Almost disjoint families and “never” cardinal invariants

Charles Morgan, Samuel Gomes da Silva

Abstract. We define two cardinal invariants of the continuum which arise naturally from combinatorially and topologically appealing properties of almost disjoint families of sets of the natural numbers. These are the *never soft* and *never countably paracompact* numbers. We show that these cardinals must both be equal to ω_1 under the effective weak diamond principle $\diamondsuit(\omega,\omega,<)$, answering questions of da Silva S.G., *On the presence of countable paracompactness, normality and property* (a) *in spaces from almost disjoint families*, Questions Answers Gen. Topology **25** (2007), no. 1, 1–18, and give some information about the strength of this principle.

Keywords: almost disjoint families, parametrized weak diamond principles, property (a), countable paracompactness

Classification: Primary 03E65, 54D20; Secondary 03E17, 54A35

1. Introduction

A family $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ of infinite subsets of ω is said to be *almost disjoint* if $A_\alpha \cap A_\beta$ is finite for all distinct $\alpha, \beta < \kappa$.

Definition 1.1. An almost disjoint family $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ *has property* (a), or $(a)_{\mathcal{A}}$ *holds*, if

$$\forall\, g \in {}^{\kappa}\omega\ \exists\, P \in [\omega]^{\omega}\ \forall\, \alpha < \kappa\ (|P \cap A_\alpha| < \omega \ \text{ and } \ P \cap A_\alpha \not\subseteq g(\alpha)).$$

The condition $(a)_{\mathcal{A}}$ is an appealing combinatorial property, strengthening non-maximality of $\mathcal{A}$, and deserves further investigation. It prompts the definition of two (uncountable) cardinal invariants of the continuum, the *non-soft* number and the *never soft* number, which help formulate concisely some of the more combinatorial results surveyed in [Sil2] and questions arising from them.

Definition 1.2. The cardinal invariants $\mathfrak{nsa}$ and $\mathfrak{vsa}$ are defined by:

$$\mathfrak{nsa} = \min\{\, |\mathcal{A}| \,:\, \neg(a)_{\mathcal{A}} \,\}.$$
$$\mathfrak{vsa} = \min\{\, \kappa \in \mathrm{Card} \,:\, \forall\, \mathcal{A}\ (|\mathcal{A}| = \kappa \implies \neg(a)_{\mathcal{A}})\,\}.$$

The first author was supported by Fapesb, the Fundação de Amparo à Pesquisa do Estado da Bahia (Grant BOL 0430/2008).

Besides its intrinsic combinatorial interest the property $(a)_{\mathcal{A}}$ also has considerable topological importance. Although not necessary for the rest of the paper we now briefly outline this.

The *Isbell-Mrówka space* $\Psi(\mathcal{A})$ derived from an almost disjoint family $\mathcal{A}$ has the elements of $\mathcal{A}\cup\omega$ as points. Its topology is that the elements of ω are isolated and the neighbourhoods of each $A \in \mathcal{A}$ are the sets $\{A\} \cup (A \setminus F)$ for each finite subset F of ω. It is striking that every Hausdorff, first countable, locally compact, separable space whose set of accumulation points is non-empty and discrete is homeomorphic to some Isbell-Mrówka space, and thus almost disjoint families are topologically natural and significant objects.

Consideration of various topological properties related to the normality of Isbell-Mrówka spaces (surveyed in [Sil2]) led to the isolation of $(a)_{\mathcal{A}}$ (see [SV]). As in this paper we focus on combinatorial aspects of almost disjoint families we will say, in a general and very natural way, that an almost disjoint family $\mathcal{A}$ is $\mathcal{P}$ if the corresponding Isbell-Mrówka space $\Psi(\mathcal{A})$ is $\mathcal{P}$ for any topological property $\mathcal{P}$.

We do not need to make explicit either the Matveev's general, topological definition of "property (a)" from [M] or the definition of the adjective *soft*, inherited from [SV], or indeed the general topological definition of countable paracompactness (see the following definition), for the purpose of this note. The definitions we give in this paper, Definition 1.1 and Definition 1.3, are immediately equivalent to these more general definitions, as can be seen from intermediate formulations given in the context of Isbell-Mrówka spaces in [Sil2] and [SV].

A property related to almost disjoint families is countable paracompactness. In the following definition, we abbreviate the formula

$$\exists\langle E_n \mid n < \omega\rangle \;\; \forall m < n < \omega \;\; E_n \subseteq E_m \subseteq \omega$$

by

$$\exists^{\searrow}\langle E_n \mid n < \omega\rangle.$$

Definition 1.3. An almost disjoint family $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa\rangle$ is *countably paracompact* if, and only if,

$$\forall g \in {}^{\kappa}\omega \;\; \exists^{\searrow}\langle E_n \mid n < \omega\rangle \;\; \exists f \in {}^{\kappa}\omega \;\; \forall \alpha < \kappa$$
$$A_\alpha \setminus E_{g(\alpha)} \text{ and } A_\alpha \cap E_{f(\alpha)} \text{ are both finite.}$$

As for "property (a)", we can immediately define two cardinal invariants, the *non-countably paracompact* and *never countably paracompact* numbers.

