Spectral analysis for rank one perturbations of diagonal operators in non-archimedean Hilbert space

Toka Diagana, George D. McNeal

Abstract. The paper is concerned with the spectral analysis for the class of linear operators $A = D_\lambda + X \otimes Y$ in non-archimedean Hilbert space, where D_λ is a diagonal operator and $X \otimes Y$ is a rank one operator. The results of this paper turn out to be a generalization of those results obtained by Diarra.

Keywords: spectral analysis, diagonal operator, rank one operator, eigenvalue, spectrum, non-archimedean Hilbert space

Classification: 44A35, 42A85, 42A75

1. Introduction

Let $(\mathbb{K}, |\cdot|)$ be a nontrivial complete non-archimedean valued field. Classic examples of such a field include, but not limited to, $(\mathbb{Q}_p, |\cdot|_p)$, the field of p-adic numbers, and $(\mathbb{C}_p, |\cdot|_p)$, the field of complex p-adic numbers [7], [14].

Fix once and for all a sequence $\omega = (\omega_j)_{j\in\mathbb{N}} \subset \mathbb{K}$ of nonzero terms. The space $c_0(\omega, \mathbb{N}, \mathbb{K})$ is defined as the set of all $u = (u_j)_{j\in\mathbb{N}}$, $u_j \in \mathbb{K}$ for all $j \in \mathbb{N}$ such that $\omega_j u_j^2$ tends to 0 in $\mathbb{K}$ as $j \to \infty$. Equivalently,

$$c_0(\omega, \mathbb{N}, \mathbb{K}) := \left\{ u = (u_j)_{j\in\mathbb{N}} : u_j \in \mathbb{K},\ \forall\, j \in \mathbb{N},\ \ \lim_{j\to\infty} |u_j||\omega_j|^{1/2} = 0 \right\}.$$

The space $c_0(\omega, \mathbb{N}, \mathbb{K})$ equipped with the norm defined for each $u = (u_j)_{i\in\mathbb{N}} \in c_0(\omega, \mathbb{N}, \mathbb{K})$ by

$$\|u\| = \sup_{j\in\mathbb{N}} |u_j||\omega_j|^{1/2}$$

is a non-archimedean Banach space [7], [14].

Similarly, an inner product (symmetric, non-degenerate, bilinear form) is also defined on $c_0(\omega, \mathbb{N}, \mathbb{K})$ for all $u = (u_j)_{j\in\mathbb{N}}$, $v = (v_j)_{j\in\mathbb{N}} \in c_0(\omega, \mathbb{N}, \mathbb{K})$ by

$$\langle u, v\rangle := \sum_{j\in\mathbb{N}}^{\infty} \omega_j\, u_j\, v_j.$$

It is routine to check that this series converges for all $u, v \in c_0(\omega, \mathbb{N}, \mathbb{K})$ and that the Cauchy-Schwarz's inequality is satisfied, that is,

$$|\langle u, v\rangle| \le \|u\| \,.\, \|v\|$$

for all $u, v \in c_0(\omega, \mathbb{N}, \mathbb{K})$.

The non-archimedean Banach space $c_0(\omega, \mathbb{N}, \mathbb{K})$ has a special base, which is denoted by $(e_j)_{j\in\mathbb{N}}$ where e_j is the sequence whose i-th term is 0 if $j \neq i$, and the j-th term is 1.

In the literature, the space $c_0(\omega, \mathbb{N}, \mathbb{K})$ endowed with its natural norm and inner product is called a p-adic (or non-archimedean) Hilbert space [8], [9]. One must point out that in contrast with the classical Hilbert spaces, the norm on $c_0(\omega, \mathbb{N}, \mathbb{K})$ does not stem from the inner product. In addition to that the space $c_0(\omega, \mathbb{N}, \mathbb{K})$ contains isotropic vectors, that is, $\langle x, x\rangle = 0$ while $0 \neq x \in c_0(\omega, \mathbb{N}, \mathbb{K})$. Indeed, an example of such vectors is given as follows: suppose that the field $\mathbb{K}$ is algebraically closed and choose $\omega = (\omega_i)_{i\in\mathbb{N}}$ so that

$$\omega_0 = 1, \ \omega_1 = 1, \quad \text{and} \quad \left|\omega_j\right| < 1 \quad \text{for} \quad j \geq 2.$$

If we consider the nonzero vector $\widehat{x} = (x_j)_{j\in\mathbb{N}} \in c_0(\omega, \mathbb{N}, \mathbb{K})$, where

$$x_0 = 1, \quad x_1 = \sqrt{-1} \in \mathbb{K}, \quad x_j = 0, \ j \geq 2,$$

then $\langle \widehat{x}, \widehat{x}\rangle = 0$, while $\|\widehat{x}\| = 1$.

For more properties on p-adic or non-archimedean Hilbert spaces and related recent developments, we refer the reader to [6], [7], [8], [9].

Consider the linear operator $\widehat{B}$ defined on $c_0(\omega, \mathbb{N}, \mathbb{K})$ by

$$\widehat{B}e_j = e_j + \sum_{i\neq j} \frac{e_i}{\omega_i}\,, \tag{1.1}$$

under the following assumptions:

(i) $\omega_0 = 1$ and $|\omega_j| > 1$ for all $j \geq 1$;
(ii) $\lim_{j\to\infty} \frac{1}{|\omega_j|} = 0$; and
(iii) $(|\omega_j|)_{j\in\mathbb{N}}$ is strictly increasing, that is, $|\omega_{j+1}| > |\omega_j|$ for all $j \in \mathbb{N}$.

Under assumptions (i)–(ii)–(iii), the spectral analysis for $\widehat{B}$ was studied by Diarra [9]. In particular, Diarra has shown that $\widehat{B}$ has eigenvalues of the form $\lambda = 1 + \alpha$, where α runs over the collection of all zeros of the function defined by

$$\varphi(\alpha) = 1 - \sum_{j\in\mathbb{N}} \frac{1}{1 + \alpha\omega_j}\,.$$

Finally, Diarra made extensive use of the classical p-adic analytic functions theory to locate all the zeros of φ. In particular, it was shown that

$$\sigma(\widehat{B}) = \{1\} \cup \sigma_p(\widehat{B}),$$

where $\sigma_p(\widehat{B})$ is the collection of eigenvalues of $\widehat{B}$ also called the point spectrum.

