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Linear inessential operators and generalized inverses

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Abstract. The space of inessential bounded linear operators from one Banach space X into another Y is introduced. This space, $I(X, Y)$, is a subspace of $B(X, Y)$ which generalizes Kleinecke's ideal of inessential operators. For certain subspaces W of $I(X, Y)$, it is shown that when $T \in B(X, Y)$ has a generalized inverse modulo W, then there exists a projection $P \in B(X)$ such that $T(I - P)$ has a generalized inverse and $TP \in W$.

Keywords: inessential operator, Fredholm operator, generalized inverse

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1. Introduction

In 1963, in his classic paper [K], D. Kleinecke introduced the ideal of inessential bounded linear operators on a Banach space X, denoted $I(X)$. Let $B(X)$ be the algebra of all bounded linear operators on X, and let $K(X)$ be the ideal of all compact operators on X. Let $\pi : B(X) \to B(X)/K(X)$ be the usual embedding map: $\pi(T) = T + K(X)$, $T \in B(X)$. Kleinecke defined $I(X) = \{T \in B(X) : \pi(T) \in \mathrm{rad}(B(X)/K(X))\}$ where $\mathrm{rad}(B(X)/K(X))$ is the Jacobson radical of the Calkin algebra. It is proved in [K] that if $T \in I(X)$ and $S \in \Phi(X)$ (the Fredholm operators), then $S + T \in \Phi(X)$ and $\mathrm{ind}(S + T) = \mathrm{ind}(S)$ [K, Theorem 6]. Set $\mathrm{Per}(\Phi(X)) = \{T \in B(X) : \text{for all } S \in \Phi(X),\ S + T \in \Phi(X)\}$. $\mathrm{Per}(\Phi(X))$ is called the *perturbation ideal* of $\Phi(X)$; see Sections 5.5 and 5.6 of [CPY] for an introduction to perturbation ideals and their properties. Kleinecke's original results show that $I(X) \subseteq \mathrm{Per}(\Phi(X))$. In fact, $\mathrm{Per}(\Phi(X)) = I(X)$ [CPY, Theorem (5.5.9), p. 98].

In the first section of this paper we introduce $I(X, Y)$, the space of all inessential bounded linear operators defined on a Banach space X with values in a Banach space Y. We prove that when $\Phi(X, Y)$ is nonempty, then $I(X, Y) = \mathrm{Per}(\Phi(X, Y))$.

Throughout, X, Y, and Z are Banach spaces, and $B(X, Y)$ denotes the space of all bounded linear operators defined on X with values in Y. For $T \in B(X, Y)$, the null space of T is denoted by $\mathbf{N}(T)$, and the range of T by $\mathbf{R}(T)$. If for an operator $T \in B(X, Y)$ there exists $G \in B(Y, X)$ such that $TGT - T = 0$, then G is called a *g-inverse* (generalized inverse) for T. An important fact when T has a g-inverse G as above, is that $\mathbf{R}(T)$ is closed and G acts as a bounded right inverse for T on $\mathbf{R}(T)$, that is, $T(Gy) = y$ for all $y \in \mathbf{R}(T)$; see [LT, p. 251]. Also in [LT], Theorem 12.9 gives the basic characterization concerning the existence of g-inverses (called pseudoinverses in [LT]).

The definition of a g-inverse is algebraic, so it extends naturally to elements of an algebra: When A is an algebra and $t \in A$, then $g \in A$ is a g-inverse of t if $tgt - t = 0$. The monograph [C] by S. Caradus is an excellent source for information concerning all aspects of the theory and practice of g-inverses of linear operators and the general algebraic properties of g-inverses. The existence of g-inverses in certain algebras of bounded linear operators, is studied in the author's paper [B]. All bounded linear operators which are Fredholm have g-inverses. This fact carries over to Fredholm theory in algebras of operators; see K. Jörgens' book [J].

In his paper [R], V. Rakočević proves that when $T \in B(X)$ and T has a g-inverse modulo $K(X)$, that is, there exists $G \in B(X)$ such that $TGT - T \in K(X)$, then there exists $J \in K(X)$ such that $T + J$ has a g-inverse in $B(X)$. In the last section of this paper we extend this result to certain subspaces of $I(X, Y)$.