Definition 1.4. The uncountable cardinal invariants $\mathfrak{ncp}$ and $\mathfrak{vcp}$ are defined by:

$$\mathfrak{ncp} = \min\{\,|\mathcal{A}| \,:\, \mathcal{A} \text{ is not countably paracompact}\}.$$

$$\mathfrak{vcp} = \min\{\,\kappa \in \mathrm{Card} \,:\, \text{no } \mathcal{A} \text{ of size } \kappa \text{ is countably paracompact}\}.$$

The cardinal invariants $\mathfrak{nsa}$ and $\mathfrak{ncp}$ were first defined, respectively, in [Sze] and [Sil2]. In this paper, we introduce the "never" cardinal invariants $\mathfrak{vsa}$ and $\mathfrak{vcp}$. Clearly $\omega_1 \leq \mathfrak{nsa} \leq \mathfrak{vsa}$ and $\omega_1 \leq \mathfrak{ncp} \leq \mathfrak{vcp}$.

Fact 1.5. $\mathfrak{vsa}$, $\mathfrak{vcp} \leq 2^\omega$.

PROOF: These inequalities are consequences of the books of Matveev ([M]) and Fleissner ([F]), respectively. See [Sil2, pp. 5–7]. □

Thus $\mathfrak{vsa}$ and $\mathfrak{vcp}$ are well defined cardinal invariants. Moreover, "not having property (a)" and "not being countably paracompact" are coherent notions of "large" for subsets of $[\omega]^\omega$, broadening simply having maximal cardinality, that is, cardinality 2^ω. This is in contrast, for example, with being "maximal almost disjoint", since (trivially) for any maximal almost disjoint family there are non-maximal almost disjoint families of the same size.

The next remark and corollary give a unified (and mildly generalized) view of observations of the second author [Sil1] and (in a specific case) of Watson [W].

Remark 1.6. *If $\kappa < \max\{\mathfrak{vsa}, \mathfrak{vcp}\}$ there is a dominating family of size 2^ω in $({}^\kappa\omega, <)$.*

PROOF: As $\kappa < \max\{\mathfrak{vsa}, \mathfrak{vcp}\}$ there is some almost disjoint $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ such that either $(a)_\mathcal{A}$ or $\mathcal{A}$ is countably paracompact. For each $P \subseteq \omega$, define a partial function h_P from κ into ω by $h_P(\alpha) = \max(P \cap A_\alpha)$ if it exists, and leave it undefined otherwise. Similarly, for every descending sequence $\mathcal{E} = \langle E_n \mid n < \omega \rangle$ of elements of $[\omega]^\omega$ let $h_\mathcal{E}$ be the partial function given by $h_\mathcal{E}(\alpha) =$ the least n such that $A_\alpha \cap E_n$ is finite if such an n exists, and undefined otherwise. Let $g \in {}^\kappa\omega$.

If $(a)_\mathcal{A}$ then there is some $P \subseteq \omega$ such that h_P is total and for every $\alpha < \kappa$ one has $g(\alpha) < h_P(\alpha)$. So $\{h_P : P \subseteq \omega\}$ is a dominating family.

If $\mathcal{A}$ is countably paracompact there is a descending ω-sequence $\mathcal{E}$ such that $h_\mathcal{E}$ is total and for every $\alpha < \kappa$ one has $g(\alpha) < h_\mathcal{E}(\alpha)$. It follows that the collection of all such $h_\mathcal{E}$ is a dominating family. □

Corollary 1.7. *If $\omega_1 < \max\{\mathfrak{vsa}, \mathfrak{vcp}\}$ and $2^\omega < 2^{\omega_1}$ then $\aleph_{\omega_1} < 2^\omega$ and if 2^ω is also regular then there are inner models in which there are measurable cardinals.*

PROOF: By Remark 1.6 the first hypothesis gives that there is a dominating family of size 2^ω in ${}^{\omega_1}\omega$, and the work of Jech-Prikry ([JP]) then gives the conclusions. □

The paper is organized as follows. In Section 2 we recall the definition of a certain category, $\mathcal{PV}$. Each object in this category has an associated "parametrized weak diamond principle", and we detail basic facts showing that these principles are closely related to set-theoretic hypotheses such as $\diamondsuit$ and $2^\omega < 2^{\omega_1}$ and so merit their name. In Section 3 we study $(\omega, \omega, <)$, one of the objects of $\mathcal{PV}$, establishing various equivalences and giving some information about the strength of the corresponding *effective* weak diamond principle $\diamondsuit(\omega, \omega, <)$. (For instance, we prove that the latter is independent of CH, and that it is consistent with ¬CH independently of whether $2^\omega < 2^{\omega_1}$ or not.) In Section 4 we prove that $\diamondsuit(\omega, \omega, <)$ implies $\mathfrak{vsa} = \mathfrak{vcp} = \omega_1$. In Section 5 we present a number of new questions including some touching on cardinal invariants related directly to normality.

Our notation is standard, but for clarity we run over some of the more frequently used terms. $|X|$ denotes the cardinality of a set X. ω is the set of all natural numbers and the first infinite cardinal. ω_1 is the first uncountable cardinal and $\mathfrak{c}$, the cardinality of the continuum, is the cardinal 2^ω. CH denotes the statement "$\omega_1 = \mathfrak{c}$" (the *Continuum Hypothesis*). $[\omega]^\omega$ is the family of all infinite subsets of ω and $[\omega]^{<\omega}$ is the family of all finite subsets of ω. ${}^\alpha A$ is the family of functions from α to A, but we write $2^{<\alpha}$ for $\bigcup_{\beta<\alpha} {}^\beta 2$. *Inclusion mod finite*, $\subseteq^*$, is the quasi-order (i.e., reflexive, transitive relation) defined by $A \subseteq^* B$ if $A \setminus B$ is finite. We assume that the reader is familiar with the standard definitions of "club" (closed and unbounded) and "stationary" subsets of ω_1. The set of all countable limit ordinals is denoted by $\lim(\omega_1)$.

