VNR rings, Π-regular rings and annihilators

Roger Yue Chi Ming

Dedicated to Aurélie Fhal.

Abstract. Von Neumann regular rings, hereditary rings, semi-simple Artinian rings, self-injective regular rings are characterized. Rings which are either strongly regular or semi-simple Artinian are considered. Annihilator ideals and Π-regular rings are studied. Properties of WGP-injectivity are developed.

Keywords: von Neumann regular, Π-regular, annihilators, p-injective, YJ-injective, WGP-injective, semi-simple Artinian

Classification: 16D40, 16D50, 16E50, 16P20

Introduction

This paper is motivated by generalizations of injectivity, namely, p-injectivity and YJ-injectivity. Recall that

(a) a left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M ([7, p. 122], [21, p. 277], [22, p. 340] and [26]). p-injectivity is extended to YJ-injectivity in [34], [35];

(b) ${}_AM$ is YJ-injective if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and every left A-homomorphism of Aa^n into M extends to one of A into M ([5], [23], [35], [43]). YJ-injectivity is also called GP-injectivity in [14], [16].

We call here a left A-module M WGP-injective (weak GP-injective) if, for any $a \in A$, there exists a positive integer n such that every left A-homomorphism of Aa^n into M extends to one of A into M. (Here a^n may be zero.)

WGP-injectivity is studied in connection with VNR rings, strongly regular rings and Π-regular rings. YJ-injectivity is also considered in connection with hereditary rings and semi-simple Artinian rings.

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. A is called semi-primitive or semi-simple [15] (resp. (a) left non-singular; (b) right non-singular) if $J = 0$ (resp. (a) $Z = 0$; (b) $Y = 0$). For any left A-module M, $Z(M) = \{y \in M \setminus l(y)$ is an essential left ideal of $A\}$ is called the left singular submodule of M. Right singular submodules are defined similarly. ${}_AM$ is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$). A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal of A. Thus J, Z, Y are ideals of A.

A is called fully (resp. (a) fully left; (b) fully right) idempotent if every ideal (resp. (a) left ideal; (b) right ideal) of A is idempotent.

Recall that

(1) A is von Neumann regular if, for every $a \in A$, $a \in aAa$;
(2) A is Π-regular (resp. strongly Π-regular) if, for every $a \in A$, there exists a positive integer n such that $a^n \in a^n A^n a^n$ (resp. $a^n \in a^{n+1}A$);
(3) A is a P.I.-ring if A satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible.

Following C. Faith [7], A is called a VNR ring if A is von Neumann regular ring. A well-known theorem of E.P. Armendariz and J.W. Fisher asserts that a P.I.-ring is VNR if and only if it is fully idempotent.

A VNR ring is also called an absolutely flat ring in the sense that all left (right) A-modules are flat (M. Harada–M. Auslander). This characterization may be weakened as follows: A is VNR if and only if every cyclic singular left A-module is flat [30, Theorem 5] (cf. G.O. Michler's comment in Math. Reviews 80i#16021).

In [26], p-injective modules are introduced to study VNR rings and associated rings. Indeed, A is VNR if and only if every left (right) A-module is p-injective ([2], [23], [24], [26]). Flatness and p-injectivity are distinct concepts.

A is called left YJ-injective if ${}_AA$ is YJ-injective. YJ-injectivity is defined similarly on the right side. If A is right YJ-injective, then $Y = J$ [34, Proposition 1] (this is the origin of our notation). Also, A is right YJ-injective if and only if for every $0 \neq a \in A$ there exists a positive integer n such that Aa^n is a non-zero left annihilator [35, Lemma 3] (cf. also [16, Lemma 1], [23, p. 31], [43, Corollary 2]). In recent years, p-injectivity and YJ-injectivity have drawn the attention of many authors (cf. [2], [5], [7], [10], [14], [16], [18], [21], [22], [23], [24], [29], [43]).

A is called a left WGP-injective ring if ${}_AA$ is WGP-injective. WGP-injectivity is defined similarly on the right side. Note that [43, Theorem 3] ensures that A is a Π-regular ring if and only if every left (right) A-module is WGP-injective.

