Interpolation of κ-compactness and PCF

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Abstract. We call a topological space κ-compact if every subset of size κ has a complete accumulation point in it. Let $\Phi(\mu,\kappa,\lambda)$ denote the following statement: $\mu < \kappa < \lambda = \mathrm{cf}(\lambda)$ and there is $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ such that $|\{\xi : |S_\xi \cap A| = \mu\}| < \lambda$ whenever $A \in [\kappa]^{<\kappa}$. We show that if $\Phi(\mu,\kappa,\lambda)$ holds and the space X is both μ-compact and λ-compact then X is κ-compact as well. Moreover, from PCF theory we deduce $\Phi(\mathrm{cf}(\kappa),\kappa,\kappa^+)$ for every singular cardinal κ. As a corollary we get that a linearly Lindelöf and $\aleph_\omega$-compact space is uncountably compact, that is κ-compact for all uncountable cardinals κ.

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We start by recalling that a point x in a topological space X is said to be a *complete accumulation point* of a set $A \subset X$ iff for every neighbourhood U of x we have $|U \cap A| = |A|$. We denote the set of all complete accumulation points of A by A°.

It is well-known that a space is compact iff every infinite subset has a complete accumulation point. This justifies to call a space κ-compact if its every subset of cardinality κ has a complete accumulation point. Now, let κ be a singular cardinal and $\kappa = \sum\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ with $\kappa_\alpha < \kappa$ for each $\alpha < \mathrm{cf}(\kappa)$. Clearly, if a space X is both κ_α-compact for all $\alpha < \mathrm{cf}(\kappa)$ and $\mathrm{cf}(\kappa)$-compact then X is κ-compact as well. This trivial "extrapolation" property of κ-compactness (for singular κ) implies that in the above characterization of compactness one may restrict to subsets of regular cardinality.

The aim of this note is to present a new "interpolation" result on κ-compactness, i.e. one in which $\mu < \kappa < \lambda$ and we deduce κ-compactness of a space from its μ- and λ-compactness. Again, this works for singular cardinals κ and the proof uses non-trivial results from Shelah's PCF theory.

Definition 1. Let κ,λ,μ be cardinals, then $\Phi(\mu,\kappa,\lambda)$ denotes the following statement: $\mu < \kappa < \lambda = \mathrm{cf}(\lambda)$ and there is $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ such that $|\{\xi : |S_\xi \cap A| = \mu\}| < \lambda$ whenever $A \in [\kappa]^{<\kappa}$.

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As we can see from our next theorem, this property Φ yields the promised interpolation result for κ-compactness.

Theorem 2. *Assume that $\Phi(\mu,\kappa,\lambda)$ holds and the space X is both μ-compact and λ-compact. Then X is κ-compact as well.*

Proof: Let Y be any subset of X with $|Y| = \kappa$ and, using $\Phi(\mu,\kappa,\lambda)$, fix a family $\{S_\xi : \xi < \lambda\} \subset [Y]^\mu$ such that $|\{\xi : |S_\xi \cap A| = \mu\}| < \lambda$ whenever $A \in [Y]^{<\kappa}$. Since X is μ-compact we may then pick a complete accumulation point $p_\xi \in {S_\xi}^\circ$ for each $\xi < \lambda$.

Now we distinguish two cases. If $|\{p_\xi : \xi < \lambda\}| < \lambda$ then the regularity of λ implies that there is $p \in X$ with $|\{\xi < \lambda : p_\xi = p\}| = \lambda$. If, on the other hand, $|\{p_\xi : \xi < \lambda\}| = \lambda$ then we can use the λ-compactness of X to pick a complete accumulation point p of this set. In both cases the point $p \in X$ has the property that for every neighbourhood U of p we have $|\{\xi : |S_\xi \cap U| = \mu\}| = \lambda$.

Since $S_\xi \cap U \subset Y \cap U$, this implies using $\Phi(\mu,\kappa,\lambda)$ that $|Y \cap U| = \kappa$, hence p is a complete accumulation point of Y, hence X is indeed κ-compact. $\square$

Our following result implies that if $\Phi(\mu,\kappa,\lambda)$ holds then κ must be singular.

Theorem 3. *If $\Phi(\mu,\kappa,\lambda)$ holds then we have* $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$.

