A universal property of C_0-semigroups

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Abstract. Let $T : [0,\infty) \to L(E)$ be a C_0-semigroup with unbounded generator $A : D(A) \to E$. We prove that $(T(t)x - x)/t$ has generically a very irregular behaviour for $x \notin D(A)$ as $t \to 0+$.

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1. Introduction

Let $(E, \|\cdot\|)$ be a complex Banach space, $L(E)$ the Banach algebra of all bounded endomorphisms of E, and $T : [0,\infty) \to L(E)$ a C_0-semigroup with generator $A : D(A) \to E$ defined as

$$Ax = \lim_{t\to 0+} \frac{T(t)x - x}{t} \tag{1}$$

with $D(A)$ the set of all $x \in E$ where this limit exists. It is well known that A is closed, $D(A)$ is a dense subset of E, and that $D(A) = E$ if and only if A is bounded [5]. *Throughout the paper let us assume that A is unbounded.* Motivated by the very irregular behaviour of difference quotients of continuous functions (see [2] and the references given there), first discovered in Marcinkiewicz's famous result on the existence of universal primitives [4], we will prove in this paper that in the frame above $(T(t)x - x)/t$ has generically a chaotic behaviour for $x \notin D(A)$ as $t \to 0+$.

2. Main result

Let $(E^*, \|\cdot\|)$ denote the topological dual space of E and let ω denote the Fréchet space of all complex sequences $(z_k)_{k\in\mathbb{N}}$ endowed with the topology of coordinatewise convergence. We will prove the following result:

Theorem 1. *Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ with limit 0. Then there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in E^* such that for each sequence $(c_k)_{k\in\mathbb{N}}$ in $\mathbb{C}\setminus\{0\}$ the set of all $x\in E$ with the property*

$$\left\{\left(c_k\varphi_k\left(\frac{T(t_n)x-x}{t_n}\right)\right)_{k\in\mathbb{N}} : n\in\mathbb{N}\right\} \quad \textit{is dense in } \omega,$$

is a dense G_δ subset of E.

3. Universal elements

We will make use of the following Universality Criterion of Grosse-Erdmann [2, Theorem 1]:

Let X, Y be topological spaces with X a Baire space and Y second countable. Let $L_j : X \to Y$ $(j \in J)$ be a family of continuous mappings. An element $x \in X$ is called universal for this family if $\{L_j x : j \in J\}$ is dense in Y. Let U denote the set of all universal elements.

Proposition 1 (Universality Criterion)**.** *The following conditions are equivalent.*

1. *The set U is a dense G_δ-subset of X.*
2. *The set U is dense in X.*
3. *The set $\{(x, L_j x) : x \in X, j \in J\}$ is dense in $X \times Y$.*

Now, consider the case that specifically $(E, \|\cdot\|)$ is a Banach space, and that F is a metrizable separable topological vector space. Let d be a translation-invariant metric on F defining its topology. Let $L_n : E \to F$ $(n \in \mathbb{N})$ be a sequence of continuous linear operators, let $B : D \to F$ be the linear operator defined by

$$Bx = \lim_{n\to\infty} L_n x$$

on

$$D = \{x \in E : (L_n x) \ \text{ is convergent}\},$$

and assume that D is a dense subset of E. The following criterion is an adaptation of Proposition 1 to this case (see also [3]):

Proposition 2. *Under the conditions and notations above, assume that*

$$\{Bx : x \in D,\ \|x\| \le 1\} \tag{2}$$

is dense in F. Then U is a dense G_δ-subset of E.