C. Faith proved that if every cyclic left A-module is either isomorphic to ${}_AA$ or injective, then A is either semi-simple Artinian or a left semi-hereditary simple domain [7, p. 65]. In [31, Theorem 1.5], the "p-injective analogue" of Faith's result is proposed (cf. [7, p. 65]). Following [31], we write "A is left PCP" if every cyclic left A-module is either isomorphic to ${}_AA$ or p-injective. Recall that a left ideal I of A is a maximal left annihilator if $I = l(S)$ for some non-zero subset S of A and for any left annihilator K which strictly contains I, $K = A$. In that case, for any $0 \neq s \in S$, $I = l(s)$. A maximal right annihilator is similarly defined.

1. WGP-injectivity, VNR rings and annihilators

K. Goodearl's book [9] has motivated a large number of papers on von Neumann regular rings and associated rings. Our first result extends semi-prime self-injective case.

Proposition 1.1. *Let A be a semi-prime right WGP-injective ring. Then C, the center of A, is VNR.*

Proof: If $u \in C$, $u^2 = 0$, then $(Au)^2 = Au^2 = 0$ implies that $u = 0$ (A being semi-prime), whence C must be reduced. Now for any $0 \neq c \in C$, since A is right WGP-injective, there exists a positive integer n such that every right A-homomorphism of c^nA into A extends to an endomorphism of A_A. Since C is reduced, we have $c^n \neq 0$. For any $v \in l(r(Ac^n))$, since $r(c^n) = r(l(r(c^n))) \subseteq r(v)$, we may define a right A-homomorphism $h : c^nA \to A$ by $h(c^na) = v(a)$ for all $a \in A$. Then there exists $y \in A$ such that $v = h(c^n) = yc^n \in Ac^n$. We have shown that $Ac^n = l(r(Ac^n))$. Clearly, $r(Ac) \subseteq r(Ac^n)$. If $w \in r(Ac^n)$, $(Acw)^n \subseteq (Ac)^n w = Ac^n w = 0$ which implies that $Acw = 0$ (A being semi-prime). Therefore $r(Ac^n) \subseteq r(Ac)$ which yields $r(Ac) = r(Ac^n)$. Then $c \in l(r(Ac)) = l(r(Ac^n)) = Ac^n$. If $n > 1$, $c = cdc$ for some $d \in A$. If $n = 1$, Ac is a left annihilator. In any case, Ac must be a left annihilator for each $c \in C$. Since $c^2 = 0$, Ac^2 is a left annihilator and we have just seen that, in that case, $c \in Ac^2$. Therefore $c = cbc$ for some $b \in A$. Now set $z = c^2b^3$. Then $czc = (cbc)bcbc = (cbc)bc = c$ and $c^2b = bc^2 = cbc = c$. For every $a \in A$, $bc^2a = ca = ac = abc^2 = c^2ab$ and hence $b^3c^2a = c^2ab^3$. Therefore $za = c^2b^3a = b^3c^2a = c^2ab^3 = ac^2b^3 = az$ which shows that $z \in C$. We have proved that C is VNR. □

An interesting corollary follows.

Corollary 1.1.1. *If A is a semi-prime* Π*-regular ring, then the center of A is VNR.*

Theorem 1.2. *The following conditions are equivalent for a ring A with center C:*

(1) *A is VNR;*
(2) *A is a semi-prime ring such that for each non-zero ideal T of C, A/AT is a VNR ring;*
(3) *A is a semi-prime right WGP-injective ring such that for each maximal left ideal M of C, A/AM is a VNR ring;*
(4) *A is a* Π*-regular left PCP ring;*
(5) *A is a left PCP ring containing a non-zero WGP-injective left ideal;*
(6) *A is a left PCP ring containing a non-zero WGP-injective right ideal;*
(7) *A is a left non-singular ring such that every proper finitely generated left ideal is either a maximal left annihilator or a flat left annihilator of an element of A.*

PROOF: It is clear that (1) implies (2) through (7).
Assume (2). We know that C is a reduced ring. For any $0 \neq t \in C$, $ACt^2 = At^2$ and since A/At^2 is VNR by hypothesis, then $t + At^2 = (t + At^2)(a + At^2)(t + At^2)$ for some $a \in A$ and $t - tat \in At^2$. Since $tat = at^2 \in At^2$, then $t \in At^2$ which yields $t = tdt$ for some $d \in A$. As in Proposition 1.1, with $z = t^2d^3$, we have $z \in C$ and $t = tzt$. Therefore C is VNR and for any maximal ideal M of C, A/AM is a VNR ring by hypothesis. Thus (2) implies (1) by [1, Theorem 3].