We thank the referee for their very useful comments and suggestions.

2. The category $\mathcal{PV}$ and parametrized $\diamondsuit$'s

The category $\mathbf{Dial}_2(\mathbf{Sets})^{op}$ is the dual of the fundamental (and simplest) example of a Dialectica category ([P2]). As surveyed in the Blass's interesting paper [Bla1], it has proven useful in linear logic, the study of cardinal invariants of the continuum and complexity theory. Its *objects* are triples $o = (A, B, E)$ consisting of sets A and B and a relation $E \subseteq A \times B$ such that

$$\forall\, a \in A \;\; \exists b \in B \;\; a\, E\, b \;\text{ and }\; \forall\, b \in B \;\; \exists a \in A \;\; \neg a\, E\, b.$$

(ϕ, ψ) is a *morphism* from $o_2 = (A_2, B_2.E_2)$, to $o_1 = (A_1, B_1, E_1)$, if $\phi : A_1 \longrightarrow A_2$, $\psi : B_2 \longrightarrow B_1$ and

$$\forall\, a \in A_1 \;\; \forall\, b \in B_2 \;\; \phi(a)\, E_2\, b \;\longrightarrow\; a\, E_1\, \psi(b).$$

(The '**Sets**' part of the name comes from the fact that if $o = (A, B, E)$ is an object in the category then A and B are sets, while the '2' comes from the equivalence of such an $E \subseteq A \times B$ with its characteristic function — whose range is $\{0, 1\} = 2$.)

The category is partially ordered by $o_1 \leq_{GT} o_2$ if there is a morphism from o_2 to o_1. Two objects are Galois-Tukey equivalent, $o_1 \sim_{GT} o_2$, if $o_1 \leq_{GT} o_2$ and $o_2 \leq_{GT} o_1$.

We now restrict attention to the small subcategory whose objects have constituent sets A and B with $|A|$, $|B| \leq 2^\omega$. We use Blass's notation $\mathcal{PV}$ (after de Paiva ([P1]) and Vojtáš ([V]), its introducers) for this category. Note that Vojtáš uses order-theoretic vocabulary and refers to morphisms in $\mathcal{PV}$ as *generalized Galois-Tukey connections*.

The following fact is well known and easy to check.

Fact 2.1. $(\mathbb{R}, \mathbb{R}, \neq)$ *is minimal and* $(\mathbb{R}, \mathbb{R}, =)$ *is maximal in the restriction of* $\leq_{GT}$ *to* $\mathcal{PV}$. □

Next, we recall the parametrized diamond principles, defined in [MHD], associated to objects in $\mathcal{PV}$.

Definition 2.2 ([MHD])**.** Let $o = (A, B, E)$ be an object in $\mathcal{PV}$. Then $\Phi(o)$ is the statement:

$$\begin{aligned}&\forall F : 2^{<\omega_1} \longrightarrow A \;\; \exists g \in {}^{\omega_1}B \;\; \forall f \in {}^{\omega_1}2 \\ &\qquad\qquad \{\alpha < \omega_1 \;:\; F(f \restriction \alpha)\, E\, g(\alpha)\} \text{ is stationary in } \omega_1.\end{aligned}$$

Fact 2.3. *If $o_1 \leq_{GT} o_2$ then $\Phi(o_2)$ implies $\Phi(o_1)$. So if $o_1 \sim_{GT} o_2$ then $\Phi(o_1) \longleftrightarrow \Phi(o_2)$.*

Proof: Immediate from the definitions. □

We include some information about the $\Phi(o)$ for context; it is not used below.

Fact 2.4. *The following implications hold, for every object $o \in ob(\mathcal{PV})$:*

$$\begin{aligned}&\diamondsuit \longleftrightarrow \Phi(\mathbb{R}, \mathbb{R}, =) \longrightarrow \Phi(o) \longrightarrow \\ &\qquad \Phi(\mathbb{R}, \mathbb{R}, \neq) \longleftrightarrow \Phi(2, 2, \neq) \longleftrightarrow \Phi(2, 2, =) \longleftrightarrow 2^{\omega} < 2^{\omega_1}.\end{aligned}$$

Proof: For the first equivalence see [MHD]. The two subsequent implications follow from Facts 2.1 and 2.3. The next equivalence is due to Abraham (unpublished); a proof may be found in [MHD]. The penultimate equivalence is easy — g witnesses $\Phi(2, 2, =)$ for some F if and only if $1 - g$ witnesses $\Phi(2, 2, \neq)$. The final equivalence is due to Devlin and Shelah. (The proof is non-trivial, see [DS] or [I].) □

In this note we make use of effective versions of these parametrized diamond principles rather than the full, unrestricted principles themselves. These effective parametrized diamond principles, whose definition we now give, were also originally introduced in [MHD].

Definition 2.5.

(i) An object (A, B, E) in $\mathcal{PV}$ is *Borel* if A, B and E are Borel subsets of some Polish space.

(ii) A map $f : X \longrightarrow Y$ from a Borel subset of a Polish space to a Borel subset of another is itself *Borel* if for every Borel $Z \subseteq Y$ one has that $f^{-1}``Z$ is Borel.

