Comment.Math.Univ.Carolin. 50,4 (2009) 477–511

On elementary moves that generate all spherical latin trades

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Abstract. We show how to generate all spherical latin trades by elementary moves from a base set. If the base set consists only of a single trade of size four and the moves are applied only to one of the mates, then three elementary moves are needed. If the base set consists of all bicyclic trades (indecomposable latin trades with only two rows) and the moves are applied to both mates, then one move suffices. Many statements of the paper pertain to all latin trades, not only to spherical ones.

Keywords: latin trade, spherical latin bi-trade, planar Eulerian triangulation

Classification: 05B15

The earliest papers on latin bi-trades [7], [8] emphasized their interpretation as partial latin squares (equivalently, partial quasigroups). Only later [9], [11] the attention turned to the underlying geometric structure. Presently this approach seems to be prevalent, cf. [2], [3], [4], [21], [22], [17], [16]. Here we shall rely mainly upon the notion of the trading surface (τ_1, τ_2, τ_3) which was obtained [12], [13] as a result of an effort to get a description of latin bi-trades that is naturally connected both to algebra and geometry (a formal definition appears in Section 3).

From every trading surface one can derive a cubic 3-connected bipartite graph by regarding triangles as additional points, and turning each arrow of τ_i into a pair of edges that connect the points of the arrow with the point representing the triangle adjacent to the arrow. The cycles of τ_i become the faces of the graph. By attributing to such a face the colour i, we see that the graph is face 3-colourable. This process can be reversed when we start from a face 3-colourable cubic 3-connected bipartite graph upon a surface (the orientation of arrows depends upon the ordering of colours). This observation is the essence of a result by Cavenagh and Lisoněk [5] in which they have shown that trading spheres are equivalent to black-and-white (i.e., Eulerian) triangulations of the sphere (the dual notion to planar bipartite cubic 3-connected graphs). The only nontrivial fact needed for this connection is a classical result of Heawood [18] by which a spherical triangulation is 3-colourable if and only if all vertices are of even order.

The graphs dual to the planar Eulerian triangulations can be obtained from the octahedron by the two moves of Batagelj [1], and their generation received

Supported by institutional grant MSM 0021620839.

a lot of attention, e.g. [19]. In this paper we obtain the same kind of results in the language of latin bi-trades. What is gained when we translate our results back?

Firstly, we shall show that only one of the Batagelj's moves suffices when we start from the bicyclic bi-trades (these can be represented upon two rows by using a cyclic shift. The corresponding graph has a form of a double wheel). The configuration needed for this move contains in its graph version a central vertex. We shall show, secondly, that the move needs to be applied, up to few exceptional configurations, only to cases when the central vertex is, say, black (this assumes that the vertices are divided into black and white throughout the generation). The exceptional configurations correspond to certain subgraphs that can be obtained by a slight modification of the graphs that represent bicyclic bi-trades. Thirdly, we shall obtain new results pertaining to the structure of cycles of length four to which the reduction move can be applied (these cycles correspond to what we shall call a *seed*).

The core of the results can be found already in [10]. However, I originally thought that the three moves are indispensable. After learning through the work of Cavenagh and Lisoněk [5] about the connection to the moves of Batagelj [1], I realized that the number of moves can be diminished.

The importance of spherical latin bi-trades has been recently confirmed not only by the fact that they are equivalent to planar (i.e. spherical) Eulerian (i.e. black-and-white) triangulations [5], but also by a recent result that every spherical latin trade can be embedded into the operational table of a finite abelian group. There are two independent and different proofs of this fact [6], [14].

Recent developments in the area seem to indicate that for many tasks the permutational representation used in this paper is less economical than the standard triangulation approach. Using this approach one can present a much shorter proof that all spherical latin bi-trades can be obtained from bicyclic bi-trades by a single move [15]. Nevertheless, the permutational representation represents an important facet of theory and is essential for some connections (which is well illustrated, say, by [4]).

1. Trading surfaces

Consider two partial latin squares L_1 and L_2 that have identical *shapes* (i.e. a cell (i, j) is occupied in either both of them, or in none of them). Call L_1 and L_2 *row balanced* if the set of symbols occurring in L_1 in row i is the same as the set of symbols that appear in the row i in L_2, for every row i. If an analogous condition holds for all columns, then L_1 and L_2 are said to be *column balanced*. Call a pair (L_1, L_2) a *latin bi-trade* if L_1 and L_2 have identical shapes, are row-balanced and column-balanced, and agree in no cell of their common shape.

Latin squares can be treated as tables or as triple sets, and the same holds for latin trades. It is straightforward to see that an equivalent definition of a latin bi-trade is that of a pair (T°, T^*), where both T° and T^* are subsets of $A_1 \times A_2 \times A_3$ such that:

(R1) sets T° and T^* are disjoint;
(R2) for all $(a_1, a_2, a_3) \in T^\circ$ and all $r, s \in \{1, 2, 3\}$, $r \neq s$, there exists exactly one $(b_1, b_2, b_3) \in T^*$ with $a_r = b_r$ and $a_s = b_s$;
(R3) for all $(b_1, b_2, b_3) \in T^*$ and all $r, s \in \{1, 2, 3\}$, $r \neq s$, there exists exactly one $(a_1, a_2, a_3) \in T^\circ$ with $a_r = b_r$ and $a_s = b_s$.

Condition (R2) describes, in fact, a mapping $\beta_t : T^\circ \to T^*$, $(a_1, a_2, a_3) \mapsto (b_1, b_2, b_3)$, where $\{r, s, t\} = \{1, 2, 3\}$, and condition (R3) states that this mapping is invertible. By setting $\sigma_{s,r} = \beta_s^{-1}\beta_r$ we obtain a permutation of T°, with $\sigma_{s,r} = \sigma_{r,s}^{-1}$ and $\sigma_{t,s}\sigma_{s,r} = \sigma_{t,r}$. Thus $\tau_1\tau_2\tau_3$ is the identity mapping of T° if $\tau_1 = \sigma_{2,3}$, $\tau_2 = \sigma_{3,1}$ and $\tau_3 = \sigma_{1,2}$. One can also write $\tau_i = \sigma_{i+1,i-1}$, where the indices are computed modulo 3. The mappings τ_i will be called *structural permutations* of the latin bi-trade (T°, T^*), and (τ_1, τ_2, τ_3) will be also known as its *structural triple*. This terminology comes from [13], where one can find a somewhat more detailed exposition.

The cyclic decomposition of τ_1 yields cycles that retain the first coordinate (which identifies rows). Similarly τ_2 retains columns and τ_3 retains symbol values. In other words, if a cycle ρ of τ_i, $i \in \{1, 2, 3\}$, moves $a = (a_1, a_2, a_3) \in T^\circ$, then a_i is an invariant of ρ. We shall say that a_i is the *value* of ρ. Any two coordinates of a determine the third one uniquely, and so the structural permutations τ_i satisfy these conditions:

(P1) a cycle of τ_r has at most one point in common with a cycle of τ_s, whenever $1 \leq r < s \leq 3$; and
(P2) all permutations τ_i, $1 \leq i \leq 3$, are fixed point free and $\tau_1\tau_2\tau_3$ is the identity.