(3) implies (1) by [1, Theorem 3] and Proposition 1.1. (4) implies either (5) or (6).

Assume (5). Since A is left PCP, A is either VNR or a simple domain [31, Theorem 1.5]. In case A is a simple domain, let I be a non-zero left ideal of A which is WGP-injective. For any $0 \neq d \in I$, there exists a positive integer n such that every left A-homomorphism of Ad^n into I extends to one of A into I. Let $j : Ad^n \to I$ denote the natural injection. Then $d^n = j(d^n) = d^n y$ for some $y \in I$. Since A is a domain, $1 = y \in I$ which yields $I = A$. For any $0 \neq b \in A$, there exists a positive integer n such that every left A-homomorphism of Ab^m into A extends to an endomorphism of ${}_AA$. Define $g : Ab^m \to A$ by $g(ab^m) = a$ for all $a \in A$. Then $1 = g(b^m) = b^m z$ for some $z \in A$. This shows that every non-zero element of A is right invertible (and hence invertible) in A. In that case, A is a division ring. Thus (5) implies (1).

Similarly, (6) implies (1).

Assume (7). Suppose there exists a principal left ideal P of A which is not the flat left annihilator of an element of A. Then $P \neq 0$, $P \neq A$, and $P = l(u)$, $u \in A$, is a maximal left annihilator. P cannot be essential in A (because $Z = 0$). There exists $0 \neq c \in A$ such that $P \cap Ac = 0$ and $F = P \oplus Ac$ is a finitely generated left ideal of A. If $F \neq A$, then F is a proper left annihilator of an element in any case. Now $P \subset F \subset A$ (strict inclusion) which contradicts the maximality of P. Therefore $F = A$ and P is a direct summand of ${}_AA$ which contradicts our original hypothesis. We have proved that every principal left ideal of A must be a flat left annihilator of an element of A. For any $0 \neq a \in A$, $Aa = l(v)$, $v \in A$, in any case. Now $Av \approx A/l(v)$ implies that A/Aa is a finitely related flat left A-module and hence projective [4, p. 459]. Therefore ${}_AAa$ is a direct summand of ${}_AA$. Thus (7) implies (1). □

Singular modules play an important role in ring theory [7, p. 180]. For an exhaustive study of non-singular rings and modules, consult the standard reference [8]. Rings whose singular right modules are injective (noted right SI-rings) are introduced and studied by K. Goodearl who proved that right SI-rings are right hereditary (cf. for example [2]).

Indeed, it is sufficient that all divisible singular right A-modules are injective for A to be right hereditary (cf. [31, Theorem 2.4]). We know that if A is right non-singular, for any injective right A-module M, the singular submodule $Z(M)$ is injective [25, Theorem 4]. Also if A is right self-injective regular, for any essentially finitely generated right A-module M, $Z(M)$ is a direct summand of M [41, Corollary 10].

We now give two examples of quasi-Frobenius rings which are neither hereditary nor VNR.

Example 1. If A denotes the rings of integers modulo 4, then A is also a commutative principal ideal quasi-Frobenius ring which is not hereditary, VNR.

Example 2. Let K denote a field, A the commutative K-algebra with the basis 1, a, b, c and the multiplication $1r = r1 = r$ for all $r \in A$, $ab = ba = 0$, $a^2 = b^2 = c$, $ac = ca = bc = cb = c^2 = 0$. If J stands for the Jacobson radical of A, we have $J^2 = \mathrm{Soc}(A) = cA$ and A is a quasi-Frobenius ring but A/J^2 is not quasi-Frobenius. Consequently, A is not a principal ideal, hereditary, VNR ring.

