On quasi-uniform space valued semi-continuous functions

Tomasz Kubiak, María Angeles de Prada Vicente

Abstract. F. van Gool [Comment. Math. Univ. Carolin. **33** (1992), 505–523] has introduced the concept of lower semicontinuity for functions with values in a quasi-uniform space $(R,\mathcal{U})$. This note provides a purely topological view at the basic ideas of van Gool. The lower semicontinuity of van Gool appears to be just the continuity with respect to the topology $T(\mathcal{U})$ generated by the quasi-uniformity $\mathcal{U}$, so that many of his preparatory results become consequences of standard topological facts. In particular, when the order induced by $\mathcal{U}$ makes R into a continuous lattice, then $T(\mathcal{U})$ agrees with the Scott topology $\sigma(R)$ on R and, thus, the lower semicontinuity reduces to a well known concept.

Keywords: lower semi-continuity, quasi-uniformity, continuous lattice

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1. Introduction

There is a quite extensive literature devoted to the concept of a lower semicontinuous function with the range different than the reals. In particular, F. van Gool [4] has introduced the notion of a lower semicontinuous function from a topological space to a quasi-uniform space $(R,\mathcal{U})$. Crucial for this approach are certain filter bases defined in terms of the operators $(\cdot)^\circ$ and $(\cdot)^\bullet$ from $\mathcal{U}$ to $\mathcal{U}$. In his report on [4], Watson [9] has pointed out that one may have troubles with the "very easy" list of facts related to $(\cdot)^\circ$ and $(\cdot)^\bullet$ which are used throughout van Gool's paper.

In this note, we shall show that $(\cdot)^\circ$ and $(\cdot)^\bullet$ take elements of $\mathcal{U}$ into, respectively, open and closed sets of $R\times R$ endowed, respectively, with the product topologies $T(\mathcal{U}^{-1})\times T(\mathcal{U})$ and $T(\mathcal{U})\times T(\mathcal{U}^{-1})$, where $T(\mathcal{U})$ is the topology generated by $\mathcal{U}$, and $\mathcal{U}^{-1}$ is the dual of $\mathcal{U}$. This observation provides a topological view at the basic ideas of van Gool. The list of facts related to $(\cdot)^\circ$ and $(\cdot)^\bullet$ are now easy consequences of the properties of the interior and closure operators. The preorder induced by the quasi-uniformity $\mathcal{U}$ (resp. $\mathcal{U}^{-1}$) is proved to agree with the specialization preorder with respect to the topology $T(\mathcal{U})$ (resp. $T(\mathcal{U}^{-1})$), which allows us to give a topological interpretation of the basic relations introduced by

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van Gool. Then most of the results of Sections 3 and 4 of [4] become simple consequences of standard topological facts. In particular, when R is a continuous lattice, which happens if and only if $T(\mathcal{U})$ is exactly the Scott topology $\sigma(R)$ (see Proposition 5.4 and Proposition 6.1), lower semicontinuous functions in the sense of [4] are precisely the Scott continuous functions.

We shall use the term quasi-uniformity (as in [1] or [8]) instead of semi-uniformity as used in [4] following Nachbin [7]. In fact, the paper [4] has changed the original terminology of Nachbin [7], and it is obscure why, in [4], semi-uniform spaces (= quasi-uniform spaces) have been called uniform ordered spaces with a semi-uniformity. We shall also omit the adjective "preordered", because the preorder has always been understood in [4] as the one induced by the quasi-uniformity (unless the preorder will be required to be an order). Also, unlike [4], we shall use standard notation as far as possible. In particular, we use the symbol $\ll$ to denote the way-below relation of [2].

2. Terminological background

As a general reference to quasi-uniformities we suggest Fletcher and Lindgren [1] or Page [8] (cf. also [6] and [7]).

A *quasi-uniform space* $(R,\mathcal{U})$ is a set R together with a filter $\mathcal{U}$ on $R\times R$ such that for any $A\in\mathcal{U}$ the following hold:

(1) $\{(x,x) : x\in R\}\subset A$,
(2) there exists a $B\in\mathcal{U}$ such that $B\circ B\subset A$ (where $\circ$ stands for the usual composition of relations).