Latin trades are often studied as fragments of a latin square of a given order. That is not our case, since here their intrinsic structure is the main subject of our interest. In such a context it seems reasonable to assume, as in [13], two additional conditions:

(R4) the sets A_1, A_2 and A_3 are pairwise disjoint; and
(R5) for all $\alpha \in \bigcup A_i$, $1 \leq i \leq 3$, there exists $(a_1, a_2, a_3) \in T^\circ$ with $a_i = \alpha$.

It is easy to construct latin bi-trades where there exist two distinct cycles of a structural permutation that yield the same value. If this never occurs, for all three structural permutations, then the bi-trade is called *separated*. In Section 2 we shall see that from every latin bi-trade one can derive essentially only one separated latin bi-trade.

Separated latin bi-trades can be equivalently considered as triples of permutations that satisfy conditions (P1) and (P2). This is proved in [13] and we shall restate the result in Section 2. Conditions (P1) and (P2) seem to be the best starting point to realize that with each latin bi-trade one can associate an oriented combinatorial surface (or a union of oriented combinatorial surfaces). The vertices of such a surface are points of the set X that is permuted by permutations τ_i (we have $X = T^\circ$ in our original setting). The edges are all pairs $\{x, \tau_i(x)\}$,

$x \in X$ and $i \in \{1,2,3\}$, and there are two kinds of 2-cells (which will be called *faces* here, for simplicity):

Cyclic faces correspond to the cycles of τ_i, $i \in \{1,2,3\}$, and *triangular faces* correspond to all triples (x_1, x_2, x_3), where $\tau_i(x_i) = x_{i-1}$. The orientation will be considered fixed by the mappings τ_i; each pair $(x, \tau_i(x))$ thus yields an *oriented edge*.

This geometric interpretation of bi-trades was known to the author since the early nineties, but the first published source seems to be [11]. A more detailed exposition of the basic idea can be found in [20], and [13] is probably the first paper where the connection is used to obtain structural results. The bulk of these results comes from the thesis [10], and this is true also for this paper. However, when [10] was being prepared, I did not realize how simply one can define the corresponding combinatorial surface, and that made the original text very formal and nearly impenetrable, even for the author.

2. Separated latin bi-trades

Let (T°, T^*) be a latin bi-trade, and let (τ_1, τ_2, τ_3) be its structural triple. Our first lemma is in fact a restatement of the definition of τ_i.

Lemma 2.1. *Assume $i \in \{1,2,3\}$. Then τ_i sends $(a_1, a_2, a_3) \in T^\circ$ to $(a'_1, a'_2, a'_3) \in T^\circ$ if and only if there exists $(b_1, b_2, b_3) \in T^*$ such that $b_j \neq a_j$ exactly when $j \equiv i - 1 \bmod 3$, and $b_k \neq a'_k$ exactly when $k \equiv i + 1 \bmod 3$.*

Lemma 2.2. *For each $i \in \{1,2,3\}$ let ρ_i be a cycle of τ_i, and let $b_i \in A_i$ be the value if ρ_i. Then:*

(i) *cycles ρ_1, ρ_2 and ρ_3 meet in a common point $b \in T^\circ$ if and only if $b = (b_1, b_2, b_3)$;*
(ii) *there exist points $x_1, x_2, x_3 \in T^\circ$ such that $\rho_i(x_{i-1}) = x_{i+1}$ for every $i \in \{1,2,3\}$ if and only if $(b_1, b_2, b_3) \in T^*$.*

Proof: Any triple of T° is determined by any two of its coordinates, and gives rise to the three cycles that move it. The part (i) is hence clear. To get the converse implication of part (ii), consider $b = (b_1, b_2, b_3) \in T^*$. By (R3) there exist triples $x_i \in T^\circ$, $i \in \{1,2,3\}$, that differ from b only in the ith coordinate. But then τ_i sends x_{i-1} to x_{i+1}, by Lemma 2.1. To see the direct implication, consider $i \in \{1,2,3\}$, set $x_{i-1} = (a_1, a_2, a_3)$, $x_{i+1} = \rho_i(x_{i-1}) = (a'_1, a'_2, a'_3)$ and determine $b = (b_1, b_2, b_3)$ in the same way as in Lemma 2.1. Both x_{i-1} and x_{i+1} agree with b in two coordinates, and hence the choice of b does not depend upon the choice of i. The value of ρ_i is clearly equal to $b_i = a_i = a'_i$. $\square$

In addition to (T°, T^*) consider now a latin bi-trade (S°, S^*). Assume that A_i and B_i, $i \in \{1,2,3\}$, are the projections of T° and S°, respectively. (Thus both T° and T^* are subsets of $A_1 \times A_2 \times A_3$, while S° and S^* are subsets of $B_1 \times B_2 \times B_3$.) A triple $(\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_i : A_i \to B_i$ are bijections, $1 \le i \le 3$, is called an *isotopy* when every $(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$ belongs to T° (or to T^*) if and only $(\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3))$ belongs to S° (or to S^*).

A more general notion is obtained when mappings $\varphi_i : A_i \to B_i$ are assumed to fulfil conditions

$$(a_1, a_2, a_3) \in T^\circ \Rightarrow (\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3)) \in S^\circ \text{ and}$$
$$(a_1, a_2, a_3) \in T^* \Rightarrow (\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3)) \in S^*.$$

Such a triple of mappings will be called for the purposes of this section a *homotopy*. Homotopies in this sense can be regarded as morphisms in the category of latin bi-trades, and then isotopies become isomorphisms. Since at this point there is no clear benefit from such an abstract approach, we shall not develop theory further in this direction. Similarly, we shall leave aside a general treatment of coverings, and will directly define an injective covering:

A homotopy $(\varphi_1, \varphi_2, \varphi_3) : (T^\circ, T^*) \to (S^\circ, S^*)$ is called an *injective covering* if for all $(b_1, b_2, b_3) \in S^\circ$ (or S^*) there exists exactly one $(a_1, a_2, a_3) \in T^\circ$ (or T^*) such that

$$(\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3)) = (b_1, b_2, b_3).$$

Every homotopy $(\varphi_1, \varphi_2, \varphi_3) : (T^\circ, T^*) \to (S^\circ, S^*)$ yields a mapping $\varphi : T^\circ \to S^\circ$ that sends each $(a_1, a_2, a_3) \in T^\circ$ to $(\varphi_1(a_1), \varphi_2(a_2), \varphi_3(a_3)) \in S^\circ$. This mapping will be called the *point mapping* of the homotopy $(\varphi_1, \varphi_2, \varphi_3)$.

Lemma 2.3. *Let $(\varphi_1, \varphi_2, \varphi_3) : (T^\circ, T^*) \to (S^\circ, S^*)$ be a homotopy of latin bi-trades, and let $\varphi : T^\circ \to S^\circ$ be the associated point mapping. The homotopy is an injective covering if and only if φ is a bijection. Furthermore,*

$$\varphi\tau_i = \sigma_i\varphi \text{ for all } i \in \{1, 2, 3\},$$

where τ_i and σ_i are the structural permutations of (T°, T^*) and (S°, S^*), respectively.

