paracompact and in [7] that a GO-space of countable extent is dually discrete. It turns out that the requirement of countable extent can be omitted; the following theorem answers Problems 4.1 and 4.2 from [7].

Theorem 4.1. *Each GO-space is dually discrete.*

Proof: Suppose that X is a GO-space and K is the ordered compactification of X as defined in the preceding paragraphs. We consider the subspace $Y \subseteq K$ defined by $Y = X \cup L_N \cup R_N$. We first show that every pseudogap of Y is a Q-pseudogap and hence by Theorem E of [15], Y is paracompact. To this end, suppose that $p \in K \setminus Y$ is a pseudogap of Y and hence is a Q-pseudogap of X; we assume without loss of generality that p is a left Q-pseudogap of X. Then for some regular cardinal κ, there is a closed (in X) and discrete, strictly increasing transfinite sequence $D = \{d_\alpha : \alpha < \kappa\} \subseteq (\leftarrow, p)_K \cap X$, such that $p = \sup_K(D)$. Since D is closed in X, it follows that for each limit ordinal $\lambda < \kappa$, $q_\lambda = \sup_K\{d_\alpha : \alpha < \lambda\} \notin X$ and hence is a pseudogap of X; furthermore, q_λ is a Q-pseudogap of X since $\{d_\alpha : \alpha < \lambda\}$ is a strictly increasing transfinite sequence which is closed and discrete in X and hence $q_\lambda = \sup_K\{d_\alpha : \alpha < \lambda\} \in K\backslash Y$. Thus $\{d_\alpha : \alpha < \kappa\}$ is also closed and discrete in Y, showing that p is a Q-pseudogap of Y, completing the proof that Y is paracompact.

Let $\mathcal{O} = \{O_x : x \in X\}$ be an arbitrary neighbourhood assignment in X where, without loss of generality, we assume that each set O_x is convex. We will extend the family $\mathcal{O}$ to a neighbourhood assignment in Y. To this end, suppose that $y \in Y\backslash X$; the point y corresponds to an N-pseudogap of X and again without loss of generality we assume that y is a left N-pseudogap and hence $y \notin \mathrm{cl}_K((y, \to)_K)$.

We claim that there is a point $a_y \in (\leftarrow, y)_X$ and a discrete cofinal subset $D_y \subseteq (\leftarrow, y)_X$ such that $(a_y, z] \subseteq O_z$ for all $z \in D_y$. For if to the contrary, no such a_y and D_y exist then, since each member of $\mathcal{O}$ is convex, for any $x \in (\leftarrow, y)_X$ there is a point $b \in (x, y)_X$ such that $(x, z) \nsubseteq O_z$ (that is $O_z \subseteq (x, \to)$) for each $z \in (b, y)_X$.

Now, since y is a left N-pseudogap of X, $\chi(y, (\leftarrow, y)_X \cup \{y\})) > \omega$ and hence no countable set is cofinal in $(\leftarrow, y)_X$; thus for some cardinal κ we can construct recursively a strictly increasing transfinite sequence $B = \{b_\alpha : \alpha < \kappa\} \subseteq (\leftarrow, y)_X$ such that $O_z \subseteq (b_\alpha, \to)_X$ for each $\alpha < \kappa$ for any $z \in (b_\beta, y)_X$. Now since y is a left N-pseudogap of X, there is no strictly increasing, transfinite sequence which is closed and discrete subset of $(\leftarrow, y)_K \cap X$ whose supremum in K is y. Thus the set B must have a cofinal set of cluster points B^d in $(\leftarrow, y)_K \cap X$. Now if $x \in B^d$, then since B is a strictly increasing sequence, $x \in \mathrm{cl}_X(\to, x)_X$ and hence there are $\alpha < \beta < \kappa$ such that $\{b_\alpha, b_\beta\} \subseteq O_x$. However, by the recursive hypothesis, $O_x \subseteq (b_\alpha, \to)_X$, which is a contradiction.

Analogously, if the point y is a right N-pseudogap, then we can choose a discrete subspace $E_y \subseteq (y, \to)_X$ and $b_y \in (y, \to)_X$ such that y is the infimum of E_y and $[x, b_y) \subseteq O_x$ for each $x \in E_y$.

The proof now proceeds exactly as in Theorem 2.23 of [7] using the fact that Y is paracompact and hence is a D-space (see [8]). □

5. Open problems

The problem of whether the union of two D-spaces is a D-space has been posed previously. Neither is it known whether the union of two dually discrete spaces is dually discrete. (If one of the subspaces is closed, then a positive answer is provided by Theorem 2.4.)