The relation $\leq_{\mathcal{U}} = \leq := \bigcap\mathcal{U}$ is reflexive and transitive, and is called the *preorder* associated with $\mathcal{U}$. Given $(R,\mathcal{U})$, $A\in\mathcal{U}$ and $x\in R$, let $A(x)=\{y\in R : (x,y)\in A\}$. Then the collection $\{A(x) : A\in\mathcal{U}\}$ is a neighborhood system of x for a topology that will be called the *lower topology generated by the quasi-uniformity* $\mathcal{U}$ and denoted by $T(\mathcal{U})$. Each quasi-uniformity $\mathcal{U}$ on R has its dual quasi-uniformity $\mathcal{U}^{-1}=\{A^{-1} : A\in\mathcal{U}\}$, where $A^{-1}=\{(y,x) : (x,y)\in A\}$.

We thus have another preorder $\leq_{\mathcal{U}^{-1}}$ as well as another topology $T(\mathcal{U}^{-1})$ on R that will be referred to as the *upper topology generated by the quasi-uniformity* $\mathcal{U}$.

For a quasi-uniformity $\mathcal{U}$ on R, the family $\{A\cap A^{-1} : A\in\mathcal{U}\}$ is a base for a uniformity, denoted $\mathcal{U}^*$, (non-Hausdorff in general) which is the smallest amongst all the uniformities containing $\mathcal{U}$. We have $T(\mathcal{U}^*)=T(\mathcal{U})\vee T(\mathcal{U}^{-1})$, the supremum of $T(\mathcal{U})$ and $T(\mathcal{U}^{-1})$, called the *topology generated by the quasi-uniformity* $\mathcal{U}$.

We shall need the following facts (cf. Exercise 1.B in [8] and Propositions 1.17 and 1.19 in [1]).

Lemma 2.1. *For a quasi-uniform space $(R,\mathcal{U})$ and $M\subset R\times R$, the following hold:*

(1) *for every $A,B\in\mathcal{U}$, $A\circ M\circ B$ is a neighborhood of M in the product topology $T(\mathcal{U}^{-1})\times T(\mathcal{U})$;*

(2) $\mathrm{Cl}_{T(\mathcal{U})\times T(\mathcal{U}^{-1})} M = \bigcap\{A\circ M\circ B^{-1} : A\in\mathcal{U}, B\in\mathcal{U}^{-1}\}$;
(3) $\mathrm{Int}_{T(\mathcal{U}^{-1})\times T(\mathcal{U})} A\in\mathcal{U}$ *for every* $A\in\mathcal{U}$.

3. The operators $(\cdot)^\circ$ and $(\cdot)^\bullet$

Given a quasi-uniform space $(R,\mathcal{U})$, van Gool [4] defined two operators which served him to have two filter bases for $\mathcal{U}$. These are the operators

$$(\cdot)^\circ : \mathcal{U}\to\mathcal{U} \quad \text{and} \quad (\cdot)^\bullet : \mathcal{U}\to\mathcal{U}$$

defined for each $A\in\mathcal{U}$ as follows:

$$A^\circ = \bigcup\{B\in\mathcal{U} : C\circ B\circ C\subset A \ \text{ for some } \ C\in\mathcal{U}\}$$

and

$$A^\bullet = \bigcap\{B\circ A\circ B : B\in\mathcal{U}\}.$$

Those two filter bases are then the following (cf. [4]; also see Exercise 1B(c), (d) of [8]):

$$\mathcal{U}^\circ = \{A\in\mathcal{U} : A = A^\circ\} \ \text{ and } \ \mathcal{U}^\bullet = \{A\in\mathcal{U} : A = A^\bullet\}.$$

The next observation is essential to what follows. It will enable us to describe in topological terms the two filter bases just mentioned. Having this at hand, some saved effort will be achieved in providing arguments for some results of [4] (see [9] again). In particular, results which are topological in nature will be proved in topological rather than uniformity terms.

Proposition 3.1. *Let $(R,\mathcal{U})$ be a quasi-uniform space. For $A\in\mathcal{U}$, the following statements hold:*

(1) $A^\circ = \mathrm{Int}_{T(\mathcal{U}^{-1})\times T(\mathcal{U})}(A^\circ)$,
(2) $A^\bullet = \mathrm{Cl}_{T(\mathcal{U})\times T(\mathcal{U}^{-1})}(A)$.