2. Inessential operators

Definition 1. A linear operator $T \in B(X, Y)$ is *inessential* if for every $S \in B(Y, X)$, $ST \in I(X)$ and $TS \in I(Y)$. We denote the set of all inessential operators in $B(X, Y)$ by $I(X, Y)$.

Since $I(X)$ is an ideal in $B(X)$, for every $T \in I(X)$ and every $S \in B(X)$, ST and TS are both in $I(X)$. But also, if $ST \in I(X)$ for all $S \in B(X)$, then taking S to be the identity operator, we have $T \in I(X)$. This verifies that $I(X, X) = I(X)$.

Proposition 2. *The following are equivalent for an operator $T \in B(X, Y)$:*

(i) $T \in I(X, Y)$;
(ii) *for every* $S \in B(Y, X)$, $ST \in I(X)$;
(iii) *for every* $S \in B(Y, X)$, $TS \in I(Y)$.

Proof: We verify that (ii)$\Longrightarrow$(iii); a similar argument shows (iii)$\Longrightarrow$(ii). Assume that (ii) holds. Then for every $S \in B(Y, X)$, $\sigma(ST)$ is either a finite set or a sequence converging to zero. As is well known, $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$. It follows that every operator in the right ideal $T(B(Y, X))$ of $B(Y)$ has spectrum that is either a finite set or a sequence converging to zero. Then by [BMSW, Theorem R.2.6, p. 58], $T(B(Y, X)) \subseteq I(Y)$. This proves (iii). $\square$

Proposition 3. (i) *$I(X, Y)$ is a closed subspace of $B(X, Y)$.*
(ii) *If $T \in I(X, Y)$, $R \in B(Y, Z)$, then $RT \in I(X, Z)$.*
(iii) *If $T \in I(X, Y)$, $R \in B(Z, X)$, then $TR \in I(Z, X)$.*

Proof: Statement (i) is easily verified (using the fact that $I(X)$ is closed). We prove (ii); the proof of (iii) is similar.

Assume that $T \in I(X,Y)$ and $R \in B(Y,Z)$. Let S be arbitrary in $B(Z,X)$. Since $SR \in B(Y,X)$, $S(RT) = (SR)T \in I(X)$ by Definition 1. Then Proposition 2 implies that $RT \in I(X,Z)$. □

We use $\operatorname{def}(T)$ to denote the defect of $T \in B(X,Y)$. As is well known, when $\operatorname{def}(T) = \dim(Y/\mathbf{R}(T))$ is finite, then $\mathbf{R}(T)$ is closed [AA, Corollary 2.17].

Proposition 4. *If* $T \in I(X,Y)$ *and* $S \in \Phi(X,Y)$, *then* $T + S \in \Phi(X,Y)$.

Proof: There exists an operator $R \in B(Y,X)$ such that $RS = I - E$ and $SR = I - F$ where both E and F have f.d. range [AA, Theorem 4.46, p. 161]. Then by that same theorem, $R \in \Phi(Y,X)$. Note that $RS \in \Phi(X)$ and $SR \in \Phi(Y)$. Since $RT \in I(X)$ and $TR \in I(Y)$, we have that $R(T+S) = RT + RS \in \Phi(X)$ and $(T+S)R = TR + SR \in \Phi(Y)$. Then $\mathbf{R}((T+S)R) \subseteq \mathbf{R}(T+S)$ and $\operatorname{def}((T+S)R) < \infty$, it follows that $\operatorname{def}(T+S) < \infty$. Also, $\mathbf{N}(T+S) \subseteq \mathbf{N}(R(T+S))$ which is f.d. This proves that $T + S \in \Phi(X,Y)$. □

Notes. (1) If $V \in B(X)$ and $W \in \Phi(X,Y)$ with $WV \in \Phi(X,Y)$, then $V \in \Phi(X)$.