Proof: Choose $i \in \{1, 2, 3\}$ and $a = (a_1, a_2, a_3) \in T^\circ$. Let $a' = (a'_1, a'_2, a'_3)$ equal $\tau_i(a_1, a_2, a_3)$, and let $(b_1, b_2, b_3) \in T^*$ be as in Lemma 2.1. Triples $\varphi(a)$ and $\varphi(a')$ belong to S° and agree in the ith coordinate. The triple $(\varphi_1(b_1), \varphi_2(b_2), \varphi_3(b_3))$ belongs to S^*, and agrees with $\varphi(a)$ in the $(i-1)$th coordinate and with $\varphi(a')$ in the $(i+1)$th coordinate. Hence $\sigma_i\varphi(a) = \varphi(a') = \varphi\tau_i(a)$, by Lemma 2.1 again.

If $(\varphi_1, \varphi_2, \varphi_3)$ is a covering, then φ is a bijection, by definition. Suppose that φ is a bijection, and consider $(c_1, c_2, c_3) \in S^*$. For each $i \in \{1, 2, 3\}$ choose a cycle γ_i of σ_i in such a way that c_i is the value of γ_i, and put $\rho_i = \varphi^{-1}\gamma_i\varphi$. We have $\varphi^{-1}\sigma_i\varphi = \tau_i$, by the preceding part of the proof, and hence ρ_i has to be a cycle of τ_i. Define b_i as the value of ρ_i, and note that $\varphi_i(b_i) = c_i$, $1 \le i \le 3$. Since the triangle configurations of Lemma 2.2, part (ii), are transferred by the mapping φ^{-1}, there must be $(b_1, b_2, b_3) \in T^*$. This triple is in T^* the unique one with

$$(\varphi_1(b_1), \varphi_2(b_2), \varphi_3(b_3)) = (c_1, c_2, c_3),$$

since the triangle configurations are transferred in a one-to-one manner. $\square$

Lemma 2.4. *Let (T°, T^*) and (S°, S^*) be latin bi-trades with structural permutations τ_i and σ_i, respectively, and suppose that the bi-trade (T°, T^*) is separated. Then for every bijection $\varphi : T^\circ \to S^\circ$ such that $\varphi\tau_i = \sigma_i\varphi$, $1 \le i \le 3$, there exists exactly one injective covering $(\varphi_1, \varphi_2, \varphi_3) : (T^\circ, T^*) \to (S^\circ, S^*)$ such that φ is its point mapping.*

Proof: Let ρ be a cycle of τ_i, $i \in \{1,2,3\}$. Then $\varphi\rho\varphi^{-1}$ is a cycle of σ_i. Denote by α the value of ρ, by β the value of $\varphi\sigma\varphi^{-1}$, and define φ_i so that $\varphi_i(\alpha) = \beta$. This makes φ well defined, since α determines ρ in a unique way. The bijection φ transfers intersections of cycles and their triangle configurations in a one-to-one manner, and so the rest follows from Lemma 2.2. □

Proposition 2.5. *Let $(\varphi_1, \varphi_2, \varphi_3) : (S^\circ, S^*) \to (U^\circ, U^*)$ and $(\psi_1, \psi_2, \psi_3) : (T^\circ, T^*) \to (U^\circ, U^*)$ be injective coverings of latin bi-trades. Suppose that the bi-trade (S°, S^*) is separated. Then there exists a unique injective covering $(\gamma_1, \gamma_2, \gamma_3) : (S^\circ, S^*) \to (T^\circ, T^*)$ such that*

$$(\psi_1, \psi_2, \psi_3)(\gamma_1, \gamma_2, \gamma_3) = (\varphi_1, \varphi_2, \varphi_3).$$

If the bi-trade (T°, T^*) is also separated, then the triple $(\gamma_1, \gamma_2, \gamma_3)$ yields an isotopy of (S°, S^*) and (T°, T^*).

Proof: Denote by $\varphi : S^\circ \to U^\circ$ and $\psi : T^\circ \to U^\circ$ the associated point mappings, respectively, and put $\gamma = \psi^{-1}\varphi$. From Lemmas 2.3 and 2.4 we see that γ is a point mapping of some injective covering $(\gamma_1, \gamma_2, \gamma_3)$. There is only one such covering, by the uniqueness clause of Lemma 2.4. The composition $(\psi_1, \psi_2, \psi_3)(\gamma_1, \gamma_2, \gamma_3)$ yields a covering $(S^\circ, S^*) \to (U^\circ, U^*)$ with the point mapping equal to $\psi\gamma = \varphi$. Since the point mapping of an injective covering determines the covering uniquely, by Lemma 2.4, the composition has to agree with $(\varphi_1, \varphi_2, \varphi_3)$.

If $(S^\circ, S^*) = (T^\circ, T^*)$ and $\varphi_i = \psi_i$ for all $i \in \{1,2,3\}$, then each γ_i has to be equal to the identity mapping, because it is determined uniquely. It follows, in a standard way, that each γ_i is invertible when (T°, T^*) is also a separated latin bi-trade. □

Proposition 2.5 shows that if a latin bi-trade (T°, T^*) can be injectively covered by a separated latin bi-trade, then this covering is unique up to isotopism. To prove the existence of the covering we first show that conditions (P1) and (P2) suffice to construct a separated latin bi-trade. This is proved in [13] as Proposition 1.4. We repeat the proof since our statement is slightly more general.

Proposition 2.6. *Let τ_i, $1 \le i \le 3$, be permutations of a set X that satisfy* (P1) *and* (P2). *Denote by A_i, $1 \le i \le 3$, the set of all cycles of the permutation τ_i. Define $T^\circ, T^* \subseteq A_1 \times A_2 \times A_3$ by*

$(\rho_1, \rho_2, \rho_3) \in T^\circ \Leftrightarrow$ *all ρ_i, $1 \le i \le 3$, are incident to some $x \in X$, and*
$(\rho_1, \rho_2, \rho_3) \in T^* \Leftrightarrow$ *there exist elements $x_i \in X$, $i \in \{1,2,3\}$, such that ρ_i moves x_{i-1} to x_{i+1}.*

Then (T°, T^*) is a separated latin bi-trade with structural permutations equal to $\psi\tau_i\psi^{-1}$, $1 \le i \le 3$, where $\psi : X \to T^\circ$ sends $x \in X$ to the triple (ρ_1, ρ_2, ρ_3) consisting of those ρ_j which move x, $1 \le j \le 3$.

Proof: Conditions (R4) and (R5) are obvious. Choose $i \in \{1,2,3\}$ and consider an element $x \in X$. For $j \in \{1,2,3\}$, $j \ne i$, denote by ρ_j the cycle of τ_j that moves x. Put $x_{i-1} = \rho_{i+1}(x)$, $x_{i+1} = \rho_{i-1}^{-1}(x)$ and $x_i = x$. Since $\tau_{i-1}\tau_i\tau_{i+1}(x) = x$, there must exist unique $\rho_i \in A_i$ that maps x_{i-1} to x_{i+1}, and we see that $(\rho_1, \rho_2, \rho_3) \in T^*$, and that this triple is fully determined by ρ_{i-1} and ρ_{i+1}.