The present paper is aimed at studying properties and the spectral analysis to rank one perturbations of diagonal operators on $c_0(\omega, \mathbb{N}, \mathbb{K})$, which turns out to be more general than that of $\widehat{B}$. More precisely, under some reasonable assumptions, we compute the spectrum for elements of the class of linear operators on $c_0(\omega,\mathbb{N},\mathbb{K})$ denoted $\mathcal{D}_{per}(c_0(\omega,\mathbb{N},\mathbb{K}))$, consisting of all bounded linear operators on the non-archimedean Hilbert space $c_0(\omega,\mathbb{N},\mathbb{K})$ of the form

$$A = D_\lambda + X\otimes Y, \tag{1.2}$$

where D_λ is the diagonal operator defined by $D_\lambda e_j = \lambda_j e_j$ with $\lambda = (\lambda_j)_{j\in\mathbb{N}} \subset \mathbb{K}$ being a sequence and $X\otimes Y$ for each $X = (\alpha_j)_{j\in\mathbb{N}}$, $Y = (\beta_j)_{j\in\mathbb{N}} \in c_0(\omega,\mathbb{N},\mathbb{K})$, is the rank one linear operator defined by

$$(X\otimes Y)W = \langle Y, W\rangle X$$

for each $W \in c_0(\omega,\mathbb{N},\mathbb{K})$.

Namely, it is shown (Theorem 4.5) that if $A = D_\lambda + X\otimes Y$, then its spectrum, under some suitable assumptions, is given by

$$\sigma(A) = \{\theta_j\}_{j\geq 0} \cup \sigma_p(A)$$

where $\sigma_p(A)$ is the set of all eigenvalues of the bounded linear operator A and $\theta = (\theta_j)_{j\in\mathbb{N}}$ with

$$\theta_j = \lambda_j + \omega_j\alpha_j\beta_j$$

for all $j \in \mathbb{N}$.

To deal with these spectral issues, we will make extensive use of the techniques initiated and developed in Diarra's work [8], [9] with some slight adjustments.

Note that the spectral theory of the class of linear operators appearing in (1.2) in the classical context has been extensively studied by several authors, see. e.g., Ionascu [11]. In particular, in the recent paper by Foias et al. [10], the existence of a hyperinvariant subspace for the class of linear operators appearing in (1.2) has been established.

It should be noted that $Ae_j = D_\lambda e_j + X\otimes Y(e_j)$ can be written as

$$Ae_j = \big(\lambda_j + \omega_j\alpha_j\beta_j\big)e_j + \omega_j\beta_j\sum_{i\neq j}\alpha_i e_i. \tag{1.3}$$

for all $j \in \mathbb{N}$.

The operator $\widehat{B}$ given in (1.1) is a particular element of $\mathcal{D}_{per}(c_0(\omega,\mathbb{N},\mathbb{K}))$. Indeed, assuming that

$$\lim_{j\to\infty}\frac{1}{\omega_j} = 0,$$

take $\widehat{X} = \widehat{Y} = (\frac{1}{\omega_j})_{j\in\mathbb{N}} \in c_0(\omega,\mathbb{N},\mathbb{K})$ and let

$$\widehat{D}_\lambda e_j = \left(1 - \frac{1}{\omega_j}\right)e_j \quad \text{for} \quad j\in\mathbb{N}$$

in (1.2). Doing so, one can see that $\widehat{B} = \widehat{D}_\lambda + \widehat{X} \otimes \widehat{Y}$ with $\lambda = (1 - \frac{1}{\omega_j})_{j\in\mathbb{N}}$.

Throughout the rest of the paper we omit most of the pathological cases related to the choices of the vectors X and Y. Precisely, we will only consider the case when $X = (\alpha_j)_{j\in\mathbb{N}}$, $Y = (\beta_j)_{j\in\mathbb{N}} \in c_0(\omega, \mathbb{N}, \mathbb{K})$ with $\alpha_j, \beta_j \in \mathbb{K} - \{0\}$ for each $j \in \mathbb{N}$.

2. Bounded linear operators on $c_0(\omega, \mathbb{N}, \mathbb{K})$

Following the work of Diarra [6], [7], [8], [9], it is well-known that under suitable assumptions, every linear operator A on $c_0(\omega, \mathbb{N}, \mathbb{K})$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ and $Ae_j = \sum_{i\geq 0} a_{ij} e_i$.

A linear operator $A : c_0(\omega, \mathbb{N}, \mathbb{K}) \mapsto c_0(\omega, \mathbb{N}, \mathbb{K})$ is said to be bounded whenever its norm $\|A\|$ defined by

$$\big\|A\big\| := \sup_{u\neq 0} \frac{\|Au\|}{\|u\|}$$

is finite. In that case, it can be easily shown that

$$\big\|A\big\| = \sup_{i,j\in\mathbb{N}} \frac{|a_{ij}|\|e_i\|}{\|e_j\|}.$$

The collection of all bounded linear operators on $c_0(\omega, \mathbb{N}, \mathbb{K})$ will be denoted by $B(c_0(\omega, \mathbb{N}, \mathbb{K}))$.

An example of a bounded linear operator on $c_0(\omega, \mathbb{N}, \mathbb{K})$ is for instance the rank one operator $X \otimes Y$ defined above. Indeed,

$$\begin{aligned} \big\|X \otimes Y\big\| &= \sup_{j\in\mathbb{N}} \frac{\|(X \otimes Y)(e_j)\|}{|\omega_j|^{1/2}} \\ &= \big\|X\big\| \cdot \sup_{j\in\mathbb{N}} \big|\omega_j\big|^{1/2} \big|\beta_j\big| \\ &= \big\|X\big\| \cdot \big\|Y\big\| \\ &< \infty. \end{aligned}$$

3. Useful properties of elements of $\mathcal{D}^0_{per}(c_0(\omega, \mathbb{N}, \mathbb{K}))$

Let $\mathcal{D}^0_{per}(c_0(\omega, \mathbb{N}, \mathbb{K}))$ denote all elements $A = D_\lambda + X \otimes Y$ of $\mathcal{D}_{per}(c_0(\omega, \mathbb{N}, \mathbb{K}))$ such that $X = (\alpha_j)_{j\in\mathbb{N}}$, $Y = (\beta_j)_{j\in\mathbb{N}} \in c_0(\omega, \mathbb{N}, \mathbb{K})$ with $\alpha_j, \beta_j \in \mathbb{K} - \{0\}$ for each $j \in \mathbb{N}$.

Proposition 3.1. *Let $A = D_\lambda + X \otimes Y \in \mathcal{D}^0_{per}(c_0(\omega, \mathbb{N}, \mathbb{K}))$. Then the following hold:*

(i) *the adjoint A^* of A exists and $A^* = D_\lambda + Y \otimes X$;*
(ii) *the operator A is self-adjoint, that is, $A = A^*$, if and only if there exists a nonzero $\kappa \in \mathbb{K} - \{0\}$ such that $Y = \kappa X$.*

Proof: (i) Since the operator $X \otimes Y$ is bounded on $c_0(\omega, \mathbb{N}, \mathbb{K})$, it is then clear that if it exists, the adjoint $A^* = (D_\lambda + X\otimes Y)^* = D_\lambda^* + (X\otimes Y)^* = D_\lambda + (X\otimes Y)^*$. Now, it is routine to check that $(X\otimes Y)^* = Y\otimes X$, and therefore $A^* = D_\lambda + Y\otimes X$.