Proof: Assume that $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ witnesses $\Phi(\mu,\kappa,\lambda)$ and fix a strictly increasing sequence of ordinals $\eta_\alpha < \kappa$ for $\alpha < \mathrm{cf}(\kappa)$ that is cofinal in κ. By the regularity of $\lambda > \kappa$ there is an ordinal $\xi < \lambda$ such that $|S_\xi \cap \eta_\alpha| < \mu$ holds for each $\alpha < \mathrm{cf}(\kappa)$. But this S_ξ must be cofinal in κ, hence from $|S_\xi| = \mu$ we get $\mathrm{cf}(\mu) \le \mathrm{cf}(\kappa) \le \mu$.

Now assume that we had $\mathrm{cf}(\mu) < \mathrm{cf}(\kappa)$ and set $|S_\xi \cap \eta_\alpha| = \mu_\alpha$ for each $\alpha < \mathrm{cf}(\kappa)$. Our assumptions then imply $\mu^* = \sup\{\mu_\alpha : \alpha < \mathrm{cf}(\kappa)\} < \mu$ as well as $\mathrm{cf}(\kappa) < \mu$, contradicting that $S_\xi = \bigcup\{S_\xi \cap \eta_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ and $|S_\xi| = \mu$. This completes our proof. $\square$

According to theorem 3 the smallest cardinal μ for which $\Phi(\mu,\kappa,\lambda)$ may hold for a given singular cardinal κ is $\mathrm{cf}(\kappa)$. Our main result says that this actually does happen with the natural choice $\lambda = \kappa^+$.

Theorem 4. *For every singular cardinal κ we have* $\Phi(\mathrm{cf}(\kappa),\kappa,\kappa^+)$.

Proof: We shall make use of the following fundamental result of Shelah from his PCF theory: There is a strictly increasing sequence of length $\mathrm{cf}(\kappa)$ of regular cardinals $\kappa_\alpha < \kappa$ cofinal in κ and such that in the product

$$\mathbb{P} = \prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$$

there is a scale $\{f_\xi : \xi < \kappa^+\}$ of length κ^+. (This is Main Claim 1.3 on p. 46 of [2].)

Spelling it out, this means that the κ^+-sequence $\{f_\xi : \xi < \kappa^+\} \subset \mathbb{P}$ is increasing and cofinal with respect to the partial ordering $<^*$ of eventual dominance on $\mathbb{P}$. Here for $f, g \in \mathbb{P}$ we have $f <^* g$ iff there is $\alpha < \mathrm{cf}(\kappa)$ such that $f(\beta) < g(\beta)$ whenever $\alpha \leq \beta < \mathrm{cf}(\kappa)$.

Now, to show that this implies $\Phi(\mathrm{cf}(\kappa), \kappa, \kappa^+)$, we take the set $H = \bigcup\{\{\alpha\} \times \kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ as our underlying set. Note that then $|H| = \kappa$ and every function $f \in \mathbb{P}$, construed as a set of ordered pairs (or in other words: identified with its graph) is a subset of H of cardinality $\mathrm{cf}(\kappa)$.

We claim that the scale sequence $\{f_\xi : \xi < \kappa^+\} \subset [H]^{\mathrm{cf}(\kappa)}$ witnesses $\Phi(\mathrm{cf}(\kappa), \kappa, \kappa^+)$. Indeed, let A be any subset of H with $|A| < \kappa$. We may then choose $\alpha < \mathrm{cf}(\kappa)$ in such a way that $|A| < \kappa_\alpha$. Clearly, then there is a function $g \in \mathbb{P}$ such that we have $A \cap (\{\beta\} \times \kappa_\beta) \subset \{\beta\} \times g(\beta)$ whenever $\alpha \leq \beta < \mathrm{cf}(\kappa)$. Since $\{f_\xi : \xi < \kappa^+\}$ is cofinal in $\mathbb{P}$ w.r.t. $<^*$, there is a $\xi < \kappa^+$ with $g <^* f_\xi$ and obviously we have $|A \cap f_\eta| < \mathrm{cf}(\kappa)$ whenever $\xi \leq \eta < \kappa^+$. □

Note that the above proof actually establishes the following more general result: If for some increasing sequence of regular cardinals $\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ that is cofinal in κ there is a scale of length $\lambda = \mathrm{cf}(\lambda)$ in the product $\prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ then $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$ holds.