Proposition 1.3. *The following conditions are equivalent:*

(1) *A is a right hereditary ring;*
(2) *any right ideal of A is either projective or a p-injective right annihilator;*
(3) *any right ideal of A is either projective or a YJ-injective right annihilator.*

Proof: It is clear that (1) implies (2) while (2) implies (3).
Assume (3). Suppose that $Y \neq 0$. If $0 \neq y \in Y$, there exists a complement right ideal K of A such that $L = yA \oplus K$ is an essential right ideal of A. If L_A is projective, then so is yA_A which implies that $r(y)$ is a direct summand of A_A. But $r(y)$ is an essential right ideal of A which yields $r(y) = A$, whence $y = 0$, a contradiction! Therefore L is YJ-injective right annihilator. Then yA_A is YJ-injective (being a direct summand of L_A). There exists a positive integer n such that $y^n \neq 0$ and any right A-homomorphism of y^nA into yA extends to one of A into yA. Let $j : y^nA \to yA$ be the inclusion map. There exists $w \in A$ such that $y^n = j(y^n) = ywy^n$, $w \in A$. Now $y^nA \cap r(yw) = 0$ which implies that $y^n = 0$ (because $yw \in Y$). This contradiction proves that $Y = 0$. For any right ideal R of A, there exists a complement right ideal C of A such that $E = R \oplus C$ is an essential right ideal of A. If E is a YJ-injective right annihilator, we have $E = A$ (in as much as $Y = 0$). In any case, R_A is projective and (3) implies (1). □

The next result seems to be new.

Proposition 1.4. *If A is left duo, then either A is a left non-singular ring or $Z \cap J \neq 0$*

Proof: Suppose that $Z \neq 0$ and $Z \cap J = 0$. Since $Z \neq 0$, there exists $0 \neq z \in Z$ such that $z^2 = 0$ [29, Lemma 2.1]. Then $(Az)^2 = AzAz \subseteq Az^2 = 0$ implies that $Az \subseteq J$ (every nil left ideal of A is contained in J). Therefore $z \in Z \cap J = 0$ a contradiction! We have shown that either $Z = 0$ or $Z \cap J \neq 0$. □

Corollary 1.4.1. *If A is left duo, left WGP-injective, and $Z \cap J = 0$, then A is strongly regular.*

PROOF: By Proposition 1.4, $Z = 0$. Since A is left duo, A is reduced (cf. [28, Lemma 1]). Then, A being left WGP-injective, it is left YJ-injective and we know that a reduced left YJ-injective ring is strongly regular [35, Lemma 5]. □

A P.I.-ring whose essential left ideals are idempotent needs not be even semi-prime, as shown by the following example.

Example 3. If A denotes the 2×2 upper triangular matrix ring over a field, A is Π-regular, P.I.-ring whose essential one-sided ideals are idempotent but A is not semi-prime (the Jacobson radical J of A is non-zero with $J^2 = 0$).

Proposition 1.5. *Let A be a P.I.-ring whose essential left ideals are idempotent. Then every prime factor ring of A is simple Artinian.*

PROOF: Let B denote a prime factor ring of A. Then every essential left ideal of B is idempotent. For any $0 \neq b \in B$, set $T = BbB$. Let K be a complement left subideal of T such that $L = Bb \oplus K$ is essential in ${}_BT$. Since ${}_BT$ is essential in ${}_BB$ (B being prime), then ${}_BL$ is essential in ${}_BB$. Now $L = L^2$ and $b \in L^2$. If

$$b = \sum_{i=1}^{n} (b_i b + k_i)(d_i b + c_i), \ b_i, d_i \in B, \ k_i, c_i \in K,$$

then

$$b - \sum_{i=1}^{n} (b_i b + k_i) d_i b = \sum_{i=1}^{n} (b_i b + k_i) c_i \in Bb \cap K = 0.$$

Now $b \in T$, $k_i \in T$ and since T is an ideal of B, then $b \in Tb$ and hence $Bb = (Bb)^2$ which proves that B is a fully left idempotent ring and hence A is a strongly Π-regular ring which is therefore Π-regular [20, Proposition 23.4]. Then every non-zero-divisor of B is invertible in B and B coincides with its classical left (and right) quotient ring, whence B is a simple Artinian ring by a theorem of E.C. Posner [17, Theorem]. □

As usual, A is called a right Kasch ring if every maximal right ideal of A is a right annihilator. We propose some characterizations of semi-simple Artinian rings.