(ii) We have $A = D_\lambda + (X\otimes Y) = D_\lambda + (Y\otimes X) = A^*$ if and only if $X\otimes Y = Y\otimes X$, equivalently, X and Y must be linearly dependent. □

Corollary 3.2. *Suppose* $\lim_{j\to\infty} \frac{1}{|\omega_j|^{1/2}} = 0$. *Then, the operator* $\widehat{B}$ *given by* $\widehat{B}e_j = e_j + \sum_{i\neq j} \frac{e_i}{\omega_i}$, $j \in \mathbb{N}$, *is selfadjoint.*

Proof: Notice that $\widehat{B} = D_\lambda + X\otimes Y$ where $X = Y = (\frac{1}{\omega_j})_{j\in\mathbb{N}} \in c_0(\omega, \mathbb{N}, \mathbb{K})$. Consequently, using (ii) of Proposition 4.2 it follows that $\widehat{B}$ is selfadjoint. □

4. Spectral analysis for $D_\lambda + X \otimes Y$

Throughout the rest of the paper, if B is a bounded linear operator on $c_0(\omega, \mathbb{N}, \mathbb{K})$, then the symbols $\sigma(B)$, $\sigma_p(B)$, $N(B) = \{x \in c_0(\omega, \mathbb{N}, \mathbb{K}) : Bx = 0\}$, and $R(B) = \{Bx : x \in c_0(\omega, \mathbb{N}, \mathbb{K})\}$ stand for the spectrum, the set of eigenvalues, the kernel and image of B, respectively.

For $X = \sum_{j\geq 0} \alpha_j e_j$, $Y = \sum_{j\geq 0} \beta_j e_j$ and $\lambda = (\lambda_j)_{j\in\mathbb{N}}$, we set $\theta = (\theta_j)_{j\in\mathbb{N}}$ where $\theta_j = \lambda_j + \omega_j\alpha_j\beta_j$ for all $j \in \mathbb{N}$.

The present setting requires the following assumptions:

(iv) $|\alpha_0| = |\beta_0| = |\omega_0| = 1$ and $\omega_0\alpha_0\beta_0 = 1$;

(v) $\lambda_0 = 0$, $\lambda_j \neq 1$ and there exist two constants $\widehat{m}, \widehat{M} > 0$ with $0 < \widehat{m} < 1$ such that $\widehat{m} < |\lambda_j| \leq \widehat{M}$ for all $j \geq 1$;

(vi) $0 < m_\alpha := \inf_{j\in\mathbb{N}} |\alpha_j||\omega_j|^{1/2} \leq \|X - \alpha_0 e_0\| = \sup_{j\geq 1} |\alpha_j||\omega_j|^{1/2} < \widehat{m}$, where $\widehat{m}$ is the constant appearing in (v);

(vii) the sequence $(|\omega_j|)_{j\geq 1}$ is strictly increasing with $|\omega_j| > 1$ for all $j \geq 1$; and

(viii) $\sup_{j\in\mathbb{N}} |1 - \frac{1}{\omega_j\beta_j}| < 1$.

Remark 4.1. (1) Note that from assumption (viii), $|\omega_j\beta_j| = 1$ for all $j \in \mathbb{N}$. Using assumptions (iv)–(vii) it follows that

$$\|Y\| = \sup_{j\in\mathbb{N}} |\beta_j||\omega_j|^{1/2} = \sup_{j\in\mathbb{N}} |\omega_j|^{-1/2} = 1.$$

Similarly, from assumptions (iv)–(vi) it follows that $\|X\| = 1$.

(2) From assumption (v) it follows that the diagonal operator D_λ is bounded and therefore, $A = D_\lambda + X\otimes Y$ is bounded with

$$\|A\| \leq \max\Big(\sup_{j\geq 0} |\lambda_j|, \|X\| \cdot \|Y\|\Big).$$

(3) Assumption (viii) is compatible with the rest of the above-mentioned assumptions and is satisfied in several cases. This is in particular the case when $\beta_j = \frac{1}{\omega_j}$ for all $j \in \mathbb{N}$.

(4) Setting $\widehat{e}_j = \frac{e_j}{\omega_j\beta_j}$, one can easily see that $(\widehat{e}_j)_{j\in\mathbb{N}}$ is also an orthogonal base for $c_0(\omega,\mathbb{N},\mathbb{K})$, as

$$\sup_{j\in\mathbb{N}} \frac{\|e_j - \widehat{e}_j\|}{\|e_j\|} < 1$$

by using assumption (viii).

It should be noted that the above condition yields $\|\widehat{e}_j\| = \|e_j\| = |\omega_j|^{1/2}$ for all $j \in \mathbb{N}$.

Proposition 4.2. *Let* $A = D_\lambda + X \otimes Y \in \mathcal{D}^0_{per}(c_0(\omega,\mathbb{N},\mathbb{K}))$ *and suppose that assumptions* (iv)–(v)–(vi)–(vii)–(viii) *hold. Then* A *is invertible and its inverse* A^{-1} *is given by*

$$A^{-1}(\widehat{e}_0) = \Big[1 - \sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\Big]\widehat{e}_0 + \sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\widehat{e}_j$$

and

$$A^{-1}(\widehat{e}_j) = \frac{\widehat{e}_j - \widehat{e}_0}{\lambda_j} \quad \textit{for} \quad j \geq 1.$$

Moreover, the operator A *is isometric if and only if* $|\lambda_j| = 1$ *for all* $j \geq 1$.