(iii) If o_1 and o_2 are both Borel then $o_1 \leq^B_{GT} o_2$ if there is a morphism from o_2 to o_1 both of whose constituent maps are Borel, and $o_1 \sim^B_{GT} o_2$ if $o_1 \leq^B_{GT} o_2$ and $o_2 \leq^B_{GT} o_1$.

(iv) A map $F : 2^{<\omega_1} \longrightarrow A$ is *Borel* if it is level-by-level Borel: *i.e.*, if for each $\alpha < \omega_1$ the map $F \restriction 2^{<\alpha} : 2^{<\alpha} \longrightarrow A$ is Borel.

(v) If o is Borel we define the principle $\diamondsuit(o)$ as in [MHD]:

$$\begin{aligned}&\forall \text{ Borel } F : 2^{<\omega_1} \longrightarrow A \;\; \exists g \in {}^{\omega_1}B \;\; \forall f \in {}^{\omega_1}2 \\ &\qquad\qquad \{\alpha < \omega_1 \;:\; F(f \restriction \alpha)\, E\, g(\alpha)\} \text{ is stationary in } \omega_1.\end{aligned}$$

Fact 2.6. *If o_1, o_2 are both Borel and $o_1 \leq^B_{GT} o_2$ then $\diamondsuit(o_2) \longrightarrow \diamondsuit(o_1)$; if $o_1 \sim^B_{GT} o_2$ we have $\diamondsuit(o_2) \longleftrightarrow \diamondsuit(o_1)$.*

Proof: Again immediate from the definitions. □

3. The object $(\omega, \omega, <)$

We focus attention on the object $(\omega, \omega, <)$ of the category $\mathcal{PV}$. We give several equivalents of it and some further information about its place in the Galois-Tukey ordering. We also show that $\diamondsuit(\omega, \omega, <)$ is independent of CH, and that it is consistent with the failure of CH independently of whether $2^\omega < 2^{\omega_1}$ holds or not.

Lemma 3.1. $(\omega, \omega, <) \sim_{GT} ([\omega]^{<\omega}, [\omega]^{<\omega}, \subsetneq)$.

Proof: The pair of functions $n \mapsto \{0, \ldots, n-1\}$ and $a \mapsto \max(a)$ give morphisms in both directions. □

Lemma 3.2. *Let $E \subseteq \mathcal{P}(\omega) \times \omega$ be given by $a\,E\,n$ iff $a \in [\omega]^\omega$ or $a \subseteq n$. Then $(\omega, \omega, <) \sim_{GT} (\mathcal{P}(\omega), \omega, E)$.*

Proof: In order to show $(\omega, \omega, <) \leq_{GT} (\mathcal{P}(\omega), \omega, E)$ consider the morphism consisting of the singleton map $n \mapsto \{n\}$ from ω to $\mathcal{P}(\omega)$ and the identity map on ω. If $\{n\}\,E\,m$ then, since $\{n\}$ is finite, one has $n < m$, which is what is required.

For the reverse inequality consider the morphism consisting of the map $x : \mathcal{P}(\omega) \longrightarrow \omega$ given by $x(a) = \max(a)$ if a is finite and $x(a) = 0$ if a is infinite and the identity map on ω. If $x(a) < k$ then either a is infinite or $\max(a) < k$. But then either a is infinite or $a \subseteq k$. □

Lemma 3.3. *For every finite m we have*

$$([\omega]^m, \omega, \not\ni) \leq_{GT} (\omega, \omega, <) \sim_{GT} ([\omega]^{<\omega}, \omega, \not\ni).$$

Proof: The pair consisting of the map $mx : [\omega]^m \longrightarrow \omega$, given by $mx(a) = \max(a)$, and the identity on ω yields that $([\omega]^m, \omega, \not\ni) \leq_{GT} (\omega, \omega, <)$ for all $m \in \omega \cup \{< \omega\}$. For if $\max(a) < k$ then $k \notin a$. On the other hand, the pair consisting of the map $f : \omega \longrightarrow [\omega]^{<\omega}$, given by $f(n) = \{0, \ldots, n\}$, and the identity function on ω shows that we have $(\omega, \omega, <) \leq_{GT} ([\omega]^{<\omega}, \omega, \not\ni)$. For if $k \notin f(n)$ then $n < k$. □

Lemma 3.4. $(\omega, \omega, <) \leq_{GT} ({}^\omega\omega, {}^\omega\omega, <)$.

Proof: The maps $c : \omega \longrightarrow {}^\omega\omega$, given by $c(n) = c_n$ and $c_n(k) = n$ for all $k \in \omega$, and $\min : {}^\omega\omega \longrightarrow \omega$ given by $\min(f) = \min(\mathrm{im}(f))$ form a Galois-Tukey connection. For if $c_n \leq f$ one has $c_n(k) = n < f(k)$ for all $k \in \omega$, whence $n < \min(\mathrm{im}(f))$. □

We now discuss the relationship between $\diamondsuit(\omega, \omega, <)$ and cardinal arithmetic assumptions such as CH and $2^\omega < 2^{\omega_1}$.

First of all we show that $\diamondsuit(\omega, \omega, <)$ is independent of CH.

Definition 3.5. (i) A sequence $H = \langle \eta_\alpha \mid \alpha \in \lim(\omega_1) \rangle$ is a *ladder system* if each $\eta_\alpha : \omega \longrightarrow \alpha$ is a cofinal embedding.