Proof: (1) For the nontrivial inclusion, let $(x,y)\in A^\circ$. This means that there exist $B,C\in\mathcal{U}$ such that $(x,y)\in B$ and $C\circ B\circ C\subset A$. Also, there exists a $D\in\mathcal{U}$ such that $D\circ D\subset C$. We thus have $D\circ(D\circ B\circ D)\circ D\subset A$, and since $B\subset D\circ B\circ D$, also $D\circ B\circ D\subset A^\circ$. Now, it suffices to note that $D^{-1}(x)\times D(y)\subset D\circ B\circ D$. Indeed, if $(z,t)\in D^{-1}(x)\times D(y)$, then $(z,x)\in D$, $(x,y)\in B$, and $(y,t)\in D$, so that $(z,t)\in D\circ B\circ D$. Consequently, $(x,y)\in D^{-1}(x)\times D(y)\subset \mathrm{Int}_{T(\mathcal{U}^{-1})\times T(\mathcal{U})}(A^\circ)$.

(2) See (2) of Lemma 2.1 and consider that for any $B,C\in\mathcal{U}$, $D=B\cap C\in\mathcal{U}$. $\square$

Corollary 3.2. *Let $(R,\mathcal{U})$ be a quasi-uniform space. Then:*

(1) *if $A \in \mathcal{U}^\circ$ and $x \in R$, then $A(x) \in T(\mathcal{U})$ and, equivalently, $A^{-1}(x) \in T\left(\mathcal{U}^{-1}\right)$;*

(2) *if $A \in \mathcal{U}^\bullet$ and $x \in R$, then $A(x)$ is $T(\mathcal{U}^{-1})$-closed and, equivalently, $A^{-1}(x)$ is $T(\mathcal{U})$-closed.*

Remark 3.3. It follows that for each $x \in X$ the collection $\{A(x) : A \in \mathcal{U}^\circ\}$ is a neighborhood base of open sets in the topology $T(\mathcal{U})$.

The following properties (stated without proof in [4, pp. 508–509]) are now clear (cf. [9]):

Proposition 3.4 ([4])**.** *For $A \in \mathcal{U}$ the following hold:*

(1) *$A^\circ \subset A \subset A^\bullet$,*

(2) *$A^{\circ\circ} = A^\circ$ and $A^{\bullet\bullet} = A^\bullet$.*

With $\tau = T(\mathcal{U}^*) \times T(\mathcal{U}^*)$ one has the following:

(3) $\mathrm{Int}_\tau(A^\circ) = \mathrm{Int}_{T(\mathcal{U}^{-1})\times T(\mathcal{U})}(A^\circ) = A^\circ$,

(4) $\mathrm{Cl}_\tau(A^\bullet) = \mathrm{Cl}_{T(\mathcal{U})\times T(\mathcal{U}^{-1})}(A^\bullet) = A^\bullet$.

Below and elsewhere the sets $\uparrow x$ and $\downarrow x$ are defined in terms of the preorder $\leq_\mathcal{U}$, i.e. $\uparrow x = \{y \in R : x \leq_\mathcal{U} y\}$ and dually for $\downarrow x$. A set $A \subset R$ is said to be *increasing* (*decreasing*) if $\uparrow x \subset A$ (resp. $\downarrow x \subset A$) whenever $x \in A$.

Proposition 3.5. *Let $(R,\mathcal{U})$ be a quasi-uniform space and let $\leq_\mathcal{U}$ be the associated preorder. Then:*

(1) *$A(x)$ is increasing for each $A \in \mathcal{U}^\circ$ and $x \in R$;*

(2) *$A^{-1}(x)$ is decreasing for each $A \in \mathcal{U}^\circ$ and $x \in R$;*

(3) *U is increasing for each $U \in T(\mathcal{U})$.*

Proof: (1) Let $y \in A(x)$ and $y \leq_\mathcal{U} z \in R$. Since $A(x) \in T(\mathcal{U})$ by Corollary 3.2(1), there exists a $B \in \mathcal{U}$ such that $y \in B(y) \subset A(x)$ and $(y,z) \in B$, i.e. $z \in B(y) \subset A(x)$. Consequently, $\uparrow y \subset A(x)$.

(2) We proceed as in (1) replacing $\mathcal{U}$ by $\mathcal{U}^{-1}$ and using $A^{-1}(x) \in T(\mathcal{U}^{-1})$ by Corollary 3.2(1) again.