Problem 5.1. *Suppose that $X = X_0 \cup X_1$ and X_i is dually discrete for $i = 0, 1$. Must X be dually discrete? What happens if both sets X_0 and X_1 are dense in X?*

If X is a Lindelöf P-space then any countable subset of X is closed and discrete; this clearly implies that X is a D-space. The following problems involving continuous images of Lindelöf spaces show how little is known of this topic and point to possible future lines of research.

Problem 5.2. *Is any continuous image of a Lindelöf GO-space, dually discrete? Must it be a D-space?*

Problem 5.3. *Is any continuous image of a Lindelöf LOTS, dually discrete? Must it be a D-space?*

Problem 5.4. *Suppose that X is a Lindelöf space such that every second countable continuous image of X is countable. Must X be dually discrete? Must it be a D-space?*

Problem 5.5. *Is it true that every Lindelöf space is a continuous image of a Lindelöf GO-space?*

Problem 5.6. *Is it true that every Lindelöf space is a continuous image of a Lindelöf LOTS?*

Problem 5.7. *Is it true that every compact space is a continuous image of a Lindelöf GO-space?*

Acknowledgment. We wish to thank the referee for a careful reading of the original manuscript and for suggesting a number of improvements which have been incorporated in the text.

References

[1] Alas O.T., Junqueira L., Wilson R.G., *Dually discrete spaces*, Topology Appl. **155** (2008), no. 13, 1420–1425.

[2] Alas O.T., Sanchis M., Tkachenko M.G., Tkachuk V.V., Wilson R.G., *Irresolvable and submaximal spaces: homogeneity versus σ-discreteness and new ZFC examples*, Topology Appl. **107** (2000), 259–273.

[3] Alas O.T., Tkachuk V.V., Wilson R.G., *Covering properties and neighbourhood assignments*, Topology Proc. **30** (2006), no. 1, 25–38.

[4] Arhangel'skii A.V., *D-spaces and finite unions*, Proc. Amer. Math. Soc. **132** (2004), no. 7, 2163–2170.

[5] Arhangel'skii A.V., Buzyakova R.Z., *Addition theorems and D-spaces*, Comment. Math. Univ. Carolin. **43** (2002), 653–663.

[6] Arhangel'skii A.V., Collins P.J., *On submaximal spaces*, Topology Appl. **64** (1995), no. 3, 219–241.

[7] Buzyakova R.Z., Tkachuk V.V., Wilson R.G., *A quest for nice kernels of neighbourhood assignments*, Comment. Math. Univ. Carolin. **48** (2007), no. 4, 689–697.

[8] van Douwen E.K., Lutzer D., *A note on paracompactness in generalized ordered spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 4, 1237–1245.

[9] van Douwen E., Pfeffer W.F., *Some properties of the Sorgenfrey line and other spaces*, Pacific J. Math. **81** (1979), no. 2, 371–377.

[10] van Douwen E., Wicke H.H., *A real, weird topology on the reals*, Houston J. Math. **3** (1977), no. 1, 141–152.

[11] Eisworth T., *On D-spaces*, in Open Problems in Topology, II, ed. Elliott Pearl, Elsevier, Amsterdam, 2007.

[12] Engelking R., *General Topology*, Heldermann, Berlin, 1989.

[13] Galvin F., *Indeterminacy of point-open games*, Bull. Acad. Polon. Sci., Sér. Math. **26** (1978), no. 5, 445–449.

[14] Gruenhage G., *A note on D-spaces*, Topology Appl. **153** (2006), 2229–2240.
[15] Lutzer D., *Ordinals and paracompactness in ordered spaces*, Proceedings Topo72, General Topology and its Applications, Pittsburgh International Conference, 1972, Lecture Notes in Mathematics **372**, Springer, Berlin, 1974.
[16] van Mill J., Tkachuk V.V., Wilson R.G., *Classes defined by stars and neighbourhood assignments*, Topology Appl. **154** (2007), 2127–2134.
[17] Telgársky R., *Spaces defined by topological games*, Fund. Math. **88** (1975), 193–223.
[18] Telgársky R., *Spaces defined by topological games II*, Fund. Math. **116** (1983), 189–207.
[19] Uspenskij V.V., *The frequency spectrum of function spaces* (in Russian), Vestnik. Moskov. Univ. Ser. I Mat. **37** (1982), no. 1, 31–35.

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(*Received* May 4, 2008, *revised* December 18, 2008)