(ii) Let $n \in \omega \cup \{\omega\} \setminus 2$ and $k \in \omega \cup \{<\omega\} \setminus 1$ with $k < n$. The ladder system H can be (n,k)-*uniformized* if

$$\forall f : \omega_1 \longrightarrow n \;\; \exists g : \omega_1 \longrightarrow [n]^k \;\; \forall \alpha \in \lim(\omega_1) \quad \{i < \omega \,:\, f(\alpha) \notin g(\eta_\alpha(i))\} \text{ is finite.}$$

(iii) For n and k as in (ii), the principle $\neg\mathrm{Unif}_n(k)$ is given by the statement "No ladder system is (n,k)-uniformizable."

Proposition 3.6. *Let $0 < k \le m < n \le \omega$. Then, we have that $\diamondsuit([n]^m, n, \not\ni) \Longrightarrow \neg\,\mathrm{Unif}_n(k)$.*

Proof: Fix a coding of ${}^{\omega_1}([n]^k)$ by ${}^{\omega_1}2$ in such a way that ${}^{\delta}([n]^k)$ is coded by ${}^{\delta}2$ for all limit δ. (This is trivial as n, k are countable.)

Let $H = \langle \eta_\alpha \mid \alpha \in \lim(\omega_1) \rangle$ be a ladder system. Define a function $F : 2^{<\omega_1} \longrightarrow [n]^m$ by $F(e) = \emptyset$ for successor $\delta + 1$ and all $e \in {}^{\delta+1}2$, and

$$F(e) = \{ j < n \,:\, \{ i < \omega \,:\, j \notin e(\eta_\delta(i)) \} \text{ is finite} \}$$

for all limit $\delta < \omega_1$ and all $e \in {}^{\delta}2$. Notice that F is Borel.

By $\diamondsuit([n]^m, n, \not\ni)$ choose $g : \omega_1 \longrightarrow n$. Suppose $h : \omega_1 \longrightarrow [n]^k$ and regard it as a function with co-domain 2 by the coding fixed at the start of the proof.

Let S be a stationary set given by applying $\Phi([n]^m, n, \not\ni)$ to h and let $\delta \in S$ be a limit ordinal. Then $g(\delta) \notin F(h \restriction \delta)$. Hence, by the definition of F, $\{ i < \omega \,:\, g(\delta) \notin h(\eta_\delta(i)) \}$ is not finite. Thus h does not uniformize the coloring g. □

Fact 3.7 ([BEGPS, §2])**.** *For each $k \in \omega$ it is consistent with* CH *that every ladder system on ω_1 can be $(\omega, k+1)$-uniformized but none can be (ω, k)-uniformized.*

We remark that the authors of [BEGPS] write that a ladder system "satisfies $\mathcal{M}_k$" exactly when, in our terminology, it can be (ω, k)-uniformized.

Corollary 3.8. CH *does not imply $\diamondsuit(\omega, \omega, <)$.*

Proof: It is immediate from Proposition 3.6 and Fact 3.7 that if ZFC is consistent then for each $m \in [2, \omega]$ one has that ZFC + CH + $\neg\diamondsuit([\omega]^m, \omega, \not\ni)$ is also consistent. The corollary now follows from Lemma 3.3 and Fact 2.6. □

Next we observe that $\diamondsuit(\omega, \omega, <)$ is consistent with the failure of CH. We give two models. In the first the weak diamond, $2^\omega < 2^{\omega_1}$, is false and in the second it holds.

Proposition 3.9. *It is consistent that $\diamondsuit(\omega, \omega, <)$ holds and $2^\omega = \omega_2$.*

Proof: The cardinal $\min(\{\|X\| : X \subseteq \omega \ \& \ \forall n < \omega\, \exists m < X\ n < m\})$, the *evaluation* of $(\omega, \omega, <)$, is trivially always ω. It is then immediate from Proposition 6.6 of [MHD] that $\diamondsuit(\omega, \omega, <)$ holds in a broad collection of forcing extensions which are countable support iterations of length ω_2. The collection includes, for example, iteration of Sacks forcing. Proposition 6.6 and Remark 6.7 of [MHD] give precise details of the nature that the components of these iterations may have. $\square$

Proposition 3.10. *$\diamondsuit(\omega, \omega, <)$ is consistent with $2^\omega = \omega_{\omega_1} < 2^{\omega_1}$.*

Proof: The proof is a modest modification of the proof of Proposition 6.1 of [MHD], that $\mathcal{C}_{\omega_1}$, the Cohen algebra corresponding to the product space 2^{ω_1}, forces $\diamondsuit(\mathrm{non}(\mathcal{M}))$. We give details for completeness.

We force with $\mathbb{P} = \mathcal{C}_{\omega_{\omega_1}}$. Let $\dot{G}$ be a $\mathbb{P}$-name for the element of $2^{\omega_{\omega_1}}$ corresponding to the generic filter. Let $\dot{F}$ be a $\mathbb{P}$-name for a map from $2^{<\omega_1} \longrightarrow \omega$ and let $\dot{r}_\delta$ be a $\mathbb{P}$-name for a real such that $\dot{F} \restriction 2^\delta$ is definable from $\dot{r}_\delta$. Pick a strictly increasing function $h : \omega_1 \longrightarrow \{\omega_\alpha : \alpha < \omega_1\}$ such that $\dot{r}_\delta$ is forced to be in $V[\dot{G} \restriction h(\delta)]$. Interpret $\dot{G} \restriction [h(\delta), h(\delta) + \omega)$ canonically as a real (a subset of ω) and let $\dot{g}(\delta)$ be the first element of it.