(3) Let $x \in U$. Since $\{C(x) : C \in \mathcal{U}^\circ\}$ is a neighborhood base of x in $T(\mathcal{U})$ (see 3.3), there exists an $A \in \mathcal{U}^\circ$ such that $x \in A(x) \subset U$. Then, by (1), $\uparrow x \subset U$. □

4. Topological characterizations of various relations induced by a quasi-uniformity

First we shall show that, given a quasi-uniform space $(R,\mathcal{U})$, the preorder $\leq_\mathcal{U}$ is the specialization preorder (in the sense of [2, Definition II.3.6]) with respect to the topology $T(\mathcal{U})$, i.e.,

$$x \leq_\mathcal{U} y \quad\Leftrightarrow\quad x \in \mathrm{Cl}_{T(\mathcal{U})}\{y\}.$$

This is stated in the following proposition.

Proposition 4.1. *Let $(R,\mathcal{U})$ be a quasi-uniform space. Then for any $x \in R$:*

(1) $\downarrow x = \mathrm{Cl}_{T(\mathcal{U})}\{x\}$,
(2) $\uparrow x = \mathrm{Cl}_{T(\mathcal{U}^{-1})}\{x\}$.

Proof: We shall check (1). We calculate

$$\begin{aligned}\downarrow x &= \{y \in R : y \leq_{\mathcal{U}} x\}\\ &= \{y \in R : (y,x) \in \bigcap \mathcal{U}\}\\ &= \{y \in R : (y,x) \in A \text{ for all } A \in \mathcal{U}\}\\ &= \{y \in R : A(y) \cap \{x\} \neq \emptyset \text{ for all } A \in \mathcal{U}\}\\ &= \mathrm{Cl}_{T(\mathcal{U})}\{x\}.\end{aligned}$$

The proof of (2) is dual. □

The following provides some candidates for $(R, \leq_{\mathcal{U}})$ to play the role of the strictly "less than" relation in the reals.

Definition 4.2 ([4])**.** Let $(R,\mathcal{U})$ be a quasi-uniform space. Given $x, y \in R$, we put:

(1) $x \Subset y$ if there is an $A \in \mathcal{U}$ with $A(y) \subset \uparrow x$,
(2) $x \Supset y$ if there is an $A \in \mathcal{U}$ with $A^{-1}(y) \subset \downarrow x$,
(3) $x \lll y$ if there is an $A \in \mathcal{U}$ with $A(y) \subset \uparrow z$ for all $z \in A^{-1}(x)$.

As noted in [4], the relation $\Subset$ has the following properties (recall that we write $\leq$ for $\leq_{\mathcal{U}}$):

(a) $x \Subset y \;\Rightarrow\; x \leq y$,
(b) $z_1 \leq x \Subset y \leq z_2 \;\Rightarrow\; z_1 \Subset z_2$.

Under the assumption that $(R, \leq_{\mathcal{U}})$ is a $\vee$-semilattice with the bottom element 0, we have:

(c) $x \Subset z, y \Subset z \;\Rightarrow\; x \vee y \Subset z$,
(d) $0 \Subset x$.

Remark 4.3. In the terminology of [2], $\Subset$ is an auxiliary relation (properties (a)–(d)). When $(R, \leq)$ is a complete lattice, the relation $\Subset$ is called *approximating* if $x = \bigvee\{y \in R : y \Subset x\}$ for each $x \in R$.

The standard notation we have used in formulating Definition 4.2 already provides, in fact, topological characterizations of the relations $\Subset$, $\Supset$, and $\lll$. More precisely, yet another change of notation yields the following:

Proposition 4.4. *Let $(R,\mathcal{U})$ be a quasi-uniform space. For any $x, y \in R$, we have:*

(1) *$x \Subset y$ if and only if $y \in \mathrm{Int}_{T(\mathcal{U})}(\uparrow x)$,*
(2) *$x \Supset y$ if and only if $y \in \mathrm{Int}_{T(\mathcal{U}^{-1})}(\downarrow x)$,*
(3) *$x \lll y$ if and only if $(x, y) \in \mathrm{Int}_{T(\mathcal{U}^{-1})\times T(\mathcal{U})}(\bigcap\mathcal{U})$.*

Proof: For (1): $y \in \mathrm{Int}_{T(\mathcal{U})}(\uparrow x)$ iff $A(y) \subset \uparrow x$ for some $A \in \mathcal{U}$. This just means that $x \Subset y$. The same for (2). To see (3), $(x, y) \in \mathrm{Int}_{T(\mathcal{U}^{-1})\times T(\mathcal{U})}(\bigcap\mathcal{U})$ iff $(x, y) \in A^{-1}(x) \times A(y) \subset \bigcap\mathcal{U}$ for some $A \in \mathcal{U}$. This is equivalent to the statement that $z_1 \leq z_2$ for each $z_1 \in A^{-1}(x)$ and $z_2 \in A(y)$, i.e., $A(y) \subset \uparrow z$ for all $z \in A^{-1}(x)$. □