Before giving some further interesting application of the property $\Phi(\mu, \kappa, \lambda)$, we present a result that enables us to "lift" the first parameter $\mathrm{cf}(\kappa)$ in Theorem 4 to higher cardinals.

Theorem 5. *If $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$ holds for some singular cardinal κ then we also have $\Phi(\mu, \kappa, \lambda)$ whenever $\mathrm{cf}(\kappa) < \mu < \kappa$ with $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$.*

Proof: Let us put $\mathrm{cf}(\kappa) = \varrho$ and fix a strictly increasing and cofinal sequence $\{\kappa_\alpha : \alpha < \varrho\}$ of cardinals below κ. We also fix a partition of κ into disjoint sets $\{H_\alpha : \alpha < \varrho\}$ with $|H_\alpha| = \kappa_\alpha$ for each $\alpha < \varrho$.

Let us now choose a family $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\varrho$ that witnesses $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$. Since λ is regular, we may assume without any loss of generality that $|H_\alpha \cap S_\xi| < \varrho$ holds for every $\alpha < \varrho$ and $\xi < \lambda$. Note that this implies $|\{\alpha : H_\alpha \cap S_\xi \neq \emptyset\}| = \varrho$ for each $\xi < \lambda$.

Now take a cardinal μ with $\mathrm{cf}(\mu) = \varrho < \mu < \kappa$ and fix a strictly increasing and cofinal sequence $\{\mu_\alpha : \alpha < \varrho\}$ of cardinals below μ. To show that $\Phi(\mu, \kappa, \lambda)$ is valid, we may use as our underlying set $S = \bigcup\{H_\alpha \times \mu_\alpha : \alpha < \varrho\}$, since clearly $|S| = \kappa$.

For each $\xi < \lambda$ let us now define the set $T_\xi \subset S$ as follows:

$$T_\xi = \bigcup\{(S_\xi \cap H_\alpha) \times \mu_\alpha : \alpha < \varrho\}.$$

Then we have $|T_\xi| = \mu$ because $|\{\alpha : H_\alpha \cap S_\xi \neq \emptyset\}| = \varrho$. We claim that $\{T_\xi : \xi < \lambda\}$ witnesses $\Phi(\mu, \kappa, \lambda)$.

Indeed, let $A \subset S$ with $|A| < \kappa$. For each $\alpha < \rho$ let B_α denote the set of all first co-ordinates of the pairs that occur in $A \cap (H_\alpha \times \mu_\alpha)$ and set $B = \bigcup\{B_\alpha : \beta < \varrho\}$. Clearly, we have $B \subset \kappa$ and $|B| \le |A| < \kappa$, hence $|\{\xi : |S_\xi \cap B| = \varrho\}| < \lambda$.

Now, consider any ordinal $\xi < \lambda$ with $|S_\xi \cap B| < \varrho$. If $\langle \gamma, \delta \rangle \in (T_\xi \cap A) \cap (H_\alpha \times \mu_\alpha)$ for some $\alpha < \varrho$ then we have $\gamma \in S_\xi \cap B_\alpha$, consequently $H_\alpha \cap S_\xi \cap B \ne \emptyset$. This implies that

$$W = \{\alpha : (T_\xi \cap A) \cap (H_\alpha \times \mu_\alpha) \ne \emptyset\}$$

has cardinality $\le |S_\xi \cap B| < \varrho$. But for each $\alpha \in W$ we have

$$|T_\xi \cap (H_\alpha \times \mu_\alpha)| \le \varrho \cdot \mu_\alpha < \mu,$$

hence

$$T_\xi \cap A = \bigcup\{(T_\xi \cap A) \cap (H_\alpha \times \mu_\alpha) : \alpha \in W\}$$

implies $|T_\xi \cap A| < \mu$ as well. But this shows that $\{T_\xi : \xi < \lambda\}$ indeed witnesses $\Phi(\mu, \kappa, \lambda)$. □

Arhangel'skii has recently introduced and studied in [1] the class of spaces that are κ-compact for all uncountable cardinals κ and, quite appropriately, called them *uncountably compact*. In particular, he showed that these spaces are Lindelöf.

We recall that the spaces that are κ-compact for all uncountable *regular* cardinals κ have been around for a long time and are called linearly Lindelöf. Moreover, the question under what conditions is a linearly Lindelöf space Lindelöf is important and well-studied. Note, however, that a linearly Lindelöf space is obviously compact iff it is countably compact, i.e. ω-compact. This should be compared with our next result that, we think, is far from being obvious.