The point x determines a triple of T°, where the jth coordinate, $j \ne i$, coincides with ρ_j. Note that we have verified, in fact, condition (R2). Let ρ be the ith coordinate of the triple determined by x. The cycle ρ_{i-1} meets ρ in x and meets ρ_i in $\tau_{i-1}(x)$. These intersections cannot coincide since τ_{i-1} is fixed point free. Hence $\rho \ne \rho_i$, and this yields condition (R1).

Consider now $(\rho_1, \rho_2, \rho_3) \in T^*$. Cycles $\rho_{i-1} \in A_{i-1}$ and $\rho_{i+1} \in A_{i+1}$ move the point x_i, and there exists a unique cycle $\rho \in A_i$ that moves x_i as well. Thus by changing ρ_i to ρ we get the unique triple of T° that agrees with (ρ_1, ρ_2, ρ_3) in the coordinates $j \equiv i \pm 1 \bmod 3$, and that gives condition (R3). The ith structural permutation of (T°, T^*) clearly moves the triple determined by x_{i-1} to the triple determined by x_{i+1}, and so we see that the structural permutation is equal to $\psi\tau_i\psi^{-1}$. The cycles of $\psi\tau_i\psi^{-1}$ are of the form $\psi\rho\psi^{-1}$, where $\rho \in A_i$, and the value of $\psi\rho\psi^{-1}$ is ρ. The value hence determines each cycle uniquely, and so the latin bi-trade (T°, T^*) is really separated. $\square$

Consider now a latin bi-trade (S°, S^*). Let (τ_1, τ_2, τ_3) be its structural triple. The triple satisfies conditions (P1) and (P2), and hence we can construct a separated latin bi-trade (T°, T^*) by means of Proposition 2.6. The mapping $\psi^{-1} : T^\circ \to S^\circ$ defined in the proposition satisfies conditions of Lemma 2.4, and hence it is a point mapping of an injective covering $(T^\circ, T^*) \to (S^\circ, S^*)$. This covering is unique, by Lemma 2.4 again, and from the proof of Lemma 2.4 we see that it maps the value of a cycle upon the value of the cycle image. In our case this yields an injective covering that maps each cycle ρ of τ_i upon the value of ρ in (S°, S^*). We can hence conclude this section by the following theorem.

Theorem 2.7. *Let (T°, T^*) be a latin bi-trade with structural triple (τ_1, τ_2, τ_3). For each cycle ρ of τ_i denote by $\lambda_i(\rho)$ the value of ρ, $1 \le i \le 3$. Then there exists a unique latin bi-trade (S°, S^*) such that $(\lambda_1, \lambda_2, \lambda_3) : (S^\circ, S^*) \to (T^\circ, T^*)$ is an injective covering. This latin bi-trade is separated and is isotopic to any other separated latin bi-trade that yields an injective covering of (T°, T^*).*

3. The trading sphere

In Section 2 we have seen that the study of separated latin bi-trades can be reduced to the study of permutation triples (τ_1, τ_2, τ_3) that satisfy (P1) and (P2).

If (τ_1, τ_2, τ_3) is such a triple, then we associate with it an oriented combinatorial surface that consists of cyclic and triangular faces (see Section 1). Every point

$x \in X$ belongs exactly to three triangular faces and to three cyclic faces. These six faces form the *star* of x.

There are also other ways how to define a surface structure upon a latin bi-trade; the advantage of the present approach seems to rest in transparency of the local structure of the point stars. We shall now connect the genus of the surface with the structure of the trading permutations.

The aggregate number o of cycles in all three permutations τ_i is called the *permutational order*, and the number s of elements of the set X is called the *size*. These notions can be related directly to the triple (τ_1, τ_2, τ_3), or to a latin bi-trade which yields it. It is clear that the associated geometric structure is a combinatorial surface (and not a union of at least two combinatorial surfaces) if and only if the permutations τ_i generate on X a transitive permutation group. In such a case we shall say that (τ_1, τ_2, τ_3) is a *trading surface* on X.

The number of faces, edges and vertices of the surface can be easily expressed in terms of o and s. By doing so, the Euler identity for connected surfaces yields

$$2 + s = o + g.$$

For the detailed proof see, e.g., [13, Proposition 1.5]. A trading surface with $g = 0$ is called a *trading sphere*. We have seen that every trading sphere satisfies $o = 2 + s$.

Let us remark that trading surfaces with $g \geq 1$ can be obtained from trading spheres by a cut-and-paste construction [13].

Proposition 3.1. *Let (τ_1, τ_2, τ_3) determine a trading sphere. Then among all cycles of τ_i, $1 \leq i \leq 3$, there exist at least six that are of length two.*

Proof: Let $2 \leq \ell_1 \leq \cdots \leq \ell_o$ be the lengths of all cycles. If $\ell_6 \geq 3$, then $3s = \sum \ell_j \geq 3o - 5 = 3s + 1$, a contradiction. □

We shall describe in the next sections three different ways how to expand locally a triple (τ_1, τ_2, τ_3) into $(\tau_1', \tau_2', \tau_3')$ so that one of the triples determines a trading sphere if and only if the other does. Given a trading sphere $(\sigma_1, \sigma_2, \sigma_3)$ of size > 4 we shall then use Proposition 3.1 to identify an expansion $(\tau_1', \tau_2', \tau_3') = (\sigma_1, \sigma_2, \sigma_3)$.

4. Triangle and trapezium expansions

Let us assume that τ_i, $1 \leq i \leq 3$, are permutations of a set X that satisfy conditions (P1) and (P2). Sequence $x_3x_2x_1$ of elements of X is called a *triangle* if there exists $j \in \{1, 2, 3\}$ such that $\tau_j(x_3) = x_2$ and $\tau_{j-1}(x_2) = x_1$ (then $\tau_{j+1}(x_1) = x_3$, of course). We regard triangles as cyclic sequences, and in principle we allow for any value of j to occur. However, our default assumption is that of $j = 1$, which gives $\tau_i(x_{i-1}) = x_{i+1}$, for every $i \in \{1, 2, 3\}$. For each $i \in \{1, 2, 3\}$ there thus exists a cycle $(\dots\ x_{i-1}\ x_{i+1}\ \dots)$ of τ_i, and these three cycles will be called the *sides* of $x_3x_2x_1$.

In the constructions below we will use elements y_1, y_2 and y_3. Unless stated otherwise, they will be assumed to be pairwise distinct, and disjunct to X.

For a triangle $\mathbf{x} = x_3x_2x_1$ define $\tau_i^{\mathbf{x}}$, $i \in \{1,2,3\}$, as a permutation of $Y = X \cup \{y_1, y_2, y_3\}$ such that $\tau_i^{\mathbf{x}}(y_{i+1}) = y_{i-1}$, $\tau_i^{\mathbf{x}}(y_{i-1}) = y_{i+1}$, $\tau_i^{\mathbf{x}}(x_{i-1}) = y_i$ and $\tau_i^{\mathbf{x}}(y_i) = x_{i+1}$, and such that $\tau_i^{\mathbf{x}}$ coincides with τ_i elsewhere.