Theorem 1.6. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*
(2) *A is a right Kasch ring which is right non-singular;*
(3) *A is a right Kasch ring whose simple right modules are either YJ-injective or projective;*

(4) *A is a right Kasch ring whose simple left modules are YJ-injective;*
(5) *for every maximal right ideal M of A, $l(M) \nsubseteq J \cap Y$;*
(6) *A is a left p-injective ring whose maximal left ideals are principal projective.*

Proof: (1) implies (2) through (6) evidently.
If A is right Kasch, then for any maximal right ideal M of A, $l(M) \neq 0$. Then (2) implies (5) evidently.

Assume (3). Since every simple right A-module is either YJ-injective or projective, then $Y \cap J = 0$ [37, Propositon 8(1)]. Therefore (3) implies (5).

Assume (4). Since every simple left A-module is YJ-injective, then $J = 0$ [39, Lemma 1]. Therefore (4) implies (5).

Assume (5). Let M be a maximal right ideal of A. Since $l(M) \nsubseteq J \cap Y$, then either $l(M) \nsubseteq J$ or $l(M) \nsubseteq Y$. First suppose that $l(M) \nsubseteq J$. Then $l(M)$ contains a non-nilpotent element v. Now $M = r(v)$ and $vA \approx A/r(v)$ is a minimal right ideal of A. Since v is non-nilpotent, vA is a direct summand of A_A. Therefore vA is a projective right A-module which implies that $M = r(v)$ is a direct summand of A_A. Now suppose that $l(M) \nsubseteq Y$. Then there exists $u \in l(M)$, $u \notin Y$. Therefore $r(u)$ is not an essential right ideal of A and $M = r(u)$ is a direct summand of A_A. In any case, every maximal right ideal of A is a direct summand of A_A and hence (5) implies (1).

Assume (6). Let M be a maximal left ideal of A. Then $M = Ab$, $b \in A$ and $l(b)$ is a direct summand of ${}_AA$. Now $l(b) = Ae$, $e = e^2 \in A$, $Ae = l(u)$, where $u = 1 - e$. But $M \approx A/l(b) = A/l(u) \approx Au$ and since A is left p-injective, any left ideal of A which is isomorphic to a direct summand of ${}_AA$ is itself a direct summand of ${}_AA$. It follows that ${}_AM$ is a direct summand of ${}_AA$. Thus (6) implies (1). □

We now give conditions for Π-regularity.

Proposition 1.7. *Let A be a ring satisfying the following conditions:* (a) *every simple right A-module is flat;* (b) *for every $a \in A$, there exists a positive integer n such that Aa^n is a projective left A-module* (a^n *may be zero*). *Then A is* Π*-regular.*

Proof: Let $F = \sum_{i=1}^{n} y_i A$, $y_i \in A$, be a finitely generated proper right ideal of A, M a maximal right ideal of A containing F. Since $0 \to M \to A \to A/M \to 0$ is an exact sequence of right A-modules with A free and A/M_A is flat, there exists a right A-homomorphism $g : A \to M$ such that $g(y_i) = y_i$ for all i, $1 \leq i \leq n$ [4, Proposition 2.2]. If $g(1) = u \in M$, then for every $b \in F$, $b = \sum_{i=1}^{n} y_i b_i$, $b_i \in A$, $g(b) = g(1)b = ub$ and $g(b) = \sum_{i=1}^{n} g(y_i) b_i = \sum_{i=1}^{n} y_i b_i = b$. Therefore $(1-u)b = 0$ which yields $(1-u)F = 0$, whence F has a non-zero left annihilator (because $M \neq A$). By [3, Theorem 5.4], any finitely generated projective submodule of a projective left A-module is a direct summand. By hypothesis, for every $a \in A$, there exists a positive integer m such that Aa^m is a projective left A-module.

Therefore Aa^m is a direct summand of ${}_AA$. In that case, every left A-module is WGP-injective by definition. By [43, Theorem 3], A is Π-regular. $\square$

The proof of Proposition 1.7 together with [43, Theorem 9] ensures the validity of the following result.

Proposition 1.8. *A is VNR if and only if every simple right A-module is flat and for each $a \in A$, $a \neq 0$, there exists a positive integer n such that Aa^n is a non-zero projective left A-module.*

The next result connects injectivity and projectivity.