(For we can choose an operator $R \in \Phi(Y,X)$ such that $RW = I - E$ where E has f.d. range [AA, Theorem 4.46, p. 161]. Then $V - EV = RWV \in \Phi(X)$. It follows that $V \in \Phi(X)$.)

(2) Assume that $\Phi(X,Y)$ is nonempty. If $T \in \operatorname{Per}(\Phi(X,Y))$, $R \in B(X)$, and $S \in B(Y)$, then $STR \in \operatorname{Per}(\Phi(X,Y))$. (This follows from the proof of [CPY, Lemma (5.5.5), p. 96].)

Theorem 5. *Assume that* $\Phi(X,Y)$ *is nonempty. Then*

$$\begin{aligned} I(X,Y) &= \operatorname{Per}(\Phi(X,Y)) \\ &= \{T \in B(X,Y) : T + S \in \Phi(X,Y) \ \text{ for all } \ S \in \Phi(X,Y)\}. \end{aligned}$$

Proof: By Proposition 4, $I(X,Y) \subseteq \operatorname{Per}(\Phi(X,Y))$. Now we prove the reverse inclusion. Assume that $T \in \operatorname{Per}(\Phi(X,Y))$. Let $S \in B(Y,X)$ and $R \in \Phi(X)$. We show that $R + ST \in \Phi(X)$. Assume that $W \in \Phi(X,Y)$. By Note (2) above, $WST \in \operatorname{Per}(\Phi(X,Y))$. Since $WR \in \Phi(X,Y)$, $W(R+ST) = WR + WST \in \Phi(X,Y)$. Then by the Note (1) above, $R + ST \in \Phi(X)$. This proves that $ST \in \operatorname{Per}(\Phi(X)) = I(X)$. It follows from Proposition 2 that $T \in I(X,Y)$. □

Let $F(X)$ denote the space of all operators in $B(X)$ with f.d. (finite dimensional) range.

Proposition 6. *Assume that* $T \in I(X,Y)$ *and* $S \in \Phi(X,Y)$. *Then* $\operatorname{ind}(T+S) = \operatorname{ind}(S)$.

Proof: There exists an operator $R \in \Phi(Y,X)$ with $SR = I - E$ where $E \in F(Y)$. Note that $SR \in \Phi(Y)$ and $\operatorname{ind}(SR) = 0$. Now by definition $TR \in I(Y)$, so $\operatorname{ind}(TR + SR) = \operatorname{ind}(SR) = 0$. Also, $\operatorname{ind}(TR + SR) = \operatorname{ind}(T+S) + \operatorname{ind}(R) = \operatorname{ind}(T+S) - \operatorname{ind}(S)$. □

3. G-inverses modulo an ideal

In order to prove our result on g-inverses modulo certain subspaces of $I(X,Y)$, we need some preliminary results; some of these are of interest in there own right. The first two results are presented in the setting of a unital Banach algebra A. For $u \in A$, $\sigma(u;A)$ denotes the usual spectrum of u relative to A. For operators $T \in B(X)$, we use the notation $\sigma(T)$ for the usual operator spectrum of T relative to $B(X)$.

For $u \in A$, $\{u\}''$ is the second commutant of u in A, $\{u\}'' = \{a \in A :$ whenever $b \in A$ and $bu = ub$, then $ab = ba\}$.

We use the holomorphic functional calculus in this setting. In this regard, a cycle γ is a formal sum of closed piecewise continuously differentiable paths in $\mathbf{C}$; γ^* denotes the image of γ in $\mathbf{C}$. For $z \in \mathbf{C}\backslash\gamma^*$, $\mathrm{Ind}_\gamma(z)$ is the index of z with respect to γ.

Results similar to Theorem 7 are known. This particular version contains useful details.