It is easy to verify that the triple $(\tau_1^{\mathbf{x}}, \tau_2^{\mathbf{x}}, \tau_3^{\mathbf{x}})$ satisfies conditions (P1) and (P2). We can also refer to Section 10 of [13], where this construction is being considered with more details.

The cycle $(\dots\ x_{i-1}\ x_{i+1}\ \dots)$ of τ_i that passes through x_{i-1} and x_{i+1} is changed by $\tau_i^{\mathbf{x}}$ to $(\dots\ x_{i-1}\ y_i\ x_{i+1}\ \dots)$, and there appears a new cycle $(y_{i-1}\ y_{i+1})$. The other cycles of $\tau_i^{\mathbf{x}}$ overlap with those of τ_i. Both the permutational order and the size thus increase by 3, and hence the genus does not change.

We shall now give conditions for the reverse construction:

Proposition 4.1. *Let $(\sigma_1, \sigma_2, \sigma_3)$ determine a trading surface on Y, $|Y| > 4$, and let $y_3y_2y_1$ be a triangle such that $(y_{i-1}\ y_{i+1})$ is a cycle of σ_i for each $i \in \{1,2,3\}$. Set $x_{i+1} = \sigma_i(y_i)$, $\mathbf{x} = x_3x_2x_1$, and define a permutation τ_i of $X = Y \setminus \{y_1, y_2, y_3\}$ so that $\tau_i(x) = \sigma_i(x)$ for every $x \in X \setminus \{x_{i-1}\}$ and $\tau_i(x_{i-1}) = x_{i+1}$. Then $\sigma_i = \tau_i^{\mathbf{x}}$, for each $i \in \{1,2,3\}$.*

Proof: We have $\sigma_i(x_{i-1}) = \sigma_{i-1}^{-1}\sigma_{i+1}^{-1}(x_{i-1}) = \sigma_{i-1}^{-1}(y_{i+1}) = y_i$, and $\sigma_i(y_i) = x_{i+1}$, for every $i \in \{1,2,3\}$. Hence τ_i is a permutation of X that is obtained from σ_i by removing the cycle $(y_{i-1}\ y_{i+1})$, and by replacing the cycle $(\dots x_{i-1}\ y_i\ x_{i+1} \dots)$ with $(\dots\ x_{i-1}\ x_{i+1}\ \dots)$.

The definition of $\tau_i^{\mathbf{x}}$ describes the reverse process, and so $\tau_i^{\mathbf{x}} = \sigma_i$. However, we need to show that (τ_1, τ_2, τ_3) fulfils conditions (R1) and (R2). The cycles of τ_i are subsets of the cycles of σ_i, and that makes (R1) immediately clear.

To prove (R2) consider $u \in X$, put $u_2 = u$, $u_3 = \sigma_1^{-1}(u)$ and $u_1 = \sigma_3(u)$. Then $\sigma_i(u_{i-1}) = u_{i+1}$ for all $i \in \{1,2,3\}$, and $\tau_1\tau_2\tau_3(u) = \sigma_1\sigma_2\sigma_3(u) = u$ if $u_{i-1} \neq x_{i-1}$ for every $i \in \{1,2,3\}$. However, in such a case we get the identity as well since each τ_i sends x_{i-1} to x_{i+1}.

So it remains to show that τ_i is fixed point free. If this is not true, than there must be $x_{i-1} = x_{i+1}$. We have $x_i = \tau_{i-1}(x_{i+1}) = \tau_{i+1}^{-1}(x_{i-1})$, and so $x_{i-1} = x_{i+1}$ implies $x_i = x_{i-1}$, by (R1). Thus $x_1 = x_2 = x_3$ if some τ_i is not fixed point free. But when $x = x_1 = x_2 = x_3$, then τ_i acts on $Y \cup \{x\}$ for all $i \in \{1,2,3\}$, contrary to the assumptions of the proposition. $\square$

The construction $\tau_i \to \tau_i^{\mathbf{x}}$ will be called the *triangular expansion*. Its reverse form, the *triangular reduction*, is possible, by Proposition 4.1, whenever X is large enough and there exists a triangle with all three sides of length 2.

We shall now introduce the *trapezium expansion*, the reverse of which assumes a triangle with exactly two sides of length 2.

Assume again that τ_i, $1 \le i \le 3$, are permutations of X that fulfil (P1) and (P2). Fix $j \in \{1,2,3\}$ and $u \in X$. Put $u_j = u$, $u_{j-1} = \tau_{j+1}(u)$ and $u_{j+1} = \tau_{j-1}^{-1}(u)$. That makes $u_3u_2u_1$ a triangle. We shall define permutations $\tau_i' = \tau_i^{j,u}$ on $Y = (X \setminus \{u\}) \cup \{y_1, y_2, y_3\}$ by

$\tau'_j(u_{j-1}) = y_j$, $\tau'_j(y_j) = u_{j+1}$, $\tau'_j(\tau_j^{-1}(u)) = y_{j-1}$, $\tau'_j(y_{j-1}) = y_{j+1}$ and $\tau'_j(y_{j+1}) = \tau_j(u)$, with $\tau'_j(x) = \tau_j(x)$ in all other cases;

and by

$(\tau'_{j\pm1})^{\pm1}(y_{j\pm1}) = u_{j\mp1}$, $(\tau'_{j\pm1})^{\mp1}(y_{j\pm1}) = (\tau_{j\pm1})^{\mp1}(u)$, $\tau'_{j\pm1}(y_{j\mp1}) = y_j$ and $\tau'_{j\pm1}(y_j) = y_{j\mp1}$, with $\tau'_{j\pm1}(x) = \tau_{j\pm1}(x)$ in the other cases.

The latter two equalities express the fact that τ'_{j+1} contains the cycle $(y_{j-1}\ y_j)$, and τ'_{j-1} contains the cycle $(y_{j+1}\ y_j)$. To determine the other cycles of $\tau'_{j\pm1}$ we shall first express the former two equalities separately for τ'_{j-1} and τ'_{j+1}. We obtain

$\tau'_{j+1}(y_{j+1}) = u_{j-1}$ and $\tau'_{j+1}(\tau_{j+1}^{-1}(u)) = y_{j+1}$, together with $\tau'_{j-1}(y_{j-1}) = \tau_{j-1}(u)$ and $\tau'_{j-1}(u_{j+1}) = y_{j-1}$.

We see that the cyclic structure of τ'_{j+1} differs from that of τ_{j+1} by

replacing the cycle $(\dots\ u\ \dots)$ with $(\dots\ y_{j+1}\ \dots)$, and by adding the cycle $(y_j\ y_{j-1})$.

Similarly, the cyclic structure of τ'_{j-1} differs from that of τ_{j-1} by

replacing the cycle $(\dots\ u\ \dots)$ with $(\dots\ y_{j-1}\ \dots)$, and by adding the cycle $(y_j\ y_{j+1})$.

Finally note that τ'_j differs from τ_j so that

the cycle $(\dots\ u\ \dots)$ is replaced by $(\dots\ y_{j-1}\ y_{j+1}\ \dots)$, and the cycle $(\dots\ u_{j-1}\ u_{j+1}\ \dots)$ is replaced by $(\dots\ u_{j-1}\ y_j\ u_{j+1}\ \dots)$.