Theorem 1.9. *The following conditions are equivalent:*

(1) *A is a left self-injective VNR ring;*
(2) *every simple right A-module is flat and for each finitely generated left A-module M, $M/Z(M)$ is a projective left A-module.*

Proof: Assume (1). Since $Z = 0$, we have $Z(M/Z(M)) = 0$ for each finitely generated left A-module M by [25, Theorem 4]. Therefore $M/Z(M)$ is a finitely generated non-singular left A-module and by [41, Corollary 6], ${}_AM/Z(M)$ is projective. Therefore (1) implies (2).

Assume (2). Then every finitely generated proper right ideal of A has a non-zero left annihilator as in Proposition 1.8. Since ${}_AA/Z$ is projective, ${}_AZ$ is a direct summand of ${}_AA$, whence $Z = 0$ (in as much as Z cannot contain a non-zero idempotent). Let E denote the injective hull of ${}_AA$. Then E is the maximal left quotient ring of A and E is a left self-injective regular ring. If $y \in E$, then $C = A + Ay$ is a finitely generated non-singular left A-module which is projective by hypothesis. By [3, Theorem 5.4], ${}_AA$ is a direct summand of ${}_AC$. Since ${}_AA$ is essential in ${}_AC$, then $A = C$ which proves that $A = E$ is a left self-injective regular ring and hence (2) implies (1). $\square$

2. CM-rings, ELT and MELT rings

Recall that (1) A is a left CM-ring if, for any maximal essential left ideal M of A (if it exists), every complement left subideal of M is an ideal of M; (2) A is ELT (resp. MELT) if every essential left ideal (resp. maximal essential left ideal, if it exists) of A is an ideal of A. ERT and MERT rings are similarly defined on the right side. If A is a VNR ring, then the above four conditions are equivalent (cf. [2]). Also a MELT fully left idempotent ring is VNR [2, Theorem 3.1]. Note that A is ELT left self-injective if and only if every left ideal of A is quasi-injective [11, Theorem 2.3].

Left CM-rings generalize left uniform rings, Cozzen's domains, left PCI rings [7, p. 65] and semi-simple Artinian rings.

The rings considered in the next two propositions need not be VNR.

Proposition 2.1. *Let A be a left CM-ring whose simple singular left modules are YJ-injective. Then $Y = J = 0$.*

PROOF: Suppose that A is not semi-prime. Then there exists $0 \neq t \in A$ such that $(AtA)^2 = 0$. Let C be a complement left ideal of A such that $L = AtA \oplus C$ is an essential left ideal of A. If $L = A$, $AtA = (AtA)^2 = 0$ which contradicts $t \neq 0$. Therefore $L \neq A$. Let M be a maximal left ideal of A containing L. Then $CM \subseteq C$ (since A is left CM) which implies that $Ct \subseteq C \cap AtA = 0$ and hence $C \subseteq l(t)$ which yields $L \subseteq l(t)$. Therefore $t \in Z$. Now $Ata \subseteq J$ (AtA being a nil ideal of A) which implies that $AtA \subseteq J \cap Z$. Since every simple singular left A-module is YJ-injective, by [37, Proposition 8], $Z \cap J = 0$. Therefore $t = 0$, again a contradiction! This proves that A must be semi-prime. Now a semi-prime ring whose singular simple left modules are YJ-injective must be semi-primitive and right non-singular (cf. [40, Proposition 2]). □

Proposition 2.2. *Let A be a left CM-ring whose simple singular one-sided modules are YJ-injective. Then A is a biregular ring.*

PROOF: By Proposition 2.1, A is a semi-prime ring. Since every simple singular right A-module is YJ-injective, then $Z = 0$ [40, Proposition 2]. Since A is left non-singular, left CM, by [32, Lemma 1.1], A is either semi-simple Artinian or reduced. In case A is reduced, by [40, Proposition 3], A is biregular. Therefore A must be a biregular ring. □

Proposition 2.3. *The following conditions are equivalent:*

(1) *A is either strongly regular or semi-simple Artinian;*
(2) *A is a MELT, left CM-ring whose simple singular left and right modules are YJ-injective;*
(3) *A is a semi-prime ELT left YJ-injective left CM-ring;*
(4) *A is a semi-prime ELT right YJ-injective left CM-ring.*