Theorem 7. *Assume that $u \in A$ with $u^2 - u = r$. Also assume that Δ is a compact and relatively open subset of $\sigma(u;A)$ with $0 \notin \Delta$ and $1 \in \Delta$. Then there exists $e = e^2 \in \{u\}''$ and $h \in \{u\}''$ such that $rh = hr$ and $e = u + hr$.*

PROOF: First we show that when $(\lambda - u)^{-1}$ exists, $\lambda \neq 0$, $\lambda \neq 1$, then

$$(1) \qquad (\lambda - u)^{-1} = \left(\frac{1}{\lambda - 1}\right)u + \frac{1}{\lambda}(1-u) + \left(\frac{1}{\lambda(\lambda-1)}\right)(\lambda - u)^{-1}r.$$

For

$$\begin{aligned}
&(\lambda - u)\left[\left(\frac{1}{\lambda-1}\right)u + \frac{1}{\lambda}(1-u)\right] \\
&\quad = \left(\frac{\lambda}{\lambda-1}\right)u - \left(\frac{1}{\lambda-1}\right)u^2 + (1-u) - \frac{1}{\lambda}(u - u^2) \\
&\quad = \left(\frac{\lambda}{\lambda-1}\right)u - \left(\frac{1}{\lambda-1}\right)(u+r) + (1-u) + \frac{1}{\lambda}r \\
&\quad = \left(\frac{\lambda}{\lambda-1}\right)u - \left(\frac{1}{\lambda-1}\right)u + (1-u) + \left(\frac{1}{\lambda} - \left(\frac{1}{\lambda-1}\right)\right)r \\
&\quad = u + (1-u) - \left(\frac{1}{\lambda(\lambda-1)}\right)r = 1 - \left(\frac{1}{\lambda(\lambda-1)}\right)r.
\end{aligned}$$

Multiplying this equality through by $(\lambda - u)^{-1}$ verifies (1).

Now let V be an open set in $\mathbf{C}$ with $V \cap \sigma(u;A) = \Delta$ and $0 \notin V$. Let γ be a cycle with $\gamma^* \subseteq V\backslash\Delta$ such that $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in \Delta$ and $\mathrm{Ind}_\gamma(z) = 0$ for all $z \notin V$; note that in particular, $\mathrm{Ind}_\gamma(0) = 0$.

Let e be the spectral idempotent, $e = \frac{1}{2\pi i}\int_\gamma (\lambda - u)^{-1}\, d\lambda$. Using (1) we have,

$$\begin{aligned} e &= \left(\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda - 1}\, d\lambda\right)u + \left(\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}\, d\lambda\right)(1-u) \\ &\quad + \left(\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{\lambda(\lambda-1)}\right)(\lambda - u)^{-1}\, d\lambda\right)r \\ &= \operatorname{Ind}_\gamma(1)u + \operatorname{Ind}_\gamma(0)(1-u) + hr = u + hr, \end{aligned}$$

where $h = \frac{1}{2\pi i}\int_\gamma (\frac{1}{\lambda(\lambda-1)})(\lambda - u)^{-1}\, d\lambda$. □

Let $\operatorname{rad}(A)$ denote the Jacobson radical of the algebra A. We use the standard fact that for $u \in A$, $\sigma(u; A) = \sigma(u + \operatorname{rad}(A); A/\operatorname{rad}(A))$. Part (i) of Corollary 8 is a well known result from Banach algebra theory [P, Proposition 4.3.12]. Part (ii) shows that if $t + \operatorname{rad}(A)$ has a g-inverse in the quotient algebra $A/\operatorname{rad}(A)$, then for some $s \in \operatorname{rad}(A)$, $t + s$ has a g-inverse in A.

Corollary 8. *Let A be a unital Banach algebra.*

(i) *If $u \in A$, $u \notin \operatorname{rad}(A)$, with $u^2 - u \in \operatorname{rad}(A)$, then there exists $e = e^2 \in \{u\}''$ such that $e - u \in \operatorname{rad}(A)$.*
(ii) *If $t, g \in A$, $t \notin \operatorname{rad}(A)$, with $tgt - t \in \operatorname{rad}(A)$, then there exists $p = p^2 \in A$ such that $t(1-p)$ has a g-inverse in A and $tp \in \operatorname{rad}(A)$.*