Now let $\dot{f} : \omega_1 \longrightarrow 2$ be a $\mathbb{P}$-name. Let X be the collection of all δ for which it is forced that $\dot{f} \restriction \delta \in V[\dot{G} \restriction \omega_\delta]$. Because $\mathbb{P}$ is ccc one has that X is closed and unbounded. Let $Y \subseteq X$ be forced to be closed and unbounded. Since $\dot{G}$ is generic it follows that there are $\delta \in Y$ such that it is forced that $\dot{F}(\dot{f} \restriction \delta) < \dot{g}(\delta)$. $\square$

We thank Justin Moore for drawing our attention to this example in an email discussion of the possible consistency of $\Phi(\omega, \omega, <)$ together with $\neg$CH and $2^\omega < 2^{\omega_1}$. This latter problem is still an open question as far as we know. See Section 5 of this paper.

4. $\diamondsuit(\omega, \omega, <)$ and the "never" cardinal invariants

Proposition 4.1. *$\diamondsuit(\omega, \omega, <)$ implies $\mathfrak{vsa} = \omega_1$.*

The preceding proposition is an immediate corollary of the following result.

Proposition 4.2. *$\diamondsuit(\omega, \omega, <)$ implies that for every almost disjoint family $\mathcal{A} = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ there is some $g \in {}^{\omega_1}\omega$ such that, for every $P \in [\omega]^\omega$,*

$$\text{either } \{\alpha < \omega_1 : P \cap A_\alpha \text{ is infinite}\}$$
$$\text{or } \{\alpha < \omega_1 : P \cap A_\alpha \subseteq g(\alpha)\} \quad \text{is a stationary subset of } \omega_1.$$

Proof: Let $\mathcal{A} = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ be an almost disjoint family. Define $F : 2^{<\omega_1} \longrightarrow \mathcal{P}(\omega)$ by $F(h) = A_{\mathrm{dom}(h)} \cap X_{h \restriction \omega}$ for $h \in 2^{<\omega_1}$, where $k \in X_{h\restriction\omega}$ if and only if $h(k) = 1$ for $k \in \omega$. Notice that F is Borel. Apply (by Lemma 3.2 and Fact 2.6), $\diamondsuit(\mathcal{P}(\omega), \omega, E)$ to F and obtain a function $g : \omega_1 \longrightarrow \omega$.

Let $P \in [\omega]^\omega$. If $\{\alpha < \omega_1 : P \cap A_\alpha \text{ is infinite}\}$ is not stationary in ω_1 then its complement, $\{\alpha < \omega_1 : |P \cap A_\alpha| < \omega\}$, contains a closed and unbounded set, say, C. In this case choose some $f \in {}^{\omega_1}2$ such that $X_{f\restriction\omega} = P$. By the property of g given by $\diamondsuit(\mathcal{P}(\omega), \omega, E)$ one has that the set

$$S = \{\, \alpha < \omega_1 \;:\; A_\alpha \cap P \text{ is infinite or } A_\alpha \cap P \subseteq g(\alpha) \,\}$$

is stationary in ω_1. Since C is club and S is stationary one has that $\{\, \alpha < \omega_1 \;:\; A_\alpha \cap P \subseteq g(\alpha) \,\} \supseteq C \cap S$ is stationary in ω_1. $\square$

Proposition 4.3. *$\diamondsuit(\omega, \omega, <)$ implies $\mathfrak{vcp} = \omega_1$.*

In a similar manner to the proof of Proposition 4.1, Proposition 4.3 is an immediate corollary of the following result.

Proposition 4.4. *$\diamondsuit(\omega, \omega, <)$ implies that for every almost disjoint family $\mathcal{A} = \langle A_\alpha \,|\, \alpha < \omega_1 \rangle$ there is some $g \in {}^{\omega_1}\omega$ such that, for every sequence $\langle E_n \,|\, n < \omega \rangle$ in $[\omega]^\omega$ descending under inclusion and $f \in {}^{\omega_1}\omega$,*

$$\begin{aligned} &\textit{either } \{\, \alpha < \omega_1 \;:\; A_\alpha \cap E_{f(\alpha)} \textit{ is infinite} \,\} \\ &\qquad \textit{or } \{\, \alpha < \omega_1 \;:\; A_\alpha \not\subseteq^* E_{g(\alpha)} \,\} \quad \textit{is a stationary subset of } \omega_1. \end{aligned}$$

Proof: Let $\mathcal{A} = \langle A_\alpha \,|\, \alpha < \omega_1 \rangle$ be an almost disjoint family. Fix some canonical way in which functions in ${}^\omega\omega$ code sequences in ${}^\omega\mathcal{P}(\omega)$. If e is such a function we write $\langle X^e_m \,|\, m \in \omega \rangle$ for associated sequence.

Define $F : 2^{<\omega_1} \longrightarrow \omega$ by $F(e) =$ least m such that $A_{\mathrm{dom}(e)} \setminus X^{e\restriction\omega}_{m+1}$ is infinite for $e \in 2^{<\omega_1}$, if such an m exists, and $= 0$ otherwise. Apply $\diamondsuit(\omega, \omega, <)$ to F and obtain a function $g : \omega_1 \longrightarrow \omega$.