Proof: Utilizing the orthogonal base $(\widehat{e}_j)_{j\in\mathbb{N}}$, one can easily see that the operator A is defined as follows:

$$A\widehat{e}_j = \theta_j\widehat{e}_j + \sum_{i\neq j}\big(\theta_i - \lambda_i\big)\widehat{e}_i \quad \text{for all} \quad j \in \mathbb{N}.$$

Now from

$$\begin{aligned}\sum_{k\neq i,j}\big(\theta_k - \lambda_k\big)\widehat{e}_k &= \big(A\widehat{e}_j - \theta_j\widehat{e}_j\big) - \omega_i\alpha_i\beta_i\widehat{e}_i \\ &= \big(A\widehat{e}_i - \theta_i\widehat{e}_i\big) - \omega_j\alpha_j\beta_j\widehat{e}_j\end{aligned}$$

for all $i, j \in \mathbb{N}$ it follows that

$$A\widehat{e}_i - A\widehat{e}_j = \lambda_i\widehat{e}_i - \lambda_j\widehat{e}_j.$$

In particular, letting $j = 0$ in the previous equation and setting

$$E_i := \frac{\widehat{e}_i - \widehat{e}_0}{\lambda_i}$$

for all $i \geq 1$, we obtain that $AE_i = \widehat{e}_i$ for all $i \geq 1$. Furthermore, it is easy to see that

$$\big\|E_i\big\| = \frac{\|\widehat{e}_i\|}{|\lambda_i|} \quad \text{for all} \quad i \geq 1.$$

Taking into account the fact that $\theta_0 = \lambda_0 + \omega_0\alpha_0\beta_0 = 1$ (assumptions (iv)–(v)), one can easily rewrite $A\widehat{e}_0$ as follows

$$A\widehat{e}_0 = \theta_0\widehat{e}_0 + \sum_{j\geq 1}(\theta_j - \lambda_j)\widehat{e}_j = \widehat{e}_0 + \sum_{j\geq 1} A\big[(\theta_j - \lambda_j)E_j\big].$$

Now we claim that the series $\sum_{j\geq 1}(\theta_j - \lambda_j)E_j$ is convergent. To see that, using assumption (v), it suffices to see that

$$\begin{aligned}\big\|(\theta_j - \lambda_j)E_j\big\| &= \frac{|\omega_j|^{1/2}|\alpha_j|}{|\lambda_j|}\\ &< \frac{|\omega_j|^{1/2}|\alpha_j|}{\widehat{m}}\\ &\to 0 \ \text{ as } \ j\to\infty,\end{aligned}$$

as $X = (\alpha_j)_{j\in\mathbb{N}}$ belongs to $c_0(\omega, \mathbb{N}, \mathbb{K})$.

Consequently, setting $E_0 := \widehat{e}_0 - \sum_{j\geq 1}(\theta_j - \lambda_j)E_j$, one can easily see that $AE_0 = \widehat{e}_0$. Moreover,

$$\big\|E_0\big\| = \max\Big(\big\|\widehat{e}_0\big\|, \Big\|\sum_{j\geq 1}(\theta_j - \lambda_j)E_j\Big\|\Big) = \big\|\widehat{e}_0\big\| = 1,$$

as

$$\Big\|\sum_{j\geq 1}(\theta_j - \lambda_j)E_j\Big\| \leq \sup_{j\geq 1}\big\|(\theta_j - \lambda_j)E_j\big\| = \sup_{j\geq 1}\frac{|\omega_j|^{1/2}|\alpha_j|}{|\lambda_j|} \leq \frac{\|X - \alpha_0 e_0\|}{\widehat{m}} < 1,$$

by using assumption (vi).

In the new base $(\widehat{e}_j)_{j\in\mathbb{N}}$, the vector E_0 can be rewritten as follows:

$$E_0 = \Big[1 - \sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\Big]\widehat{e}_0 + \sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\widehat{e}_j.$$

Note that the series $\sum_{j\geq 1}\big(1 - \frac{\theta_j}{\lambda_j}\big)$ appearing in the expression of E_0 is convergent. Indeed,

$$\Big|1 - \frac{\theta_j}{\lambda_j}\Big| = \frac{|\omega_j\alpha_j\beta_j|}{|\lambda_j|} = \frac{|\alpha_j|}{|\lambda_j|} < \frac{(|\omega_j|^{1/2}|\alpha_j|)}{\widehat{m}} \to 0 \quad \text{as} \quad j\to\infty.$$

Similarly, the series $\sum_{j\geq 1}(1 - \frac{\theta_j}{\lambda_j})\widehat{e}_j$ is convergent; indeed

$$\Big|1 - \frac{\theta_j}{\lambda_j}\Big| \cdot \big\|\widehat{e}_j\big\| = \frac{|\omega_j\alpha_j\beta_j|}{|\lambda_j|} \cdot \big\|\widehat{e}_j\big\| = \frac{|\omega_j|^{1/2}|\alpha_j|}{|\lambda_j|} < \frac{|\omega_j|^{1/2}|\alpha_j|}{\widehat{m}} \to 0 \quad \text{as} \quad j\to\infty.$$

Let $E_j = S\widehat{e}_j$ for each $j \in \mathbb{N}$. If $x = \sum_{j\in\mathbb{N}} x_j\widehat{e}_j \in c_0(\omega,\mathbb{N},\mathbb{K})$, one has $\lim_{j\to\infty} \|x_j\widehat{e}_j\| = 0$ and, thus,

$$Sx = \sum_{j\in\mathbb{N}} x_j S\widehat{e}_j = \sum_{j\in\mathbb{N}} x_j E_j$$

is well-defined, because $|x_j|\cdot\|E_j\| = |x_j|\cdot\frac{\|\widehat{e}_j\|}{|\lambda_j|} < \frac{\|x_j\widehat{e}_j\|}{\widehat{m}} \to 0$ as $j\to\infty$. Moreover, one can easily check that $SA(\widehat{e}_j) = \widehat{e}_j = AS(\widehat{e}_j)$ for all $j\in\mathbb{N}$, and hence $SA(x) = AS(x) = x$ for each $x = \sum_{j\in\mathbb{N}} x_j\widehat{e}_j \in c_0(\omega,\mathbb{N},\mathbb{K})$, that is, $SA = AS = I$ and therefore, $A^{-1} = S$.

Now

$$\|S\widehat{e}_j\| = \|E_j\| = \frac{1}{|\lambda_j|}\|\widehat{e}_j - \widehat{e}_0\| = \frac{\|\widehat{e}_j\|}{|\lambda_j|} \quad \text{for all} \quad j \geq 1$$

and

$$\|S\widehat{e}_0\| = \|E_0\| = \|\widehat{e}_0\| = 1.$$

Consequently, the operator A is isometric if and only if $|\lambda_j| = 1$ for $j\geq 1$. □

Proposition 4.3. *Let $A = D_\lambda + X\otimes Y \in \mathcal{D}^0_{per}(c_0(\omega,\mathbb{N},\mathbb{K}))$. Under assumptions* (iv)–(v)–(vi)–(vii)–(viii), *$\lambda\in\sigma_p(A)$ if and only if*

$$1 = \sum_{j\geq 0} \frac{\lambda_j - \theta_j}{\lambda_j - \lambda}\,. \tag{4.1}$$

Moreover, if $\lambda\in\sigma_p(A)$, then $\dim N(A-\lambda I) = 1$.