Proof of (i): Let u be as in statement (i). Note that $1-u$ is not invertible since $u(1-u) \in \operatorname{rad}(A)$, but $u \notin \operatorname{rad}(A)$. Now $1 \in \sigma(u; A) \subseteq \{0, 1\}$. In Theorem 7 take $\Delta = \{1\}$. By Theorem 7, there exists $e = e^2 \in \{u\}''$ such that $e - u \in \operatorname{rad}(A)$. □

Proof of (ii): Assume that t and g are as in (ii), so $tgt - t = r \in \operatorname{rad}(A)$. Then $tgtg - tg = rg \in \operatorname{rad}(A)$. Note that $1 - tg$ is not invertible since $(tg-1)t \in \operatorname{rad}(A)$, but $t \notin \operatorname{rad}(A)$. Thus, $1 \in \sigma(tg; A) \subseteq \{0, 1\}$. In Theorem 7 take $\Delta = \{1\}$. Applying Theorem 7, with $u = tg$, there exist $h \in A$ and $e = e^2 \in A$ such that

$$e = tg + rgh = tg + (tgt - t)gh = t[g + (gt-1)gh].$$

Set $v = g + (gt-1)gh$ and $w = (gt-1)gh$. Note that $tw \in \operatorname{rad}(A)$. Therefore,

$$e = tv = t(g + w) \quad \text{with} \quad tw \in \operatorname{rad}(A). \tag{2}$$

Now set $s = r + twt \in \operatorname{rad}(A)$. Then $et = tgt + twt = t + r + twt = t + s$. By (2), $e = tv$, so $e = etv = tv + sv = e + sv$. It follows that $e(t+s) = t + s$ and $sv = 0$. Thus,

$$(t+s)v(t+s) = tv(t+s) = e(t+s) = t + s. \tag{3}$$

Now let $1-p = v(t+s)$. Then $1-p$ is a projection, and $t(1-p) = tv(t+s) = (t+s)v(t+s) = t+s$ which has a g-inverse by (3), and $tp = -s \in \mathrm{rad}(A)$. $\square$

As before, define $\pi : B(X) \to B(X)/K(X)$ by $\pi(T) = T + K(X)$. For $T \in B(X)$, the *Fredholm spectrum* of T, $\sigma_F(T)$, is defined as

$$\sigma_F(T) = \sigma(\pi(T); B(X)/K(X)).$$

We need a fairly deep property of the Fredholm spectrum:

> *Let Ω be the unbounded component of $\mathbf{C}\backslash\sigma_F(T)$. Then $\sigma(T) \cap \Omega$ is at most countable.*

One reference for this is [BMSW, Theorem R.2.7].

From the definition of $I(X)$ and properties of the Jacobson radical, it follows that for $T \in B(X)$, $\sigma_F(T) = \sigma(T + I(X); B(X)/I(X))$.

Corollary 9. *Let M be a left or right ideal of $B(X)$ with $M \subseteq I(X)$. If $U \in B(X)$, $U \notin I(X)$, with $U^2 - U = R \in M$, then there exists $E = E^2 \in \{U\}''$ such that $E = U + HR$ and $HR = RH$. Thus, $E - U \in M$.*

Proof: Assume that $U^2 - U = R \in M$. Since $U + I(X)$ is a nonzero idempotent in $B(X)/I(X)$, $1 \in \sigma_F(U) \subseteq \{0, 1\}$. It follows from the discussion above that $\sigma(U)$ is at most countable. Therefore, there does exist a compact and relatively open subset Δ of $\sigma(U)$ with $0 \notin \Delta$ and $1 \in \Delta$. Applying Theorem 7, there exists a projection $E \in \{U\}''$ such that $E = U + HR$ where $HR = RH$. Clearly, $E - U \in M$, as claimed. $\square$

4. G-inverses and inessential perturbations

In this section we generalize Rakočević's result on generalized inverses in the Calkin algebra to generalized inverses modulo certain subspaces of $I(X,Y)$. In what follows, we assume that W is a linear subspace of $I(X,Y)$ with the bimodule property:

(bi) If $T \in W$, $R \in B(X)$, and $S \in B(Y)$, then $STR \in W$.