Remark 4.5. Notice that in the definition of $\Subset$ one can equivalently use an $A \in \mathcal{U}^\circ$. That is:

$$x \Subset y \quad \text{if there is an} \quad A \in \mathcal{U}^\circ \quad \text{with} \quad A(y) \subset\uparrow x.$$

The next observation replaces the assumption of arcwise connectedness (in Lemma 3.3 of [4]) by connectedness, thereby answering a question of the report [9].

Corollary 4.6. *Let $(R,\mathcal{U})$ be a quasi-uniform space such that $(R, T(\mathcal{U}^*))$ is connected and $x \in R$. Then $x \Subset x$ if and only if x is the bottom element of $(R, \leq_{\mathcal{U}})$.*

Proof: If $x = 0$ is the bottom element of R, then $x \Subset x$ by property (d) above. Conversely, by (1) of 4.4, $x \Subset x$ is equivalent to the statement that $\uparrow x \in T(\mathcal{U}) \subset T(\mathcal{U}^*)$. By 4.1(1) in turn, $\uparrow x$ is $T(\mathcal{U}^{-1})$-closed and then $T(\mathcal{U}^*)$-closed. By connectedness, $\uparrow x = R$. □

5. Semicontinuous functions with values in $(R,\mathcal{U})$

In [4], two concepts of a limit in $(R,\mathcal{U})$ are defined in terms of $\mathcal{U}$ and $\mathcal{U}^{-1}$. Namely, for each filter base $\mathcal{F}$ in R the following two sets have been defined in [4]:

$$\mathrm{LIM\,INF}(\mathcal{F}) = \{x \in R : \forall A \in \mathcal{U} \;\; \exists V \in \mathcal{F} \;\text{ s.t. }\; V \subset A(x)\},$$
$$\mathrm{LIM\,SUP}(\mathcal{F}) = \{x \in R : \forall A \in \mathcal{U} \;\; \exists V \in \mathcal{F} \;\text{ s.t. }\; V \subset A^{-1}(x)\}.$$

We prefer to think about those limits as the ones associated with the topologies $T(\mathcal{U})$ and $T(\mathcal{U}^{-1})$. Clearly, then:

$$\mathcal{F} \xrightarrow{T(\mathcal{U})} x \quad \Leftrightarrow \quad x \in \mathrm{LIM\,INF}(\mathcal{F}),$$
$$\mathcal{F} \xrightarrow{T(\mathcal{U}^{-1})} x \quad \Leftrightarrow \quad x \in \mathrm{LIM\,SUP}(\mathcal{F}).$$

Definition 5.1 ([4]). Let $(R,\mathcal{U})$ be a quasi-uniform space, (X,τ) a topological space, and $f: X \to R$ an arbitrary function. We say:

(1) f is lower semicontinuous in $p \in X$ if $f(p) \in \mathrm{LIM\,INF}(f(\mathcal{N}_p^\tau))$,

(2) f is upper semicontinuous in $p \in X$ if $f(p) \in \mathrm{LIM\,SUP}(f(\mathcal{N}_p^\tau))$,

where $\mathcal{N}_p^\tau$ is the filter base of all open neighborhoods of the point $p \in X$.

Proposition 5.2. *Let $(R,\mathcal{U})$ be a quasi-uniform space, (X,τ) a topological space, and let $f: X \to R$. Then:*

(1) *f is lower semicontinuous if and only if $f: (X,\tau) \to (R, T(\mathcal{U}))$ is continuous;*

(2) *f is upper semicontinuous if and only if $f: (X,\tau) \to (R, T(\mathcal{U}^{-1}))$ is continuous.*

Proof: For (1): this follows from the definition, since $f: (X,\tau) \to (R, T(\mathcal{U}))$ is continuous at $p \in X$ if and only if $\mathcal{N}_{f(p)}^{T(\mathcal{U})} \subset f(\mathcal{N}_p^\tau)$, which is equivalent to the statement that $f(p) \in \mathrm{LIM\,INF}(f(\mathcal{N}_p^\tau))$. The proof of (2) remains the same. Indeed, $f: (X,\tau) \to (R, T(\mathcal{U}^{-1}))$ is continuous at $p \in X$ if and only if $f(\mathcal{N}_p^\tau)$ converges to $f(p)$ in the topology $T(\mathcal{U}^{-1})$, which is equivalent to the statement that $f(p) \in \mathrm{LIM\,SUP}(f(\mathcal{N}_p^\tau))$. ☐