It is hence clear immediately that the triple $(\tau'_1, \tau'_2, \tau'_3)$ satisfies condition (P1) and that each τ'_i is a fixed point free permutation. To prove $\tau'_1\tau'_2\tau'_3(x) = x$ for every $x \in Y$ first observe that in every triple (x, x', i) such that $\tau'_i(x) = x'$ there must be $\{x, x'\} \cap \{y_1, y_2, y_3\} \neq \emptyset$ whenever either $\tau_i(x)$ is not defined, or it is defined, but it differs from x'. Thus $\tau'_1\tau'_2\tau'_3(x) = x$ whenever $\{y_1, y_2, y_3\} \cap \{x, \tau'_3(x), \tau'_1(x)\} = \emptyset$.

We hence need to show $\tau'_{i+1}\tau'_i\tau'_i(y_{i'}) = y_{i'}$ for all $i, i' \in \{1, 2, 3\}$. This is clear if $i' \equiv i - 1 \bmod 3$ since $\tau'_i(y_{i-1}) = y_{i+1}$ for all $i \in \{1, 2, 3\}$. We also have $\tau'_{j+1}(y_{j-1}) = y_j$, $\tau'_j(y_j) = u_{j+1}$, $\tau'_{j-1}(u_{j+1}) = y_{j-1}$ and $\tau'_{j+1}(y_{j+1}) = u_{j-1}$, $\tau'_j(u_{j-1}) = y_j$, $\tau'_{j-1}(y_j) = y_{j+1}$. That leaves open only the cases $(i, i') \in \{(j, j+1), (j-1, j-1)\}$.

However, $\tau'_{j+1}\tau'_{j-1}\tau'_j(y_{j+1}) = \tau'_{j+1}\tau'_{j-1}\tau_j(u) = \tau'_{j+1}\tau_{j-1}\tau_j(u) = \tau'_{j+1}\tau_{j+1}^{-1}(u) = \tau'_{j+1}(\tau'_{j+1})^{-1}(y_{j+1}) = y_{j+1}$. Similarly we get $\tau'_j\tau'_{j+1}\tau'_{j-1}(y_{j-1}) = \tau'_j\tau'_{j+1}\tau_j(u) = \tau'_j\tau_{j+1}\tau_j(u) = \tau'_j(\tau_j^{-1}(u)) = y_{j-1}$.

The construction removes point u and adds points y_i, $1 \le i \le 3$. Sometimes it is useful to remove no point and add only two points. There is a certain symmetry between y_{j-1} and y_{j+1}, and so we choose y_j as the point that can appear in both of the underlying sets. Thus we relax the assumption $X \cap \{y_1, y_2, y_3\} = \emptyset$ to

$$I = X \cap \{y_1, y_2, y_3\} \neq \emptyset \implies I = \{y_j\} = \{u\}.$$

This formal assumption for the construction will be used further on.

The aggregate number of cycles increases by two, and the number of points increases by two as well. Hence no change to the genus can occur, and we can state:

Lemma 4.2. *Let τ_i, $1 \le i \le 3$, be permutations of a set X that satisfy* (P1) *and* (P2)*, and generate on X a transitive permutation group. Permutations $\tau_i^{j,u}$, $1 \le i \le 3$, have these properties as well, and yield the same genus, for every $j \in \{1,2,3\}$ and every $u \in X$.*

Under the above assumptions we also get the following lemma. It expresses a necessary condition for the reverse construction.

Lemma 4.3. *Assume $\tau_i' = \tau_i^{j,u}$, $1 \le i \le 3$. Then the cycle of τ_{j+1}' incident to y_{j+1} does not intersect the cycle of τ_{j-1}' incident to y_{j-1}.*

PROOF: For $i = j \pm 1$ denote by ρ_i the cycle of τ_i incident to u. If the cycles of the lemma had intersected in a point v, then the cycles ρ_{j-1} and ρ_{j+1} would intersect in both u and v. That would be a contradiction to (P1). $\square$

The definition of permutations $\tau_i' = \tau_i^{j,u}$ is followed by a detailed description of the change in the cyclic structure. Using this description we record, for a later application, the following observation.

Lemma 4.4. *Assume $\tau_i' = \tau_i^{j,u}$, $1 \le i \le 3$. For each cycle ρ of τ_i there exists exactly one cycle ρ' of τ_i' that has at least one common point with ρ. If cycles $\rho_1 \ne \rho_2$ meet in a point $v \ne u$, then ρ_1' and ρ_2' meet in the point v as well. If cycles $\rho_1 \ne \rho_2$ meet in u and one of them is not a side of the triangle $u\tau_{j+1}(u)\tau_{j-1}^{-1}(u)$, then ρ_1' and ρ_2' meet in y_{j-1} or in y_{j+1}.*

Proposition 4.5. *Let $(\sigma_1, \sigma_2, \sigma_3)$ be a triple of permutations of a set Y that satisfies conditions* (P1) *and* (P2)*, and let $y_3y_2y_1$ be a triangle. Furthermore, let there exist $j \in \{1,2,3\}$ such that $(y_j\ y_{j-1})$ is a cycle of σ_{j+1}, $(y_j\ y_{j+1})$ is a cycle of σ_{j-1}, and the cycle of σ_j that passes through both y_{j-1} and y_{j+1} is of length ≥ 3. Suppose also that the cycle of σ_{j+1} incident to y_{j+1} does not meet the cycle of σ_{j-1} incident to y_{j-1}.*

Define permutations τ_i, $1 \le i \le 3$, of $X = (Y \setminus \{y_1, y_2, y_3\}) \cup \{u\}$, $u \notin X$ or $u = y_j$, so that

$$
\begin{aligned}
&\tau_{j-1}(\sigma_{j-1}^{-1}(y_{j-1})) = u,\ \tau_{j-1}(u) = \sigma_{j-1}(y_{j-1}),\\
&\tau_{j+1}(\sigma_{j+1}^{-1}(y_{j+1})) = u,\ \tau_{j+1}(u) = \sigma_{j+1}(y_{j+1}),\\
&\tau_j(\sigma_j^{-1}(y_{j-1})) = u,\ \tau_j(u) = \sigma_j(y_{j+1}),\ \tau_j(\sigma_j^{-1}(y_j)) = \sigma_j(y_j),\\
&\text{and } \tau_i(x) = \sigma_i(x) \text{ in all other cases.}
\end{aligned}
$$

Then $\sigma_i = \tau_i^{j,u}$, for each $i \in \{1,2,3\}$.