By Proposition 3, $I(X,Y)$ satisfies (bi). Also, let $F(X,Y)$ be the space of all operators with $E \in B(X,Y)$ such that $\mathbf{R}(E)$ is f.d. Then it is easy to see that $\overline{F(X,Y)}$ (here the closure is in the operator norm) is a subspace of $I(X,Y)$ which satisfies (bi).

Let $K(X,Y)$ denote the space of all compact operators from X into Y. Also, let $S(X,Y)$ denote the space of all strictly singular operators from X into Y. Section 4.5 of [AA] is a good source for information concerning strictly singular operators.

Proposition 10. *Both $K(X,Y)$ and $S(X,Y)$ are subspaces of $I(X,Y)$, and both satisfy* (bi).

Proof: We give the proof for $S(X,Y)$ (the proof for $K(X,Y)$ is similar). First, $S(X,Y)$ is a closed subspace of $B(X,Y)$ that satisfies (bi) [AA, Corollary 4.6.2]. Assume that $T \in S(X,Y)$ and $S \in B(Y,X)$. Then $ST \in S(X)$ and $TS \in S(Y)$ by [AA, Corollary 4.62]. Now $S(X) \subseteq I(X)$ by [CPY, Theorem (5.6.2)]. Therefore by Definition 1, $S(X,Y) \subseteq I(X,Y)$. □

Theorem 11. *Assume $T \in B(X,Y)$. The following are equivalent:*

(i) *there exists $P = P^2 \in B(X)$ such that $TP \in W$ and $T(I-P)$ has a g-inverse;*
(ii) *$T = J + S$ where $J \in W$ and $S \in B(X,Y)$ has a g-inverse;*
(iii) *there exists $G \in B(Y,X)$ and $TGT - T = R \in W$.*

Proof: (i)$\Longrightarrow$(ii) is immediate.

Assume that (ii) holds, so $T = J + S$ where $J \in W$ and for some $G \in B(Y,X)$, $SGS = S$. Then

$$\begin{aligned} TGT - T &= (J+S)G(J+S) - (J+S) = JG(J+S) + SGJ + SGS - J - S \\ &= JG(J+S) + SGJ - J \in W. \end{aligned}$$

Thus, (iii) is true.

Assume the hypotheses in (iii) and that $T \notin W$. These hypotheses imply that $TGTG - TG = RG \in I(Y)$. Now apply Corollary 3 (with $U = TG$ and RG in place of R). Therefore there exists $E = E^2 \in B(Y)$ and $H \in B(Y)$ such that $E = TG + RGH$. Since $R = TGT - T$, we have $E = TG + (TGT - T)GH = T[G + GTGH - GH]$. Setting $U = GTGH - GH$ and $V = G + U$, we have

$$E = TV = T(G+U) \quad \text{and} \quad TU \in W. \tag{4}$$

Then $ET = TVT = TGT + TUT = T + R + TUT$. Set $J = R + TUT \in W$. Thus,

$$ET = T + J \quad \text{with} \quad J \in W. \quad \text{Also,} \quad EJ = 0 \quad \text{and} \quad E(T+J) = T + J, \tag{5}$$

since $E = TV$, $E = ETV = TV + JV$. Therefore,

$$JV = 0. \tag{6}$$

Thus, $(T+J)V(T+J) = TV(T+J) = E(T+J) = T + J$ by (5). Therefore,

$$T + J \quad \text{has g-inverse} \quad V. \tag{7}$$

Set $I - P \equiv V(T+J)$. Then $I - P$ is a projection in $B(X)$. Note that $T(I-P) = TV(T+J) = (T+J)V(T+J)$ by (6). Therefore, $T(I-P) = T + J$ by (7). Thus again by (7), $T(I-P)$ has a g-inverse. Also, $TP = -J \in W$. This proves that (i) holds. □

The following statement is another condition equivalent to those listed in Theorem 11: *There exists* $P = P^2 \in B(Y)$ *such that* $PT \in W$ *and* $(I - P)T$ *has a g-inverse.*

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