Topologies on groups determined by right cancellable ultrafilters

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Abstract. For every discrete group G, the Stone-Čech compactification βG of G has a natural structure of a compact right topological semigroup. An ultrafilter $p \in G^*$, where $G^* = \beta G \setminus G$, is called right cancellable if, given any $q, r \in G^*$, $qp = rp$ implies $q = r$. For every right cancellable ultrafilter $p \in G^*$, we denote by $G(p)$ the group G endowed with the strongest left invariant topology in which p converges to the identity of G. For any countable group G and any right cancellable ultrafilters $p, q \in G^*$, we show that $G(p)$ is homeomorphic to $G(q)$ if and only if p and q are of the same type.

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A topology τ on a group G is called *left invariant* if, for every element $g \in G$, the left shift $x \mapsto gx$ is continuous in τ. Given an infinite group G, we denote by $G(p)$ the group G provided with the strongest left invariant topology in which p converges to the identity of G. By [4, Theorem 4.12], the space $G(p)$ is *strongly extremally disconnected* in the sense that, for every open non-closed subset U of $G(p)$, there exists $g \in \mathrm{cl}\ U \setminus U$ such that $\{g\} \cup U$ is a neighbourhood of g. To distinguish the spaces $G(p)$ for different ultrafilters p on G, we need some algebra in the Stone-Čech compactification of a discrete group.

Given a discrete space X, we take the points of βX, the Stone-Čech compactification of X, to be the ultrafilters on X, with the points of X identified with the principal ultrafilters, and denote by $X^* = \beta X \setminus X$ the set of all free ultrafilters on X. The topology of βX can be defined by stating that the sets of the form $\overline{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, are a base for the open sets. We shall also use the universal property of βX stating that every mapping $f : X \to Y$, where Y is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \to Y$.

Let G be a discrete group. Using the universal property of the space βG, we extend the group multiplication from G to βG in two steps. Given $g \in G$, the mapping

$$x \mapsto gx : G \to \beta G$$

extends to the continuous mapping

$$q \mapsto gq : \beta G \to \beta G.$$

Then, for each $q \in \beta G$, we extend the mapping $g \mapsto gq$, defined from G into βG, to the continuous mapping

$$p \mapsto pq : \beta G \to \beta G.$$

The product pq of ultrafilters p, q can also be defined by the rule: given a subset $A \subseteq G$,

$$A \in pq \Leftrightarrow \{g \in G : g^{-1}A \in q\} \in p.$$

It is easy to verify that the binary operation $(p, q) \mapsto pq$ is associative, so βG is a semigroup, and G^* is a subsemigroup of βG. It follows from the second step of the extension that, for every $q \in \beta G$, the mapping $p \mapsto pq$ is continuous, so the semigroup βG is right topological. For the structure of compact right topological semigroup βG and its combinatorial applications see [1].

An ultrafilter $p \in \beta G$ is called an *idempotent* if $pp = p$. By [1, Corollary 6.43], for every infinite group G, there are $2^{2^{|G|}}$ idempotents in G^*. Given an idempotent $p \in G^*$, the space $G(p)$ is Hausdorff and *maximal*, i.e. $G(p)$ has no isolated points but $G(p)$ has an isolated point in any stronger topology. The existence of maximal topological groups is consistent with ZFC [3]. For every infinite group G, in ZFC there exists an idempotent p such that $G(p)$ is regular. To my knowledge, these are the only ZFC-examples of homogeneous regular maximal spaces. For these and other results concerning the topologies on a group G determined by idempotents from βG see [3], [4], [5]. For topologies on a semigroup S determined by idempotents from βS see [2].

An ultrafilter $p \in G^*$ is called *right cancellable* if, for any $q, r \in G^*$, $qp = rp$ implies $q = r$. For every countable group G, there exists an open and dense in G^* subset consisting of right cancellable ultrafilters [1, Theorem 8.10]. For characterizations and properties of right cancellable ultrafilters see [1, Chapter 8].

In this paper, given a countable group G, we classify up to homeomorphisms the topologies on G determined by right cancellable ultrafilters. To this end, we use the spaces $\mathrm{Seq}(q), q \in \omega^*$ defined in [6].

We denote by Seq the set of all words in the alphabet $\omega = \{0, 1, \dots\}$. Every ultrafilter $q \in \omega^*$ determines a topology on Seq in the following way: a subset $U \subseteq \mathrm{Seq}$ is open if and only if

$$(\forall t \in U)\{n \in \omega : tn \in U\} \in q.$$

The set Seq endowed with this topology is denoted by $\mathrm{Seq}(q)$.