Definition 5.3. Let $(R,\mathcal{U})$ be a quasi-uniform space with $(R, \leq_{\mathcal{U}})$ a complete lattice, (X,τ) a topological space, and $f: X \to R$ an arbitrary function. We define $f_*: X \to R$ and $f^*: X \to R$ as follows:

$$f_*(p) = \bigvee \left\{ \bigwedge f(U) : p \in U \in \tau \right\},$$
$$f^*(p) = \bigwedge \left\{ \bigvee f(U) : p \in U \in \tau \right\}$$

for every $p \in X$.

Proposition 5.4. *Let $(R,\mathcal{U})$ be a quasi-uniform space with $(R, \leq_{\mathcal{U}})$ a complete lattice. The following are equivalent:*

(1) $x = \bigvee\{y \in R : x \in \mathrm{Int}_{T(\mathcal{U})}(\uparrow y)\}$ *for all* $x \in R$,

(2) *for an arbitrary topological space (X,τ), if $f: X \to R$ is lower semicontinuous, then $f = f_*$,*

(3) $x = \bigvee\{\bigwedge U : x \in U \in T(\mathcal{U})\} = \bigvee\{\bigwedge A(x) : A \in \mathcal{U}\}$ *for every* $x \in R$,

(4) $x = \bigvee\{y \in R : y \Subset x\}$ *for all $x \in R$, i.e. $\Subset$ is approximating.*

Proof: (1) $\Rightarrow$ (2) : Let $p \in f^{-1}(\mathrm{Int}_{T(\mathcal{U})}(\uparrow x))$ for some $x \in R$. Then there exists an open U, containing the point p, such that $f(U) \subset \mathrm{Int}_{T(\mathcal{U})}(\uparrow x) \subset \uparrow x$. Therefore, $x \leq_{\mathcal{U}} \bigwedge f(U) \leq_{\mathcal{U}} f_*(p)$. By (1), we get $\bigvee\{x \in R : f(p) \in \mathrm{Int}_{T(\mathcal{U})}(\uparrow x)\} = f(p) \leq_{\mathcal{U}} f_*(p)$. The reverse inequality is obvious.

(2) $\Rightarrow$ (3) : Let $X = R$ and $\tau = T(\mathcal{U})$. Since the identity map id_R is clearly lower semicontinuous, by $(\mathrm{id}_R)_* = \mathrm{id}_R$ we get (3).

(3) $\Rightarrow$ (1) : For the nontrivial inequality, let $x \in U \in T(\mathcal{U})$. Then $U \subset \uparrow (\bigwedge U)$ and, thus, $x \in \mathrm{Int}_{T(\mathcal{U})}(\uparrow (\bigwedge U))$. Therefore

$$x = \bigvee \left\{ \bigwedge U : x \in U \in T(\mathcal{U}) \right\} \leq \bigvee \left\{ y \in R : x \in \mathrm{Int}_{T(\mathcal{U})}(\uparrow y) \right\}.$$

(4) $\Leftrightarrow$ (1) is a restatement of (1) of 4.4. $\square$

Remark 5.5. We note that 5.4 has its 'dual' formulation involving the relation $\Supset$, $T(\mathcal{U}^{-1})$, and the operation $f \mapsto f^*$.

6. The case when $(R, \leq_{\mathcal{U}})$ is a continuous lattice

Let $R = (R, \leq)$ be an arbitrary complete lattice. Given $x, y \in R$, we say x is *way below* y (notation: $x \ll y$) if, whenever $y \leq \bigvee D$ with a directed $D \subset R$, there exists a $d \in D$ such that $x \leq d$. Then R is called a *continuous lattice* if for all $x \in R$ one has $x = \bigvee\{z \in R : z \ll x\}$. The Scott topology $\sigma(R)$ on a continuous lattice R is one which has $\{\twoheaduparrow x : x \in R\}$ as a base, where $\twoheaduparrow x = \{y \in R : x \ll y\}$. The Lawson topology $\lambda(R)$ on R is the topology generated by $\sigma(R) \cup \{R \setminus \uparrow x : x \in R\}$.