PROOF: We see that $\tau_{j\pm1}$ is obtained from $\sigma_{j\pm1}$ by removing the cycle $(y_j\ y_{j\mp1})$, and by replacing $(\dots\ y_{j\pm1}\ \dots)$ with $(\dots\ u\ \dots)$. Furthermore, τ_j differs from σ_j so that $(\dots\ y_{j-1}\ y_{j+1}\ \dots)$ is replaced by $(\dots\ u\ \dots)$, and $(\dots\ \sigma_j^{-1}(y_j)\ y_j\ \sigma_j(y_j)\ \dots)$ by $(\dots\ \sigma_j^{-1}(y_j)\ \sigma_j(y_j)\ \dots)$. Consider the cycles of τ_i, $1 \le i \le 3$. If two distinct cycles intersect in a point $v \neq u$, then the corresponding cycles of σ_i intersect in v as well. From our assumptions we see that any two distinct cycles of τ_i that intersect in u have no other common point. Hence permutations τ_i, $1 \le i \le 3$, fulfil (P1). They are fixed point free since the cycle $(\dots\ y_{j-1}\ y_{j+1}\ \dots)$ is assumed to be of length at least three. The equality $\sigma_i = \tau_i^{j,u}$ is clear from the cycle decompositions, and so it remains to show $\tau_1\tau_2\tau_3(x) = x$, for all $x \in X$.

We have $\tau_{j+1}(u) = \sigma_{j+1}(y_{j+1}) = \sigma_j^{-1}\sigma_{j-1}^{-1}(y_{j-1}) = \sigma_j^{-1}(y_j)$, $\tau_j(\sigma_j^{-1}(y_j)) = \sigma_j(y_j)$ and $\tau_{j-1}(\sigma_j(y_j)) = \tau_{j-1}\sigma_j\sigma_{j+1}(y_{j-1}) = \tau_{j-1}(\sigma_{j-1}^{-1}(y_{j-1})) = y_{j-1}$. Furthermore, $\tau_{j-1}(u) = \sigma_{j-1}(y_{j-1})$, $\tau_{j+1}(\sigma_{j-1}(y_{j-1})) = \sigma_{j+1}\sigma_{j-1}(y_{j-1}) = \sigma_j^{-1}(y_{j-1})$ and $\tau_j(\sigma_j^{-1}(y_{j-1})) = u$. Finally, $\tau_j(u) = \sigma_j(y_{j+1})$, $\tau_{j-1}(\sigma_j(y_{j+1})) = \sigma_{j-1}\sigma_j(y_{j+1}) = \sigma_{j+1}^{-1}(y_{j+1})$ and $\tau_{j+1}(\sigma_{j+1}^{-1}(y_{j+1})) = u$. We have in this way verified the equality $\tau_{i+1}\tau_{i-1}\tau_i(x) = x$ for nine different pairs (i, x). The considered situations include all five pairs (i, x), where $\tau_i(x)$ does not coincide with $\sigma_i(x)$. In other situations $\tau_{i+1}\tau_{i-1}\tau_i(x) = x = \sigma_{i+1}\sigma_{i-1}\sigma_i(x)$ is induced by the overlaps between the corresponding mappings. □

5. Sliding expansion

Let τ_i, $1 \le i \le 3$, be permutations of a set X that satisfy conditions (P1) and (P2). Suppose that x and z are two distinct points of X that occur in the same cycle of τ_j, for a certain $j \in \{1, 2, 3\}$. Suppose also that $\tau_j(x) \neq z$ and that the cycle of τ_{j+1} that moves x does not intersect the cycle of τ_{j-1} that moves z.

Assume $y \notin X$, put $Y = \{y\} \cup X$, and define permutations $\tau_i' = \tau_i^{x,z}$, $1 \le i \le 3$, so that τ_i' coincides with τ_i, with the exception of the following cases:

$$\begin{array}{rclrclrcl}
\tau_j'(x) &=& z, & \tau_j'(\tau_j^{-1}(z)) &=& y, & \tau_j'(y) &=& \tau_j(x), \\
\tau_{j-1}'(z) &=& y, & \tau_{j-1}'(y) &=& \tau_{j-1}(z), & \text{and} && \\
\tau_{j+1}'(\tau_{j+1}^{-1}(x)) &=& y, & \tau_{j+1}'(y) &=& x. &&&
\end{array}$$

The cyclic decomposition of τ_j' is clearly obtained from that of τ_j by replacing the cycle

$$(\dots\ x\ \tau_j(x)\ \dots\ \tau_j^{-1}(z)\ z\ \dots) \ \text{ with cycles}$$
$$(\dots\ x\ z\ \dots) \ \text{ and } \ (\tau_j(x)\ \dots\ \tau_j^{-1}(z)\ y).$$

In the cyclic decomposition of τ_{j-1}' one replaces $(\dots\ z\ \dots)$ by $(\dots\ z\ y\ \dots)$, and in τ_{j+1}' the cycle $(\dots\ x\ \dots)$ is replaced by $(\ \dots\ y\ x\ \dots)$.

We see that all τ_i', $1 \le i \le 3$, are fixed point free permutations. To verify condition (P1) first note that any intersection of cycles from τ_i', $1 \le i \le 3$, that does not involve the element y gives an intersection of corresponding cycles from τ_i. Hence it suffices to verify that those cycles of τ_i' that intersect in y have y as their only common point. This is clear if one of the cycles is from τ_j' (then it is the cycle $(\tau_j(x)\ \dots\ \tau_j^{-1}(z)\ y)$). If one of the cycles comes from τ_{j-1}' and the other from τ_{j+1}', then the assertion follows from the initial assumptions on τ_i, $1 \le i \le 3$.

To verify that $\tau_1'\tau_2'\tau_3'$ is the identity permutation of Y it suffices to show that $\tau_{i+1}'\tau_{i-1}'\tau_i'(u) = u$ whenever $u = y$, or $\tau_i'(u) \neq \tau_i(u)$, $u \in X$. We shall do so by observing the existence of three triangles:

Firstly, $\tau_j'(x) = z$, $\tau_{j-1}'(z) = y$ and $\tau_{j+1}'(y) = x$.

Secondly, $\tau_j'(\tau_j^{-1}(z)) = y$, $\tau_{j-1}'(y) = \tau_{j-1}(z)$ and $\tau_{j+1}'\tau_{j-1}(z) = \tau_{j+1}\tau_{j-1}(z) = \tau_j^{-1}(z)$, where $\tau_{j+1}'\tau_{j-1}(z) = \tau_{j+1}\tau_{j-1}(z)$ is a consequence of the fact that $\tau_{j-1}(z)$ is not in the cycle of τ_{j+1} that moves x, and hence it cannot be equal to $\tau_{j+1}^{-1}(x)$.

Finally, $\tau_{j+1}'(\tau_{j+1}^{-1}(x)) = y$, $\tau_j'(y) = \tau_j(x)$ and $\tau_{j-1}'(\tau_j(x)) = \tau_{j-1}\tau_j(x) = \tau_{j+1}^{-1}(x)$. Note that τ_{j-1}' agrees with τ_{j-1} on $\tau_j(x)$ since we assume $\tau_j(x) \neq z$.

We can hence state:

Lemma 5.1. *Let τ_i, $1 \leq i \leq 3$, be permutations of a set X that satisfy* (P1) *and* (P2)*, and generate on X a transitive permutation group. Permutations $\tau_i^{x,z}$ have these properties as well, and yield the same genus, whenever $x, z \in X$ are distinct points that fulfil:*

(1) *$z \neq \tau_j(x)$,*
(2) *there exists $j \in \{1, 2, 3\}$ such that x and z are incident to the same cycle of τ_j, and*
(3) *the cycle of τ_{j+1} incident to x does not intersect the cycle of τ_{j-1} incident to z.*

Denote by ρ the cycle of τ_j incident to x and z, by σ the cycle of τ_{j+1} incident to x, and by $\bar{\sigma}$ the cycle of τ_{j-1} incident to z. The definition of $\tau_i' = \tau_i^{x,z}$ can be interpreted so that the cycle ρ gets ruptured between x and z, with a part sliding along σ (or $\bar{\sigma}$, according to the preferred orientation), creating thus a new triangle zyx.