Lemma 1. *Let $p, q \in \omega^*$. The spaces* $\mathrm{Seq}(p)$ *and* $\mathrm{Seq}(q)$ *are homeomorphic if and only if p and q are of the same type, i.e. there exists a bijection $f : \omega \to \omega$ such that $f^\beta(p) = q$.*

Proof: This is routine using [6, Theorem 1.1]. □

Theorem 1. *For every countable group G, the following statements hold:*

(i) *for every right cancellable ultrafilter $p \in G^*$, there exist $X \in p$ and a bijection $f : X \to \omega$ such that $G(p)$ is homeomorphic to* $\mathrm{Seq}(f^\beta(p))$;

(ii) *for every ultrafilter $q \in \omega^*$, there exists an injection $h : \omega \to G$ such that $h^\beta(q)$ is right cancellable and* $\mathrm{Seq}(q)$ *is homeomorphic to $G(h^\beta(q))$.*

Theorem 2. *Let G be a countable group, p_1 and p_2 be right cancellable ultrafilters from G^*. Then $G(p_1)$ and $G(p_2)$ are homeomorphic if and only if p_1 and p_2 are of the same type.*

Proof of Theorem 1: (i) We use the following criterion [1, Theorem 8.11]: an ultrafilter $p \in G^*$ is right cancellable if and only if there exists a family $\{P_g : g \in G\}$ of members of p such that $gP_g \cap hP_h = \emptyset$ for all distinct $g, h \in G$.

We need also the following description of topology of $G(p)$ from [4, p. 12] in the form suggested by the referee. Given an indexed family $\langle P_g\rangle_{g\in G}$ of members of p and $h \in G$, let $U(\langle P_g\rangle_{g\in G}, h, 0) = \{h\} \cup hP_h$, for $n \in \omega$ let

$$U(\langle P_g\rangle_{g\in G}, h, n+1) = \bigcup_{y\in U(\langle P_g\rangle_{g\in G},h,n)} yP_y,$$

and let $U(\langle P_g\rangle_{g\in G}, h) = \bigcup_{n=0}^{\infty} U(\langle P_g\rangle_{g\in G}, h, n)$. Then $U(\langle P_g\rangle_{g\in G}, h)$ is an open neighbourhood of h and, given any neighbourhood V of h, there is a choice of $\langle P_g\rangle_{g\in G}$ such that $U(\langle P_g\rangle_{g\in G}, h) \subseteq V$.

We choose $\langle P_g\rangle_{g\in G}$ such that each $P_g \in p$, $e \notin gP_g$ where e is the identity of G, and $gP_g \cap hP_h = \emptyset$ whenever $g \neq h$. Fix a bijection $f : P_e \to \omega$, put $X = P_e$, and let $q = f^\beta(p)$. We show that if U is an open neighbourhood of e in $G(p)$ and V is an open neighbourhood of the empty sequence in $\mathrm{Seq}(q)$, then there exist a clopen subset S of U and $\varphi : S \to V$ such that $\varphi[S]$ is clopen in $\mathrm{Seq}(q)$ and φ is a homeomorphism.

Since U is an open neighbourhood of e, choose $\langle Q_g\rangle_{g\in G}$ in P such that

$$U(\langle Q_g\rangle_{g\in G}, e) \subseteq U.$$

Since V is open in $\mathrm{Seq}(q)$, if $g \in P_e$ and $f(g) \in V$, then

$$f(g)^{-1}V = \{n \in \omega : f(g)n\} \in q,$$

so pick $R_g \in p$ such that $f[R_g] \subseteq f(g)^{-1}V$ (if $g \in G \setminus P_e$ or $f(g) \notin V$, let $R_g = G$). For $g \in G$, let $P'_g = P_g \cap Q_g \cap R_g$. We put $S = U(\langle P'_g\rangle_{g\in G}, e)$. Then every element $g \in S$, $g \neq e$ can be written as $g = x_0x_1 \dots x_n$, where $x_0 \in P'_e$ and $x_{k+1} \in P'_{x_0x_1\dots x_k}$ for each $k \in \{0, \dots, n-1\}$. Since $gP'_g \cap hP'_h = \emptyset$ whenever $g \neq h$ and $e \notin gP'_g$, this representation of g is unique.

Then we extend f to an injection $\varphi : S \to \mathrm{Seq}(q)$ defined by the rule: $\varphi(e) = \emptyset$ where $\emptyset$ is an empty sequence and, for every $g \in S$, $g \neq e$, $g = x_1x_2 \dots x_k$,

$$\varphi(g) = f(x_1)f(x_2)\dots f(x_k).$$

Given any $h \in S$, we have $U(\langle P'_g\rangle_{g\in G}, h) \subseteq S$ so S is open. Assume that $h \in \mathrm{cl}\, S$ and pick $m \in \omega$ such that $U(\langle P'_g\rangle_{g\in G}, h, m) \cap S \neq \emptyset$. Then there exist $y_0, y_1, \dots, y_m$ and $x_0, x_1, \dots, x_n$ such that

$$hy_0y_1 \dots y_m = x_0x_1 \dots x_n,\ y_0 \in P'_h,\ x_0 \in P'_e,$$
$$y_{i+1} \in P'_{hy_0\dots y_i},\ x_{j+1} = P_{x_0\dots x_j}$$

for all $i \in \{0, \dots, m-1\}$, $j \in \{0, \dots, n-1\}$. By the choice of $\langle P_g\rangle_{g\in G}$, we have $hy_0 \dots y_{m-1} = x_0x_1 \dots x_{n-1}$. Repeating this argument, we conclude that $h \in S$, so S is closed. To see that $\varphi[S]$ is clopen and φ is a homeomorphism, it suffices to notice that $\varphi(gh) = \varphi(g)\varphi(h)$ whenever $g \in S$, $h \in P'_g$, and repeat above arguments.