Proof: Let $x = \sum_{j\geq 0} x_j\widehat{e}_j \in c_0(\omega,\mathbb{N},\mathbb{K})$. Computing $A^{-1}x$ one can easily see that

$$\begin{aligned} A^{-1}x &= \sum_{j\geq 0} x_j A^{-1}\widehat{e}_j \\ &= \sum_{j\geq 1} x_j\Big\{\frac{\widehat{e}_j - \widehat{e}_0}{\lambda_j}\Big\} + x_0\Big[1 - \sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\Big]\widehat{e}_0 + x_0\sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\widehat{e}_j \\ &= \Big\{x_0\Big[1 - \sum_{j\geq 1}\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\Big] - \sum_{j\geq 1}\frac{x_j}{\lambda_j}\Big\}\widehat{e}_0 + \sum_{j\geq 1}\Big[\frac{x_j}{\lambda_j} + x_0\Big(1 - \frac{\theta_j}{\lambda_j}\Big)\Big]\widehat{e}_j. \end{aligned}$$

Let $\lambda\in\sigma_p(A)$. Then there exists a nonzero $x\in c_0(\omega,\mathbb{N},\mathbb{K})$ such that $Ax = \lambda x$; equivalently, $x = \lambda A^{-1}x$, that is,

$$\begin{cases} \lambda\big\{x_0\big[1 - \sum_{j\geq 1}\big(1 - \frac{\theta_j}{\lambda_j}\big)\big] - \sum_{j\geq 1}\frac{x_j}{\lambda_j}\big\} = x_0, \\ \lambda\big\{\frac{x_j}{\lambda_j} + x_0\big(1 - \frac{\theta_j}{\lambda_j}\big)\big\} = x_j, \end{cases}$$

and hence

$$\begin{cases} \big(1-\lambda\big)x_0 = -\lambda\sum_{j\geq 1}\big[\frac{x_j}{\lambda_j} + x_0\big(1 - \frac{\theta_j}{\lambda_j}\big)\big], & (\Delta_0) \\ \big[1 - \frac{\lambda}{\lambda_j}\big]x_j = \lambda x_0\big(1 - \frac{\theta_j}{\lambda_j}\big), \quad j\geq 1. & (\Delta_j) \end{cases}$$

(i) In (Δ_j), if $1-\frac{\lambda}{\lambda_s}=0$ for some $s\geq 1$, then $\lambda=\lambda_s$, which implies that $x_0=0$ and $x_j=0$ for $j\neq s$. Using (Δ_0) it follows that $0=-\lambda_s\frac{x_s}{\lambda_s}=x_s$. In view of the above, $x=0$, which is in contradiction with the fact x is an eigenvector, that is, nonzero.

(ii) Note that $\lambda\neq 0$, as A is invertible. If $\lambda=1$ is an eigenvalue of A, we obtain from (Δ_j) that $(1-\frac{1}{\lambda_j})x_j=x_0(1-\frac{\theta_j}{\lambda_j})$ for $j\geq 1$, that is, $[\frac{\lambda_j-1}{\lambda_j-\theta_j}]x_j=x_0$ for $j\geq 1$. Using assumption (vii) it follows that

$$\begin{aligned}\Big|\Big(\frac{\lambda_j-1}{\lambda_j-\theta_j}\Big)x_j\Big| &= \Big|\Big(\frac{\lambda_j-1}{-\omega_j\alpha_j\beta_j}\Big)x_j\Big|\\ &= \frac{|\lambda_j-1|}{|\alpha_j||\omega_j|^{1/2}}\|x_j\widehat{e}_j\|\\ &\leq \frac{\max(1,\widehat{M})}{m_\alpha}\cdot\big\|x_j\widehat{e}_j\big\|.\end{aligned}$$

Now $|x_0|=\lim_{j\to\infty}|(\frac{\lambda_j-1}{\lambda_j-\theta_j})x_j|=0$, as $\lim_{j\to\infty}\|x_j\widehat{e}_j\|=0$. Therefore, $x_0=0$ and $x_j=0$ for $j\geq 1$, that is, $x=0$, which is in contradiction with the fact x is an eigenvector, that is, nonzero.

(iii) From the discussions in (i) and (ii) above, it follows that if $\lambda\in\sigma_p(A)$, then λ does not belong to the set $\{\lambda_j:\ j\in\mathbb{N}\}\cup\{1\}$, that is, $\lambda\neq\lambda_j$ for $j\in\mathbb{N}$ and $\lambda\neq 1$. If $\lambda\in\sigma_p(A)$ is associated with the eigenvector $x=\sum_{j\geq 0}x_j\widehat{e}_j\in c_0(\omega,\mathbb{N},\mathbb{K})$, it then follows from (Δ_j) that

$$x_j=\frac{\lambda(\lambda_j-\theta_j)}{\lambda_j-\lambda}x_0 \quad\text{for}\quad j\geq 1$$

with $x_0\neq 0$.

Thus from (Δ_0), we obtain

$$(1-\lambda)x_0=-\lambda x_0\sum_{j\geq 1}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\,,$$

and hence

$$\frac{1}{\lambda}=1-\sum_{j\geq 1}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\,,$$

that is,

$$1=\sum_{j\geq 0}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\,,$$

as $\frac{\lambda_0-\theta_0}{\lambda_0-\lambda}=\frac{1}{\lambda}$.

Conversely, if $\lambda \neq \lambda_j$ for $j \in \mathbb{N}$, $\lambda \neq 1$, and (4.1) holds, then the nonzero vector $\widehat{x}_\lambda$ defined by

$$\widehat{x}_\lambda = \sum_{j\geq 0}\Big(\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big)\widehat{e}_j$$

belongs to $c_0(\omega,\mathbb{N},\mathbb{K})$. Moreover,

$$\begin{aligned}
\lambda A^{-1}\widehat{x}_\lambda &= \sum_{j\geq 0}\Big(\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big)A^{-1}\widehat{e}_j\\
&= \lambda\Big\{\frac{1}{\lambda}\Big[1-\sum_{j\geq 1}\Big(1-\frac{\theta_j}{\lambda_j}\Big)\Big]-\sum_{j\geq 1}\frac{1}{\lambda_j}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big\}\widehat{e}_0\\
&\quad+\lambda\sum_{j\geq 1}\Big[\frac{1}{\lambda_j}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}+\frac{1}{\lambda}\Big(1-\frac{\theta_j}{\lambda_j}\Big)\Big]\widehat{e}_j\\
&= \Big[1-\sum_{j\geq 1}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big]\widehat{e}_0+\sum_{j\geq 1}\Big(\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big)\widehat{e}_j\\
&= \frac{1}{\lambda}\widehat{e}_0+\sum_{j\geq 1}\Big(\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big)\widehat{e}_j\\
&= \sum_{j\geq 0}\Big(\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\Big)\widehat{e}_j\\
&= \widehat{x}_\lambda,
\end{aligned}$$

and hence $A\widehat{x}_\lambda = \lambda\widehat{x}_\lambda$, therefore $\lambda \in \sigma_p(A)$.