Let $E = \langle E_m \,|\, m < \omega \rangle \in {}^\omega\mathcal{P}(\omega)$ and suppose that for every $\alpha < \omega_1$ we have $A_\alpha \subseteq^* E_0$ and that for all $m < n < \omega$ we have $E_n \subseteq E_m$. If the set $\{\, \alpha < \omega_1 \;:\; A_\alpha \cap E_{f(\alpha)} \text{ is infinite} \,\}$ is not stationary in ω_1 then the function f_E given by $f_E(\alpha) =$ "least m such that $A_\alpha \cap E_m$ is finite" is well-defined on a subset of ω_1 containing a club set, say, C. Let $h : \omega_1 \longrightarrow 2$ be any function such that $\langle X^{h\restriction\omega}_m \,|\, m \in \omega \rangle = \langle E_m \,|\, m < \omega \rangle$. Note that $F(h \restriction \alpha)$ never takes the value 0 since if $A_\alpha \cap E_m$ is finite then $A_\alpha \setminus E_m$ is infinite (and $m > 0$).

By $\diamondsuit(\omega, \omega, <)$ we have that

$$C \cap \{\, \alpha < \omega_1 \;:\; A_\alpha \not\subseteq^* E_{g(\alpha)} \text{ or } A_\alpha \cap E_{f(\alpha)} \text{ is infinite} \,\}$$

is stationary in ω_1, and hence so is $\{\, \alpha < \omega_1 \;:\; A_\alpha \not\subseteq^* E_{g(\alpha)} \,\}$. $\square$

We note that, in view of Proposition 3.7, Proposition 4.3 answers the open Questions 3.7 and 4.5 of [Sil2] as to whether it is consistent that $\mathfrak{ncp} < 2^\omega$ in a strong way since it shows that $\mathfrak{vcp} < 2^\omega$ is consistent.

5. Notes, problems and questions

We conclude by posing a handful of problems and questions arising from this work. We preface these by introducing a cardinal invariant for normality. The following definition is "folklore".

Definition 5.1. An almost disjoint family $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is *normal* if, and only if, for any disjoint subfamilies $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$ there is a set $S \in [\omega]^\omega$ such that

$$\forall F \in \mathcal{F} \ \ F \subseteq^* S \ \text{ and } \ \forall G \in \mathcal{G} \ \ |S \cap G| < \omega.$$

There is no need to define a "non-normal" number because there are almost disjoint families of size ω_1 in ZFC that are non-normal. This follows from the existence of "Luzin gaps".

However it is well-known that normal separable spaces cannot include closed discrete subsets of size 2^ω (and one can easily check that ω is dense and $\mathcal{A}$ is closed and discrete in any Isbell-Mrówka space). Therefore the following definition is justified.

Definition 5.2. *The uncountable cardinal invariant* $\mathfrak{vn}$ *is defined by:*

$$\mathfrak{vn} = \min\{\kappa \in \text{Card} : \textit{there is no } \mathcal{A} \textit{ of size } \kappa \textit{ which is normal}\}.$$

Normal Isbell-Mrówka spaces are also countably paracompact (see [Sil2]), and thus $\mathfrak{vn} \leq \mathfrak{vcp}$. Therefore, it is immediate from Proposition 4.3 that $\Phi(\omega, \omega, <)$ implies $\mathfrak{vn} = \omega_1$. Note also (as the referee reminded us) that Szeptycki and Vaughan ([SV]) showed that a normal Ψ-space of cardinality $< \mathfrak{d}$ has property (a). So $\mathfrak{vn} \leq \mathfrak{d}$ implies that $\mathfrak{vn} \leq \mathfrak{vsa}$.

Problem 5.3. *Establish upper bounds for any of the cardinals* $\mathfrak{vsa}$, $\mathfrak{vcp}$ *and* $\mathfrak{vn}$ *in terms of other cardinal invariants. Specifically, is it true that if* $\theta \in \{\mathfrak{vsa}, \mathfrak{vcp}, \mathfrak{vn}\}$ *then* $\theta \leq \mathfrak{d}$, $\mathfrak{a}$ *or even* $\mathfrak{b}$*? (Here* $\mathfrak{d}$, $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the minimal sizes of families which are dominating in* ${}^\omega\omega$*, maximal almost disjoint in* $[\omega]^\omega$ *and unbounded in* ${}^\omega\omega$*, respectively.)*

Work of Brendle, Brendle-Yatabe, and Szeptycki shows that $\mathfrak{nsa} \leq \mathfrak{b}$. However, apart from the bounds on $\mathfrak{vn}$ in terms of $\mathfrak{vcp}$ and $\mathfrak{vsa}$ mentioned in the paragraph immediately prior to this problem, we know of no better upper bound for $\mathfrak{vsa}$, $\mathfrak{vcp}$ or $\mathfrak{vn}$ than 2^ω.

Question 5.4. *Does* $2^\omega < 2^{\omega_1}$ *alone imply* $\mathfrak{vsa} = \omega_1$ *or* $\mathfrak{vcp} = \omega_1$*?*

CH and $\Phi(\omega, \omega, <)$ both imply $\mathfrak{vsa} = \mathfrak{vcp} = \omega_1$. Both also imply $2^\omega < 2^{\omega_1}$ and, as mentioned in Section 1,

$$\text{Con(ZFC + "}2^\omega < 2^{\omega_1}\text{" + "}2^\omega \text{ is regular" + "}\omega_1 < \max\{\mathfrak{vsa}, \mathfrak{vcp}\}\text{")}$$

implies

$$\text{Con(ZFC + "There is a measurable cardinal").}$$

Moreover, $2^\omega < 2^{\omega_1}$ *is* sufficient to prove that some cardinal invariants are small. For example it is well known that $2^\omega < 2^{\omega_1}$ implies $\mathfrak{vn} = \omega_1$.