Let $g \in G(p)$, $t \in \mathrm{Seq}(q)$ and U, V be open neighbourhoods of g and t. The space $G(p)$ is homogeneous by definition, $\mathrm{Seq}(q)$ is homogeneous by [6, Theorem 1.2]. Hence, we can choose the clopen homeomorphic subset S and T such that $g \in S \subseteq U$, $t \in T \subseteq V$. To conclude the proof, we partition $G(p)$ and $\mathrm{Seq}(q)$ in ω clopen subsets $\{S_i : i \in \omega\}$ and $\{T_i : i \in \omega\}$ such that S_i and T_i are homeomorphic for each $i \in \omega$. We enumerate $G(p) = \{g_n : n \in \omega\}$, $\mathrm{Seq}(q) = \{t_n : n \in \omega\}$ and choose the clopen homeomorphic neighbourhoods S_0 and T_0 of g_0 and t_0 such that $G(p) \setminus S_0$ and $\mathrm{Seq}(q) \setminus T_0$ are infinite. Assume that we have chosen the clopen subsets $S_0, \dots, S_n$ and $T_0, \dots, T_n$ of $G(p)$ and $\mathrm{Seq}(q)$ such that $G(p)\setminus(S_0\cup\dots\cup S_n)$ and $\mathrm{Seq}(q)\setminus(T_0\cup\dots\cup T_n)$ are infinite, S_i, T_i are homeomorphic for each $i \in \{0, \dots, n\}$, and $S_i\cap S_j = \emptyset$, $T_i\cap T_j = \emptyset$ for all distinct $i, j \in \{0, \dots, n\}$. We choose the minimal $k \in \omega$ and $m \in \omega$ such that $g_k \notin S_0 \cup \dots \cup S_n$, $t_m \notin T_0 \cup \dots \cup T_n$. Then we choose the clopen homeomorphic neighbourhoods S_{n+1} and T_{n+1} of g_k and t_m such that $S_{n+1} \cap S_i = \emptyset$, $T_{n+1} \cap T_i = \emptyset$ for each $i \in \{0, \dots, n\}$, and $G(p) \setminus (S_0 \cup \dots \cup S_{n+1})$, $\mathrm{Seq}(q) \setminus (T_0 \cup \dots \cup T_{n+1})$ are infinite. After ω steps we get the partition $G(p) = \bigcup_{i\in\omega} S_i$, $\mathrm{Seq}(q) = \bigcup_{i\in\omega} T_i$.

(ii) We enumerate $G = \{g_n : n \in \omega\}$ with $g_0 = e$, put $K_n = \{g_i : i \leq n\}$ and choose inductively a sequence $(x_n)_{n\in\omega}$ in G such that the subsets $\{K_nx_n : n \in \omega\}$ are pairwise disjoint. We put $X = \{x_n : n \in \omega\}$ and note that $gX \cap X$ is finite for each $g \in G$, $g \neq e$. Given any ultrafilter $r \in G^*$ with $X \in r$, we can choose inductively a sequence $\langle R_n\rangle_{n\in\omega}$ of members of r such that the subsets $\{g_nR_n : n \in \omega\}$ are pairwise disjoint. By [1, Theorem 8.11], r is right cancellable.

We fix an arbitrary bijection $h : \omega \to X$ and put $p = h^\beta(q)$. Since p is right cancellable, we can choose $\langle P_n \rangle_{n\in\omega}$ such that each $P_n \in p$, $P_0 \subseteq X$, $e \notin g_n P_n$ and $g_n P_n \cap g_m P_m = \emptyset$ whenever $n \neq m$. Put $f = h^{-1}|_{P_0}$. Then $f^\beta(p) = q$ and (see proof of (i)) $G(p)$ is homeomorphic to Seq(q). $\square$

Proof of Theorem 2: By Theorem 1(i), there exist q_1 and q_2 from ω^* such that, for $i \in \{1, 2\}$, p_i and q_i are of the same type, and $G(p_i)$ is homeomorphic to Seq(q_i). By Lemma 1, Seq(q_1) and Seq(q_2) are homeomorphic if and only if q_1 and q_2 are of the same type. $\square$

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