Theorem 6. *Every linearly Lindelöf and $\aleph_\omega$-compact space is uncountably compact hence, in particular, Lindelöf.*

PROOF: Let X be a linearly Lindelöf and $\aleph_\omega$-compact space. According to the (trivial) extrapolation property of κ-compactness that we mentioned in the introduction, X is κ-compact for all cardinals κ of uncountable cofinality. Consequently, it only remains to show that X is κ-compact whenever κ is a singular cardinal of countable cofinality with $\aleph_\omega < \kappa$.

But, according to theorems 4 and 5, we have $\Phi(\aleph_\omega, \kappa, \kappa^+)$ and X is both $\aleph_\omega$-compact and κ^+-compact, hence theorem 2 implies that X is κ-compact as well. □

Arhangel'skii gave in [1] the following surprising result which shows that the class of uncountably compact T_3-spaces is rather restricted: Every uncountably compact T_3-space X has a (possibly empty) compact subset C such that for every open set $U \supset C$ we have $|X \setminus U| < \aleph_\omega$. Below we show that in this result the T_3 separation axiom can be replaced by T_1 plus van Douwen's property wD, see e.g. 3.12 in [3]. Since uncountably compact T_3-spaces are normal, being also

Lindelöf, and the wD property is a very weak form of normality, this indeed is an improvement. For the convenience of the reader we recall that a space X has property wD iff every infinite closed discrete set A in X has an infinite subset B that expands to a discrete (in X) collection of open sets $\{U_x : x \in B\}$.

Definition 7. A topological space X is said to be κ-concentrated on its subset Y if for every open set $U \supset Y$ we have $|X \setminus U| < \kappa$.

So what we claim can be formulated as follows.

Theorem 8. *Every uncountably compact T_1 space X with the wD property is $\aleph_\omega$-concentrated on some (possibly empty) compact subset C.*

Proof: Let C be the set of those points $x \in X$ for which every neighbourhood has cardinality at least $\aleph_\omega$. First we show that C, as a subspace, is compact. Indeed, C is clearly closed in X, hence Lindelöf, so it suffices to show for this that C is countably compact.

Assume, on the contrary, that C is not countably compact. Then, as X is T_1, there is an infinite closed discrete $A \in [C]^\omega$. But then by the wD property there is an infinite $B \subset A$ that expands to a discrete (in X) collection of open sets $\{U_x : x \in B\}$. By the definition of C we have $|U_x| \geq \aleph_\omega$ for each $x \in B$.

Let $B = \{x_n : n < \omega\}$ be any one-to-one enumeration of B. Then for each $n < \omega$ we may pick a subset $A_n \subset U_{x_n}$ with $|A_n| = \aleph_n$ and set $A = \bigcup\{A_n : n < \omega\}$. But then $|A| = \aleph_\omega$ and A has no complete accumulation point, a contradiction.

Next we show that X is $\aleph_\omega$ concentrated on C. Indeed, let $U \supset C$ be open. If we had $|X \setminus U| \geq \aleph_\omega$ then any complete accumulation point of $X \setminus U$ is not in U but is in C, again a contradiction. □

The following easy result, that we add for the sake of completeness, yields a partial converse to theorem 8.

Theorem 9. *If a space X is κ-concentrated on a compact subset C then X is λ-compact for all cardinals $\lambda \geq \kappa$.*

Proof: Let $A \subset X$ be any subset with $|A| = \lambda \geq \kappa$. We claim that we even have $A^\circ \cap C \neq \emptyset$. Assume, on the contrary, that every point $x \in C$ has an open neighbourhood U_x with $|A \cap U_x| < \lambda$. Then the compactness of C implies $C \subset U = \bigcup\{U_x : x \in F\}$ for some finite subset F of C. But then we have $|A \cap U| < \lambda$, hence $|A \setminus U| = \lambda \geq \kappa$, contradicting that X is κ-concentrated on C. □

Putting all these theorems together we immediately obtain the following result.

Corollary 10. *Let X be a T_1 space with property wD that is $\aleph_n$-compact for each $0 < n < \omega$. Then X is uncountably compact if and only if it is $\aleph_\omega$-concentrated on some compact subset.*

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