A possibly related open question (due to the second author) is whether it is consistent with $2^\omega < 2^{\omega_1}$ that there is a separable space X satisfying property (a) and including an uncountable closed discrete subset ([Sil2, Question 2.5]).

Question 5.5. *Is $\Phi(\omega, \omega, <)$ consistent with the failure of* CH*?*

In Section 3 we show $\diamondsuit(\omega, \omega, <)$, and hence $\Phi(\omega, \omega, <)$ is independent of CH. We also show that $\diamondsuit(\omega, \omega, <)$ is consistent with the failure of CH. However, the proofs we give, exploiting the models of §6 of [MHD], rely heavily on the effectivity of $\diamondsuit(\omega, \omega, <)$ and we not know how to remove this dependency.

Moreover, showing that CH is independent of $\Phi(\omega, \omega, <)$ is also an interesting problem because doing so may well require new developments in the theory of iterated forcing. Specifically, it seems to be an appropriate test case for trying to develop iterated forcing mixing ω^ω-bounding forcing with $\mathbb{D}$-completeness since it appears one will not need to juggle with intricate technical details not connected with the iteration theory *per se*. (In this it contrasts, for example, with the questions arising from the work of Mildenberger on models with weak diamond principles and no Souslin trees (see, *e.g.*, [Mil]).)

Acknowledgments. This work was done during a visit of the first author to the Universidade Federal da Bahia, and he would like to thank the members of the Departamento de Matemática for their hospitality. Both authors are grateful for the financial support of Fapesb, the Fundação de Amparo à Pesquisa do Estado da Bahia.

References

[BEGPS] Balogh Z., Eisworth T., Gruenhage G., Pavlov O., Szeptycki P., *Uniformization and anti-uniformization properties of ladder systems*, Fund. Math. **181** (2004), no. 3, 189–213.

[Bla1] Blass A., *Questions and answers — a category arising in linear logic, complexity theory, and set theory*, Advances in Linear Logic (Ithaca, NY, 1993), London Math. Soc. Lecture Note Ser. **222**, Cambridge Univ. Press, Cambridge, 1995, 61–81.

[Bla2] Blass A. *Combinatorial Cardinal Characteristics of the Continuum*, to appear as a chapter in the Handbook of Set Theory (eds. M. Foreman, M. Magidor, and A. Kanamori), 104pp.
(www.math.lsa.umich.edu/∼ablass/hbk.pdf)

[Bre] Brendle J. *Dow's principle and Q-sets*, Canad. Math. Bull. **42** (1999), no. 1, 13–24.

[DS] Devlin K.J., Shelah S., *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), no. 2–3, 239–247.

[F] Fleissner W.G., *Separation properties in Moore spaces*, Fund. Math. **98** (1978), no. 3, 279–286.

[I] Ishiu T., *The Weak Diamond*, preprint, 3 pp.
(www.users.muohio.edu/ishiut/research.html)

[JP] Jech T., Prikry K., *Cofinality of the partial ordering of functions from ω_1 into ω under eventual domination*, Math. Proc. Cambridge Philos. Soc. **95** (1984), no. 1, 25–32.

[M] Matveev M.V., *Some questions on property (a)*, Questions Answers Gen. Topology **15** (1997), no. 2, 103–111.

[Mil] Mildenberger H., *Finding generic filters by playing games*, preprint. (http://www.logic.univie.ac.at/ heike/postings/reap14.pdf)

[MHD] Moore J.T., Hrušák M., Džamonja M., *Parametrized $\diamondsuit$ principles*, Trans. Amer. Math. Soc. **356** (2004), no. 6, 2281–2306.

[P1] de Paiva V.C.V., *A dialectica-like model of linear logic*, Category Theory and Computer Science (Manchester, 1989), Lecture Notes in Comput. Sci. **389** (1989), Springer, Berlin, 341-356.

[P2] de Paiva V.C.V., *Dialectica and Chu constructions: cousins?*, Theory and Applications of Categories, Vol. 17, No. 7, 2007, 127–152.

[Sil1] da Silva S.G., *Property* (a) *and dominating families*, Comment. Math. Univ. Carolin. **46** (2005), no. 4, 667–684.

[Sil2] da Silva S.G., *On the presence of countable paracompactness, normality and property* (a) *in spaces from almost disjoint families*, Questions Answers Gen. Topology **25** (2007), no. 1, 1–18.

[Sze] Szeptycki P.J., *Soft almost disjoint families*, Proc. Amer. Math. Soc. **130** (2002), no. 12, 3713–3717.

[SV] Szeptycki P.J., Vaughan J.E., *Almost disjoint families and property* (a), Fund. Math. **158** (1998), no. 3, 229–240.

[V] Vojtáš P., *Generalized Galois-Tukey-connections between explicit relations on classical objects of real analysis*, Set Theory of the Reals (Ramat Gan, 1991), Israel Math. Conf. Proc. **6**, Bar-Ilan Univ., Ramat Gan (1993), 619–643.

[W] Watson W.S., *Separation in countably paracompact spaces*, Trans. Amer. Math. Soc. **290** (1985), no. 2, 831–842.

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, United Kingdom

Email: charles.morgan@ucl.ac.uk

Instituto de Matemática, Universidade Federal da Bahia, Campus de Ondina, Av. Adhemar de Barros, S/N, Ondina, CEP 40170-110, Salvador, BA, Brazil

Email: samuel@ufba.br

(*Received* February 12, 2009, *revised* July 16, 2009)