PROOF: Since ELT or MELT left CM-rings generalize semi-simple Artinian rings and left duo rings, (1) implies (2) through (4).
Assume (2). Since A is a left CM-ring whose simple singular left modules are YJ-injective, A is a semi-prime ring by Proposition 2.1. Since every simple singular right A-module is YJ-injective and A is semi-prime, we have $Z = 0$ by [40, Proposition 2]. Now A is left non-singular left CM which implies that A is either semi-simple Artinian or reduced [32, Lemma 1.1]. We consider the case when A is a reduced ring. Since every simple left A-module is YJ-injective, A is biregular by [40, Proposition 3]. Therefore A is a MELT fully left (and right) idempotent ring which is therefore VNR by [2, Theorem 3.1]. Since A is reduced, A is strongly regular. We have proved that (2) implies (1).

Assume (3). If $Z \neq 0$, there exists $0 \neq z \in Z$ such that $z^2 = 0$ [29, Lemma 2.1]. Since $l(z)$ is an ideal of A, $A_z \subseteq l(z)$ implies that $AzA \subseteq l(z)$, whence $(Az)^2 = 0$. Since A is semi-prime, we have $z = 0$. This contradiction proves that $Z = 0$.

Then A is a left non-singular, left CM-ring which is either semi-simple Artinian or reduced [32, Lemma 1.1]. If A is reduced then, since A is left YJ-injective, A is strongly regular [34, Proposition1(2)]. Thus (3) implies (1).

Similarly, (4) implies (1). □

A well-known generalization of a right hereditary ring is a right p.p. ring (also called a right Rickartian ring). Reduced right p.p. rings are characterized in [20, Proposition 7.3].

Remark. [20, Proposition 7.3] coincides with [36, Theorem 2].

If every cyclic semi-simple left A-module is p-injective, then A is VNR [27, Theorem 9].

Question 1. *Does the above result hold if "p-injective" is replaced by "flat"?*

We know that if every simple left A-module is p-injective, then A is fully left idempotent (cf. [13, Reference [58], p. 367] or [22, p. 340]).

Question 2. *Is A fully left idempotent if every simple right A-module is flat?*

We add a weaker conjecture:

Question 3. *Is A semi-primitive if every simple right A-module is flat? (The answer is positive if "simple" is replaced by "cyclic semi-simple".)*

Acknowledgment. The author would like to thank the referee for helpful comments and suggestions leading to this improved version of the paper.

References

[1] Armendariz E.P., Fisher J.W., Steinberg S.A., *Central localizations of regular rings*, Proc. Amer. Math. Soc. **46** (1974), 315–321.
[2] Baccella G., *Generalized V-rings and von Neumann regular rings*, Rend. Sem. Mat. Univ. Padova **72** (1984), 117–133.
[3] Bass H., *Finitistic dimension and homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.
[4] Chase S.U., *Direct product of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473.
[5] Ding N.Q., Chen, J.L., *Rings whose simple singular models are YJ-injective*, Math. Japon. **40** (1994), 191–195.
[6] Faith C., *Algebra II: Ring Theory*, Grundlehren Math. Wiss. **191** (1976).
[7] Faith C., *Rings and things and a fine array of twentieth century associative algebra*, AMS Math. Survey and Monographs **65** (1999).
[8] Goodearl K.R., *Ring Theory. Nonsingular Rings and Modules*, Pure and Applied Mathematics, no. 33, Marcel Dekker, New York, 1976.
[9] Goodearl K.R., *Von Neumann Regular Rings*, Pitman, Boston, 1979.
[10] Hirano Y., *On non-singular p-injective rings*, Publ. Math. **38** (1994), 455–461.
[11] Jain S.K., Mohamed S.H., Singh S., *Rings in which every right ideal is quasiinjective*, Pacific J. Math. **31** (1969), 73–79.
[12] Jans J.P., *Projective injective modules*, Pacific J. Math. **9** (1959), 1103–1108.
[13] Kasch F., *Modules and Rings*, London Mathematical Society Monographs, 17, Academic Press, London-New York, 1982.

[14] Kim N.K., Nam S.B., Kim J.Y., *On simple singular GP-injective modules*, Comm. Algebra **27** (1999), 2087–2096.

[15] Michler, G.O., Villamayor O.E., *On rings whose simple modules are injective*, J. Algebra **25** (1973), 185–201.