To complete the proof, note that in view of the above, the eigenvector space $E_\lambda = N(A-\lambda I)$ associated with the eigenvalue λ is spanned by $\widehat{x}_\lambda$ and hence $\dim E_\lambda = 1$. □

Suppose $|\lambda_j| = 1$ for all $j \geq 1$ and set $D := \{\gamma \in \mathbb{K} : |\gamma| \leq 1\}$, and

$$\Delta := D - \Big(\{\lambda_j\}_{j\geq 1}\cup\{0\}\Big).$$

Theorem 4.4. *Let $A = D_\lambda + X\otimes Y \in \mathcal{D}^0_{per}(c_0(\omega,\mathbb{N},\mathbb{K}))$. Suppose assumptions* (iv)–(v)–(vi)–(vii)–(viii) *hold and $|\lambda_j| = 1$ for all $j \geq 1$. If $\lambda \in \mathbb{K}$, then $\lambda \in \sigma_p(A)$ if and only if $\lambda \in \Delta$ and $d(\lambda) = 0$, where the function $d : \Delta \mapsto \mathbb{K}$ is defined by*

$$d(\lambda) := 1 - \sum_{j\geq 0}\frac{\lambda_j-\theta_j}{\lambda_j-\lambda}\,. \tag{4.2}$$

PROOF: If $\lambda \in \sigma_p(A)$ then there exists $x \neq 0$ such that $Ax = \lambda x$. From the fact that A is isometric it follows that $|\lambda| = 1$. Now if $|1-\lambda| = 1 = |\lambda - \lambda_j|$, then

$$\begin{aligned} \Big|\sum_{j\geq 1} \frac{\lambda_j - \theta_j}{\lambda_j - \lambda}\Big| &\leq \sup_{j\geq 1}\Big|\frac{\lambda_j - \theta_j}{\lambda_j - \lambda}\Big| \\ &= \sup_{j\geq 1}\big|\lambda_j - \theta_j\big| \\ &< \widehat{m} \\ &< 1 \\ &= \Big|\frac{1-\lambda}{\lambda}\Big|, \end{aligned}$$

and hence $\frac{1-\lambda}{\lambda}$ is not equal to $\sum_{j\geq 1} \frac{\lambda_j - \theta_j}{\lambda_j - \lambda}$, that is, (4.2) is not satisfied.

Therefore, λ does not belong to $\sigma_p(A)$ when $|1-\lambda| = 1 = |\lambda - \lambda_j|$ according to Proposition 4.3. In conclusion if $\lambda \in \sigma_p(A)$, then $|\lambda - 1| < 1$, $|\lambda - \lambda_j| < 1$, $\lambda \neq \lambda_j$ for $j \geq 1$, $\lambda \neq 0$ and $d(\lambda) = 0$. The converse is also clear. $\square$

Theorem 4.5. *Under the previous assumptions, if*

$$A = D_\lambda + X \otimes Y \in \mathcal{D}^0_{per}(c_0(\omega, \mathbb{N}, \mathbb{K})),$$

then the spectrum of A is given by

$$\sigma(A) = \{\theta_j\}_{j\geq 0} \cup \sigma_p(A).$$

PROOF: First of all, note that if $x = \sum_{j\geq 0} x_j \widehat{e}_j \in c_0(\omega, \mathbb{N}, \mathbb{K})$, then the series $\sum_{j\geq 0} x_j$ converges because

$$\big|x_j\big| = \frac{|x_j| \cdot \|\widehat{e}_j\|}{\|\widehat{e}_j\|} < \big|x_j\big| \cdot \big\|\widehat{e}_j\big\|$$

and $|x_j| \cdot \|\widehat{e}_j\| \to 0$ as $j \to \infty$.

Now

$$\begin{aligned} Ax &= \sum_{i\geq 0} x_i A\widehat{e}_i \\ &= \sum_{i\geq 0} x_i\Big[\theta_i \widehat{e}_i + \sum_{j\neq i}\big(\theta_j - \lambda_j\big)\widehat{e}_j\Big] \\ &= \sum_{i\geq 0} x_i\theta_i\widehat{e}_i + \sum_{i\geq 0} x_i\Big[\sum_{j\neq i}\big(\theta_j - \lambda_j\big)\widehat{e}_j\Big] \\ &= \sum_{i\geq 0} x_i\theta_i\widehat{e}_i + \sum_{i\geq 0}\big(\theta_i - \lambda_i\big)\sum_{j\neq i} x_j\widehat{e}_i \\ &= \sum_{i\geq 0}\Big[x_i\theta_i + \big(\theta_i - \lambda_i\big)\sum_{j\neq i} x_j\Big]\widehat{e}_i, \end{aligned}$$

and hence for $\lambda \in \mathbb{K}$,

$$(A-\lambda I)x = \sum_{i\geq 0}\Big[x_i\big(\theta_i-\lambda\big) + \big(\theta_i-\lambda_i\big)\sum_{j\neq i} x_j\Big]\widehat{e}_i.$$

If $y = \sum_{j\geq 0} y_j\widehat{e}_j$ belongs to $R(A-\lambda I)$, then there exists $x = \sum_{j\geq 0} x_j\widehat{e}_j \in c_0(\omega,\mathbb{N},\mathbb{K})$ such that $(A-\lambda I)x = y$. Equivalently,

$$\begin{aligned} y_i &= x_i\big(\theta_i-\lambda\big) + \big(\theta_i-\lambda_i\big)\sum_{j\neq i} x_j \quad \text{for all} \quad i\geq 0 \\ &= x_i\big(\theta_i-\lambda\big) + \big(\theta_i-\lambda_i\big)\Big[x_0 - x_i + \sum_{j\geq 1} x_j\Big] \quad \text{for all} \quad i \geq 0. \end{aligned}$$

From above one easily deduces that

$$x_0(1-\lambda) + \sum_{j\geq 1} x_j = y_0 \tag{4.3}$$

and

$$x_i(\lambda_i-\lambda) + \lambda x_0(\theta_i-\lambda_i) = y_i - y_0(\theta_i-\lambda_i) \quad \text{for all} \quad i\geq 1. \tag{4.4}$$

The rest of the proof will follow along the same lines as the discussions appearing in Diarra [8], [9].