Proof: Since D is a subspace of E and B is linear, (2) implies that

$$\{Bx : x \in D,\ \|x\| \le \varepsilon\}$$

is dense in F for each $\varepsilon > 0$. But then

$$\{Bx : x \in D,\ \|x - x_0\| \le \varepsilon\}$$

is dense in F for each $\varepsilon > 0$ and each $x_0 \in E$. Indeed, fix $y \in F$ and let $\delta > 0$. Choose $x_1 \in D$ with $\|x_1 - x_0\| \le \varepsilon/2$, and $x \in D$ with $\|x\| \le \varepsilon/2$ and $d(Bx, y - Bx_1) \le \delta$. Then

$$\|(x + x_1) - x_0\| \le \|x\| + \|x_1 - x_0\| \le \varepsilon,$$

and

$$d(B(x + x_1), y) = d(Bx, y - Bx_1) \le \delta.$$

Now, let $x_0 \in E$, $y_0 \in F$, and $\varepsilon > 0$. We find $x \in D$ such that

$$\|x - x_0\| \le \varepsilon, \quad d(Bx, y_0) \le \varepsilon/2.$$

By choosing $n \in \mathbb{N}$ such that $d(L_n x, Bx) \le \varepsilon/2$ we obtain $d(L_n x, y_0) \le \varepsilon$. Thus

$$\{(x, L_n x) : x \in E, n \in \mathbb{N}\}$$

is dense in $E \times F$. An application of Proposition 1 completes the proof. □

4. Unbounded functionals

To prepare the application of Proposition 2 to our problem we first investigate unbounded functionals. Let D be any subspace of E, let B_1 denote the unit ball in D, that is

$$B_1 = \{x \in D : \|x\| \le 1\},$$

and note that ω^*, the topological dual space of ω, is the space of all finite complex sequences [7, Chapter 2–3].

Proposition 3. *Let $\Psi_k : D \to \mathbb{C}$, $k \in \mathbb{N}$, be a sequence of linearly independent linear functionals such that each*

$$\Psi \in \operatorname{span}\{\Psi_k : k \in \mathbb{N}\}, \quad \Psi \ne 0$$

is unbounded, and let $f : D \to \omega$ be defined by $f(x) = (\Psi_k(x))_{k\in\mathbb{N}}$. Then $f(B_1)$ is dense in ω.

Proof: We first consider a single unbounded functional $\Psi : D \to \mathbb{C}$ and prove that $\Psi(B_1) = \mathbb{C}$. Clearly $0 \in \Psi(B_1)$. Let $\alpha \in \mathbb{C} \setminus \{0\}$. Since $\Psi(B_1)$ is unbounded, there exists $x_0 \in B_1$ such that $|\Psi(x_0)| > |\alpha|$. Set

$$y_0 := \frac{\alpha}{\Psi(x_0)} x_0.$$

Then

$$\|y_0\| = \frac{|\alpha|}{|\Psi(x_0)|}\|x_0\| \leq 1, \;\; \Psi(y_0) = \frac{\alpha}{\Psi(x_0)}\Psi(x_0) = \alpha.$$

Next, the set $\overline{f(B_1)}$ is closed and convex. Assume, by way of contradiction, $\overline{f(B_1)} \neq \omega$, and let $(z_k)_{k\in\mathbb{N}} \notin \overline{f(B_1)}$. According to the separation theorem for closed convex sets and points, we find a functional $(\xi_k)_{k\in\mathbb{N}} \in \omega^*$ ($\xi_k = 0$ for all $k > k_0$), and $\beta \in \mathbb{R}$ such that

$$\operatorname{Re} \sum_{k=1}^{k_0} \xi_k z_k < \beta < \operatorname{Re} \sum_{k=1}^{k_0} \xi_k \Psi_k(x) \quad (x \in B_1).$$

Now $\Psi := \sum_{k=1}^{k_0} \xi_k \Psi_k \neq 0$, hence Ψ is unbounded. Therefore $\operatorname{Re}\Psi(B_1) = \mathbb{R}$, a contradiction. □

5. Closed operators

In this section we prove two propositions on general closed operators which we apply later to A.