[16] Nam S.B., Kim N.K., Kim J.Y., *On simple GP-injective modules*, Comm. Algebra **23** (1995), 5437–5444.

[17] Posner E.C., *Prime rings satisfying a polynomial identity*, Proc. Amer. Math. Soc. **11** (1960), 180–184.

[18] Puninski G., Wisbauer R., Yousif M.F., *On p-injective rings*, Glasgow Math. J. **37** (1995), 373–378.

[19] Storrer H.H., *A note on quasi-Frobenius rings and ring epimorphism*, Canad. Math. Bull. **12** (1969), 287–292.

[20] Tuganbaev A., *Rings Close to Regular*, Mathematics and its Applications, 545, Kluwer Academic Publishers, Dordrecht, 2002.

[21] Tuganbaev A., *Max rings and V-rings*, Handbook of Algebra, vol. 3, North-Holland, Amsterdam, 2003.

[22] Wisbauer R., *Foundations of Module and Ring Theory*, Gordon and Breach, New York, 1991.

[23] Xue Wei Min, *A note on* YJ-*injectivity*, Riv. Mat. Univ. Parma (6) **1** (1998), 31–37.

[24] Yousif M.F., *On SI-modules*, Math. J. Okayama Univ. **28** (1986), 133–146.

[25] Yue Chi Ming R., *A note on singular ideals*, Tôhoku Math. J. **21** (1969), 337–342.

[26] Yue Chi Ming R., *On von Neumann regular rings*, Proc. Edinburgh Math. Soc. **19** (1974), 89–91.

[27] Yue Chi Ming R., *On von Neumann regular rings II*, Math. Scan. **39** (1976), 167–170.

[28] Yue Chi Ming R., *On annihilator ideals*, Math. J. Okayama Univ. **19** (1976), 51–53.

[29] Yue Chi Ming R., *On von Neumann regular rings III*, Monatsh. Math. **86** (1978), 251–257.

[30] Yue Chi Ming R., *On generalizations of V-rings and regular rings*, Math. J. Okayama Univ. **20** (1978), 123–129.

[31] Yue Chi Ming R., *On injective and p-injective modules*, Riv. Mat. Univ. Parma (4) **7** (1981), 187–197.

[32] Yue Chi Ming R., *On regular and continuous rings II*, Kyungpook Math. J. **21** (1981), 171–178.

[33] Yue Chi Ming R., *On quasi-Frobenius rings and Artinian rings*, Publ. Inst. Math. (Beograd) **33 (47)** (1983), 239–245.

[34] Yue Chi Ming R., *On regular rings and self-injective rings II*, Glasnik Mat. **18 (38)** (1983), 221–229.

[35] Yue Chi Ming R., *On regular rings and Artinian rings II*, Riv. Mat. Univ. Parma (4) **11** (1985), 101–109.

[36] Yue Chi Ming R., *On von Neumann regular rings XI*, Bull. Math. Soc. Sci. Math. Roumanie **30 (76)** (1986), 371–379.

[37] Yue Chi Ming R., *On injectivity and p-injectivity*, J. Math. Kyoto Univ. **27** (1987), 439–452.

[38] Yue Chi Ming R., *A note on injective rings*, Hokkaido Math. J. **21** (1992), 231–238.

[39] Yue Chi Ming R., *On p-injectivity and generalizations*, Riv. Mat. Univ. Parma (5) **5** (1996), 183–188.

[40] Yue Chi Ming R., *On YJ-injectivity and VNR rings*, Bull. Math, Soc. Sci. Math. Roumanie **46 (94)** (2003), 87–97.

[41] Zelmanowitz J., *Injective hulls of torsionfree modules*, Canad. J. Math. **27** (1971), 1094–1101.

[42] Zhang J.L., *Fully idempotent rings whose every maximal left ideal is an ideal*, Chinese Sci. Bull. **37** (1992), 1065–1068.
[43] Zhang J.L., Wu J., *Generalizations of principal injectivity*, Algebra Colloq. **6** (1999), 277–282.

Université Paris VII-Denis Diderot, UFR Maths-UMR 9994 CNRS, 2, Place Jussieu, 75251 Paris Cedex 05, France

(*Received* July 7, 2008, *revised* December 4, 2008)