(i) First of all note that $N(A-\theta_j I) = \{0\}$ for all $j\geq 0$. Thus to show that $\theta_j \in \sigma(A)$ one must prove that $A-\theta_j I$ is not surjective. Indeed, note that $\widehat{e}_m \notin R(A-\theta_j I)$ for all $m\geq 1$, $j\geq 0$. If not, then (4.3) and (4.4) can be respectively rewritten as follows

$$x_0\big(1-\theta_j\big) + \sum_{i\geq 1} x_i = \delta_{0,m}$$

and

$$x_i\big(\lambda_i-\theta_j\big) + \theta_j x_0\big(\theta_i-\lambda_i\big) = \delta_{i,m} - \delta_{0,m}\big(\theta_i-\lambda_i\big) \quad \text{for all} \quad i\geq 1.$$

Consequently,

$$\big(\lambda_i-\theta_j\big)x_i = \delta_{i,m} - \delta_{0,m}\big(\theta_i-\lambda_i\big) - \theta_j x_0\big(\theta_i-\lambda_i\big) \quad \text{for all} \quad i\geq 1.$$

In the case $j=0$, which corresponds to $\theta_0 = 1$, one can mimic what Diarra did in [9]. Now if $j\geq 1$, letting $j = i \neq m$ in the previous equations, we obtain: $(\lambda_i-\theta_i)x_i = x_0\theta_i(\lambda_i-\theta_i)$. Since $\lambda_i - \theta_i \neq 0$ for each i it follows that $x_i = x_0\theta_i$ for each $i\geq 1$ and hence

$$x_0\Big[(1-\theta_i) + \sum_{i\geq 1}\theta_i\Big] = 0.$$

(ia) If $x_0 = 0$, it follows that $x_i = 0$ for each $i \geq 1$ and hence $x_j = 0$ for all $j \geq 0$, which does not make sense.

(ib) Now if $x_0 \neq 0$, then

$$\big(1-\theta_i\big) + \sum_{k\geq 1} \theta_k = 0 \quad \text{for each} \quad i \geq 1,$$

which makes sense only if $1-\theta_i$ is constant, say $\theta_i = 1+c$ where $c \in \mathbb{K}$. If the series $\sum_{k\geq 1} \theta_k$ does not converge, then we get a contradiction. Now if the series $\sum_{k\geq 1} \theta_k$ converges, equivalently, $\lim_{k\to\infty} \theta_k = 0$, then one must have $c = -1$, which yields $1 + 0 = 0$, that is a contradiction.

(ii) Suppose that $\lambda = \lambda_j$ for some $j \geq 1$. Since $\lambda_j = \lambda \notin \sigma(A)$ it follows that

$$N\big(A-\lambda I\big) = N\big(A-\lambda_j I\big) = \big\{0\big\}.$$

Letting $\lambda = \lambda_j$ in (4.4) and solving for x_0 we obtain

$$x_0 = \frac{y_j}{\lambda_j(\theta_j-\lambda_j)} - \frac{y_0}{\lambda_j} \quad \text{for} \quad j \geq 1.$$

Similarly,

$$x_i = \frac{y_i}{\lambda_i-\lambda_j} - \frac{\theta_i-\lambda_i}{(\lambda_i-\lambda_j)(\theta_j-\lambda_j)} y_j \quad \text{for} \quad i \neq j, \quad i,j \geq 1.$$

From $|\theta_i - \lambda_i| = |\alpha_i| < |\alpha_i||\omega_i|^{1/2}$ and the fact $X \in c_0(\omega, \mathbb{N}, \mathbb{K})$ it follows that $\lim_{i\to\infty}(\theta_i - \lambda_i) = 0$. Consequently, one can easily see that $\lim_{i\to\infty} x_i = 0$, that is, the series $\sum_{i\neq j} x_i$ converges in $\mathbb{K}$.

From (4.3) it follows that $x_j = y_0 - \sum_{i\neq j} x_i - x_0(1-\lambda_j)$ and hence using the expressions of x_0 and x_i given above and right after a few computations, one obtains that

$$x_j = \frac{y_0}{\lambda_j} - \sum_{i\neq j} \frac{y_i}{\lambda_i-\lambda_j} + y_j\Big[\frac{\lambda_j-1}{\lambda_j(\theta_j-\lambda_j)} + \frac{1}{\theta_j-\lambda_j} \sum_{i\neq j} \frac{\theta_i-\lambda_i}{\lambda_i-\lambda_j}\Big].$$

It can be shown that $\lim_{i\to\infty} x_i\widehat{e}_i = 0$ and hence $x = \sum_{i\geq 0} x_i\widehat{e}_i \in c_0(\omega, \mathbb{N}, \mathbb{K})$. Moreover, it is easy to see that $(A-\lambda_j I)x = y$ and hence $R(A-\lambda_j I) = c_0(\omega, \mathbb{N}, \mathbb{K})$.

(iii) Suppose $\lambda \neq \lambda_j$ for $j \geq 1$, $\lambda \notin \{\theta_k\}_{k\geq 0}$ and $\lambda \notin \sigma(A)$. This is equivalent to the fact that either

$$\frac{1-\lambda}{\lambda} + \sum_{i\geq 1} \frac{\lambda_i-\theta_i}{\lambda_i-\lambda} \neq 0$$

or $N(A-\lambda I) = \big\{0\big\}$.

Now using (4.4) it follows that

$$x_i = \frac{1}{\lambda_i-\lambda} y_i + \frac{\lambda_i-\theta_i}{\lambda_i-\lambda} y_0 + \frac{(\lambda_i-\theta_i)\lambda}{\lambda_i-\lambda} x_0 \quad \text{for} \quad i \geq 1.$$

Setting $\widehat{\alpha}_i = \frac{y_i}{\lambda_i - \lambda}$ and $\widehat{\beta}_i = \frac{\lambda_i - \theta_i}{\lambda_i - \lambda}$ it follows that the series $\widehat{\alpha} := \sum_{i \geq 1} \widehat{\alpha}_i$ and $\widehat{\beta} := \sum_{i \geq 1} \widehat{\beta}_i$ converge in $\mathbb{K}$ and the following holds:

$$\sum_{i \geq 1} x_i = \widehat{\alpha} + \widehat{\beta} y_0 + \lambda \widehat{\beta} x_0. \tag{4.5}$$

Now using (4.3) and (4.5) it follows that

$$\big[1 - \lambda(1 - \widehat{\beta})\big] x_0 = -\widehat{\alpha} + \big(1 - \widehat{\beta}\big) y_0.$$