Proposition 4 ([6, Chapter IV.5, Problem 11]). *Let $B : D(B) \to E$ be a closed and unbounded operator on E, and let V be a closed subspace of E such that $D(B) \cap V = \{0\}$. Then $D(B) \oplus V$ is not closed in E.*

Proof: Assume that $D(B) \oplus V$ is closed in E. Set

$$G(B) := \{(x, Bx) : x \in D(B)\} \subseteq E \times E.$$

Since B is closed, the set $G(B)$ is closed, and $G(B)$ becomes a Banach space when endowed with the graph norm

$$\|(x, Bx)\| = \|x\| + \|Bx\|.$$

We define $S : G(B) \to (D(B) \oplus V)/V$ by $S(x, Bx) = \widehat{x}$ with $\widehat{x} = x + V$. Then S is bijective, linear, and S is continuous since

$$\|S(x, Bx)\| = \|\widehat{x}\| \leq \|x\| \leq \|(x, Bx)\| \quad (x \in D(B)).$$

Thus, $S^{-1} : (D(B) \oplus V)/V \to G(B)$ is continuous, by the Open Mapping Theorem. Consequently,

$$\|Bx\| \leq \|(x, Bx)\| = \|S^{-1}(\widehat{x})\| \leq \|S^{-1}\|\|\widehat{x}\| \leq \|S^{-1}\|\|x\| \; (x \in D(B)).$$

Hence B is continuous, a contradiction. □

Remark. Note that Proposition 4 implies that if V is an algebraic complement of $D(B)$, then V cannot be closed and has therefore infinite dimension, in particular.

Now, let $B : D(B) \to E$ be a densely defined closed and unbounded operator on E. Then B has an adjoint

$$B^* : D(B^*) \to E^*,$$

with

$$D(B^*) = \{\varphi \in E^* : \varphi \circ B \text{ is continuous on } D(B)\}.$$

It is well known that B^* is a closed linear operator, and that $D(B^*) = E^*$ if and only if B is continuous [1, Theorem II.2.6, II.2.8].

Proposition 5. *Let $B : D(B) \to E$ be a densely defined closed and unbounded operator on E, and let W be a subspace of E^* such that $E^* = D(B^*) \oplus W$. Then W is not closed in E^* and $\dim W = \infty$.*

PROOF: We know that B^* is closed, and that $D(B) \neq E$ since B is unbounded. By means of [1, Corollary II.4.8] the operator B^* is unbounded too. Thus, the proof is finished according to the remark following Proposition 4. $\square$

6. Proof of Theorem 1

We apply Proposition 5 to $B = A$: Let W be an algebraic complement of $D(A^*)$ in E^*. Since $\dim W = \infty$ we can choose a countably infinite linear independent subset of W denoted by $\{\varphi_k : k \in \mathbb{N}\}$.

We define a sequence of continuous linear operators $L_n : E \to \omega$, $n \in \mathbb{N}$, by

$$L_n x = \left(c_k \varphi_k \left(\frac{T(t_n)x - x}{t_n} \right) \right)_{k \in \mathbb{N}}$$

and we set $\Psi_k = c_k(\varphi_k \circ A)$ $(k \in \mathbb{N})$. Since

$$D(A^*) = \{\varphi \in E^* : \varphi \circ A \text{ is continuous on } D(A)\}$$

we conclude that each

$$\Psi \in \operatorname{span}\{\Psi_k : k \in \mathbb{N}\}, \quad \Psi \neq 0$$

is an unbounded functional on $D(A)$. Next let $C : D \to \omega$ be defined by

$$Cx = \lim_{n \to \infty} L_n x$$

on

$$D := \{x \in E : (L_n x) \text{ is convergent}\},$$

and note that $D(A) \subseteq D$, hence D is dense in E, and that

$$f(x) := (\Psi_k(x))_{k\in\mathbb{N}} = Cx \quad (x \in D(A)).$$

Let B_1 denote the closed unit ball in $D(A)$. Now, $f(B_1)$ is dense in ω according to Proposition 3. Therefore

$$\{Cx : x \in D,\ \|x\| \leq 1\}$$

is dense in ω, and, according to Proposition 2 applied to $B = C$, the set of all $x \in E$ with the property

$$\{L_n x : n \in \mathbb{N}\} \text{ is dense in } \omega$$

is a dense G_δ subset of E. □

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