According to the general terminology of VI.1.2 of [2], a topology in R is said to be *compatible* if, whenever $x \in R$ is the directed join or filtered meet of a net, then the net converges to x topologically. In what follows we shall use the fact, that in each continuous lattice R the Lawson topology $\lambda(R)$ is compatible (see [2, III.2.13 and VI.1.13]) and has a closed order. It is enough to observe that for any $x, y \in R$ with $x \not\leq y$ there exists $z \in R$ and disjoint open $\twoheaduparrow z$ (increasing) and $L \setminus \uparrow z$ (decreasing) with $x \in \twoheaduparrow z$ and $y \in L \setminus \uparrow z$.

Note that for a compact Hausdorff topological space (X, τ) with a closed order $\leq$, there is a unique quasi-uniformity $\mathcal{U}$, generating both the topology ($T(\mathcal{U}^*) = \tau$) and the order ($\leq_{\mathcal{U}} = \leq$) (see [7], also Theorem 3.6 in [4]).

Proposition 6.1. *Let $(R, \leq)$ be a continuous lattice and $\mathcal{U}$ the unique quasi-uniformity generating both the Lawson topology ($T(\mathcal{U}^*) = \lambda(R)$) and the order ($\leq_{\mathcal{U}} = \leq$). Then:*

(1) *$\Subset$ is approximating,*
(2) *$x \ll y \;\Leftrightarrow\; x \Subset y$ for all $x, y \in R$,*
(3) *$T(\mathcal{U}) = \sigma(R)$.*

Proof: A proof of (1) and (2) can be found in [4].

(3) For any $A \in \mathcal{U}^\circ$ and $x \in R$ the set $A(x)$ is increasing, and for any directed $D \subset R$ with $\bigvee D = \lim_{T(\mathcal{U}^*)} D \in A(x)$ we have $D \cap A(x) \neq \varnothing$. This means that

$T(\mathcal{U}) \subset \sigma(R)$. For the reverse inclusion, given a basic open set $\Uparrow x$ of $\sigma(R)$, and using (2) and 4.4(1) we conclude that $\Uparrow x = \mathrm{Int}_{T(\mathcal{U})}(\uparrow x) \in T(\mathcal{U})$. $\square$

As a consequence of the previous proposition and Proposition 5.2 we have:

Corollary 6.2. *If $(R, \leq)$ is a continuous lattice, a function $f : X \to R$ is lower semicontinuous (in the sense of 5.1) if and only if it is continuous with respect to the Scott topology. Dually $f : X \to R$ is upper semicontinuous if and only if it is continuous with respect to the Scott topology in R^{op} (R with the opposite order, assuming it is also continuous).*

Note that this description of semicontinuity has been mentioned without proof in [10].

References

[1] Fletcher P., Lindgren W.F., *Quasi-uniform Spaces*, Marcel Dekker, New York, 1982.

[2] Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M., Scott D.S., *A Compendium of Continuous Lattices*, Springer, Berlin, Heidelberg, New York, 1980.

[3] Gierz G., Lawson J.D., *Generalized continuous and hypercontinuous lattices*, Rocky Mountain J. Math. **11** (1981), 271–296.

[4] van Gool F., *Lower semicontinuous functions with values in a continuous lattice*, Comment. Math. Univ. Carolin. **33** (1992), 505–523.

[5] Liu Y.-M., Luo M.-K., *Lattice-valued mappings, completely distributive law and induced spaces*, Fuzzy Sets and Systems **42** (1991), 43–56.

[6] Murdeshwar M.G., Naimpally S.A., *Quasi-uniform Topological Spaces*, Publ. P. Noordhoff Ltd., Groningen, 1966.

[7] Nachbin L., *Topology and Order*, Van Nostrand Mathematical Studies, 24, Princeton, New Jersey, 1965.

[8] Page W., *Topological Uniform Structures*, Dover, New York, 1989.

[9] Watson W.S., *M.R. 94j:54007*.

[10] Zhang De-Xue, *Metrizable completely distributive lattices*, Comment. Math. Univ. Carolin. **38** (1997), 137–148.

Wydział Matematyki i Informatyki, Uniwersytet im. Adama Mickiewicza, Umultowska 87, 61-614 Poznań, Poland

Departamento de Matemáticas, Universidad del País Vasco – Euskal Herriko Unibersitatea, Apdo. 644, 48080 Bilbao, Spain

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