Using the fact that

$$0 \neq \frac{1-\lambda}{\lambda} + \sum_{i \geq 1} \frac{\lambda_i - \theta_i}{\lambda_i - \lambda} = \frac{1-\lambda}{\lambda} + \widehat{\beta} = \frac{1 - \lambda(1 - \widehat{\beta})}{\lambda}$$

it follows that

$$x_0 = \frac{1}{1 - \lambda(1 - \widehat{\beta})} \big[-\widehat{\alpha} + \big(1 - \widehat{\beta}\big) y_0 \big]$$

and

$$x_i = \frac{y_i}{\lambda_i - \lambda} + \frac{(\lambda_i - \theta_i)(y_0 - \widehat{\alpha}\lambda)}{(\lambda_i - \lambda)[1 - \lambda(1 - \widehat{\beta})]} \quad \text{for} \quad i \geq 0.$$

It can be shown that $\lim_{i \to \infty} |x_i| \cdot \|\widehat{e}_i\| = 0$, that is, $x = \sum_{i \geq 0} x_i \widehat{e}_i \in c_0(\omega, \mathbb{N}, \mathbb{K})$ with $(A - \lambda I)x = y$, therefore $R(A - \lambda I) = c_0(\omega, \mathbb{N}, \mathbb{K})$, which completes the proof. $\square$

5. Concluding remarks

(1) Note that the previous spectral analysis can be applied to all linear operators of the form $A = B + U \otimes V$ where B is not only selfadjoint but also 'looks like a diagonal' operator on $c_0(\omega, \mathbb{N}, \mathbb{K})$, that is, when

$$\sigma(B) = \sigma_p(B) = \big\{\mu_j\big\}_{j \in \mathbb{N}}$$

where $\mu_j \in \mathbb{K}$ for each $j \in \mathbb{N}$. Indeed, let $(f_j)_{j \in \mathbb{N}}$ be the sequence of eigenvectors corresponding to the eigenvalues $\{\mu_j\}_{j \in \mathbb{N}}$, that is, $Bf_j = \mu_j f_j$ for each $j \in \mathbb{N}$. For the sake of simplicity, let us assume that there exists a nontrivial isometric linear bijection L such that $Le_j = f_j$ for all $j \in \mathbb{N}$. Consequently, $(f_j)_{j \in \mathbb{N}}$ is also an orthogonal base for $c_0(\omega, \mathbb{N}, \mathbb{K})$ with $\|f_j\| = \|e_j\| = |\omega_j|^{1/2}$ for all $j \in \mathbb{N}$. Clearly, each $x \in c_0(\omega, \mathbb{N}, \mathbb{K})$ can be written as $x = \sum_{j \in \mathbb{N}} x_j f_j$ with $\lim_{j \to \infty} |x_j| \cdot \|f_j\| = \lim_{j \to \infty} |x_j| \cdot \|e_j\| = 0$. Further, the operator B can be rewritten in the base $(f_j)_{j \in \mathbb{N}}$ as follows:

$$Bx = \sum_{j \in \mathbb{N}} \mu_j x_j f_j$$

for all $x = \sum_{j\in\mathbb{N}} x_j f_j$. Under previous assumptions it easily follows that

$$Af_j = \mu_j f_j + (U \otimes V)(f_j)$$

and so one retrieves the generic operators considered in our previous spectral analysis. Therefore, the same very techniques can be utilized to compute the spectrum of $A = B + U \otimes V$.

(2) To locate all elements of $\sigma_p(A)$ (and hence $\sigma(A)$ as $\sigma(A) = \{\theta_j\}_{j\geq 0} \cup \sigma_p(A)$) where A is the linear operator appearing in Theorem 4.3, we need to study the location of all the zeros of the function

$$d(z) = 1 - \sum_{j\geq 0} \frac{\lambda_j - \theta_j}{\lambda_j - z}, \quad z \in \Delta,$$

appearing in (4.2). For that, the techniques developed in Diarra's work [8] can be utilized.

(3) It would be interesting to see whether the current spectral analysis can be made in the case of a Krull valuation as it was done for the linear operator $\widehat{B}$ (see (1.1)) in Diarra [8]. Such a question will be investigated elsewhere.

Acknowledgments. The authors would like to express their thanks to the referee for careful reading of the manuscript and insightful comments.

References

[1] Basu S., Diagana T., Ramaroson F., *A p-adic version of Hilbert-Schmidt operators and applications*, J. Anal. Appl. **2** (2004), no. 3, 173–188.
[2] Attimu D., Diagana T., *Representation of bilinear forms in non-Archimedean Hilbert space by linear operators II*, Comment. Math. Univ. Carolin. **48** (2007), no. 3, 431–442.
[3] Diagana T., *Towards a theory of some unbounded linear operators on p-adic Hilbert spaces and applications*, Ann. Math. Blaise Pascal **12** (2005), no. 1, 205–222.
[4] Diagana T., *Erratum to: "Towards a theory of some unbounded linear operators on p-adic Hilbert spaces and applications"*, Ann. Math. Blaise Pascal **13** (2006), 105–106.
[5] Diagana T., *Non-Archimedean Linear Operators and Applications*, Nova Science Publishers, Huntington, NY, 2007.
[6] Diagana T., *Representation of bilinear forms in non-archimedean Hilbert space by linear operators*, Comment. Math. Univ. Carolin. **47** (2006), no. 4, 695–705.
[7] Diagana T., *An Introduction to Classical and p-adic Theory of Linear Operators and Applications*, Nova Science Publishers, New York, 2006.
[8] Diarra B., *An operator on some ultrametric Hilbert spaces*, J. Analysis **6** (1998), 55–74.
[9] Diarra B., *Geometry of the p-adic Hilbert spaces*, preprint, 1999.
[10] Fois C., Jung I.B., Ko E., Pearcy C., *On rank one perturbations of normal operators*, J. Funct. Anal. **253** (2008), 628–646.
[11] Ionascu E., *Rank-one perturbations of diagonal operators*, Integral Equations Operator Theory **39** (2001), 421–440.
[12] Keller H.A., Ochsenius H., *Bounded operators on non-archimedean orthomodular spaces*, Math. Slovaca **45** (1995), no. 4, 413–434.

[13] Ochsenius H., Schikhof W.H., *Banach spaces over fields with an infinite rank valuation*, p-adic Functional Analysis (Poznan, 1998), Lecture Notes in Pure and Appl. Mathematics, 207, Marcel Dekker, New York, 1999, pp. 233–293.
[14] van Rooij A.C.M., *Non-archimedean Functional Analysis*, Marcel Dekker, New York, 1978.

Department of Mathematics, Howard University, 2441 6th Street N.W., Washington, D.C. 20059, USA
Email: tdiagana@howard.edu

Department of Mathematics, Howard University, 2441 6th Street N.W., Washington, D.C. 20059, USA
Email: george.mcneal@navy.mil

(*Received* January 27, 2009*, revised* May 19, 2009)