This is the reason why we call this construction the *sliding expansion*. Later we shall deal mainly with a special case in which $\tau_j(z) = x$. In such a case τ_j' contains cycle $(z\ x)$, and this is why this special case will be called the 2-*sliding expansion*.

To every cycle $\alpha \neq \rho$ of τ_i, $1 \leq i \leq 3$, there clearly corresponds a unique cycle α' of τ_i' that has at least one common point with α, and $\alpha = \alpha'$ when $\alpha \notin \{\sigma, \bar{\sigma}\}$.

The cycle ρ gets divided into two cycles ρ' and ρ'', with ρ' incident to x and z, and ρ'' incident to y.

Say that a cycle α *connects* cycles β' and β'', if α has a common point with β' and β'', and differs from both β' and β''.

Using the notation above, we state a lemma, purpose of which is to point out that the lack of multiple connectedness is a necessary precondition for sliding reduction.

Lemma 5.2. *The cycles σ' and $\bar{\sigma}'$ are the only cycles of $\tau_i^{x,z}$, $i \in \{1, 2, 3\}$, that connect ρ' and ρ''.*

PROOF: Let $\alpha \notin \{\sigma', \bar{\sigma}'\}$ connect ρ' and ρ''. Then α is a cycle of $\tau_{j\pm 1}$ that moves a point x' of ρ' and a point x'' of ρ''. That contradicts (P1) since in such a situation both x' and x'' have to be incident to ρ. □

Proposition 5.3. *Let σ_i, $1 \le i \le 3$, be permutations of a set $Y = X \cup \{y\}$ that fulfil conditions* (P1) *and (*P2). *Let zyx be a triangle, and let ρ' be the cycle of σ_j, where $j \in \{1,2,3\}$, that moves x onto z. Furthermore, let ρ'' be the cycle of σ_j that moves the element y. For $i = j \pm 1$ suppose that there is only one cycle of σ_i that connects ρ' and ρ'', and let this cycle* (*which necessarily moves y*) *be of length at least three. Then there exist* (*unique*) *permutations τ_i of the set X, $1 \le i \le 3$, that fulfil* (P1) *and* (P2), *and for which $\sigma_i = \tau_i^{x,z}$.*

Proof: Derive τ_j from σ_j by replacing the cycles ρ' and ρ'' with the cycle ρ : $(\dots\ x\ \sigma_j(y)\ \dots\ \sigma_j^{-1}(y)\ z \dots)$. Furthermore, derive τ_{j+1} from σ_{j+1} by changing $(\dots\ \sigma_{j+1}^{-1}(y)\ y\ x\ \dots)$ to $(\dots\ \sigma_{j+1}^{-1}(y)\ x\ \dots)$, and τ_{j-1} from σ_{j-1} by changing $(\dots\ z\ y\ \sigma_{j+1}(y)\ \dots)$ to $(\dots\ z\ \sigma_{j+1}(y)\ \dots)$. Permutation τ_j is clearly fixed point free, and for $\tau_{j\pm1}$ this holds because of the assumption on the length of cycles passing through the element y.

A violation of (P1) by (τ_1,τ_2,τ_3) would have to involve the cycle ρ and a cycle connecting ρ' and ρ''. However, such a situation cannot occur by our assumptions, as $(\sigma_1,\sigma_2,\sigma_3)$ satisfies (P1) and (P2) as well.

The pairs $(i,u) \in \{1,2,3\} \times X$ with $\tau_i(u) \neq \sigma_i(u)$ are exactly (j,x), $(j,\sigma_j^{-1}(y))$, $(j+1,\sigma_{j+1}^{-1}(y))$ and $(j-1,\sigma_{j-1}^{-1}(y))$. To prove (P2) we need to show that u equals $\tau_{i+1}\tau_{i-1}\tau_i(u)$ for every such (i,u). We have $\tau_j(x) = \sigma_j(y) \neq \sigma_{j-1}^{-1}(y)$, $\tau_{j-1}\tau_j(x) = \sigma_{j-1}\sigma_j(y) = \sigma_{j+1}^{-1}(y)$ and $\tau_{j+1}\tau_{j-1}\tau_j(x) = \tau_{j+1}\sigma_{j+1}^{-1}(y) = x$. Furthermore, $\tau_j(\sigma_j^{-1}(y)) = z$, $\tau_{j-1}(z) = \sigma_{j-1}(y) \neq \sigma_{j+1}^{-1}(y)$ and $\tau_{j+1}(\sigma_{j-1}(y)) = \sigma_{j+1}\sigma_{j-1}(y) = \sigma_j^{-1}(y)$. We have verified that (τ_1,τ_2,τ_3) satisfies (P1) and (P2). The equality $\sigma_i = \tau_i^{x,z}$ is clear. □

Let $\mathbf{x} = x_3x_2x_1$ be a triangle in (τ_1,τ_2,τ_3) and choose $j \in \{1,2,3\}$. The trapezium expansion $(\tau_1',\tau_2',\tau_3')$ with $\tau_i' = \tau_i^{j,x_j}$, $1 \le i \le 3$, contains cycles $(y_j\ y_{j+1})$ and $(\dots\ x_{j+1}\ y_{j-1}\ \dots)$ of τ_{j-1}', $(y_j\ y_{j-1})$ and $(\dots\ y_{j+1}\ x_{j-1}\ \dots)$ of τ_{j+1}', and $(\dots\ y_{j-1}\ y_{j+1}\ \dots)$ and $(\dots\ x_{j-1}\ y_j\ x_{j+1}\ \dots)$ of τ_j'. Permutations τ_i' act on $X' = (X \setminus \{x_j\}) \cup \{y_1,y_2,y_3\}$, and so $x_j \notin X'$. We have $\tau_j'(y_{j+1}) = \tau_j(x_j) \neq y_{j-1}$, and hence the 2-sliding expansion $(\tau_i')^{y_{j+1},y_{j-1}}$ is well defined. We shall denote it by τ_i'', after renaming y to x_j. Then τ_{j-1}'' contains the cycles $(y_j\ y_{j+1})$ and $(\dots\ x_{j+1}\ y_{j-1}\ x_j\ \dots)$, where the latter cycle replaces the cycle $(\dots\ x_{j+1}\ x_j\ \dots)$ of τ_j. Similarly τ_{j+1}'' yields cycles $(y_j\ y_{j+1})$ and $(\dots x_j\ y_{j+1}\ x_{j-1}\ \dots)$, a replacement of $(\dots\ x_j\ x_{j-1}\ \dots)$. Finally, the cycle $(\dots\ x_{j-1}\ y_j\ x_{j+1}\ \dots)$ is copied to τ_j'' from τ_j', and in τ_j'' there appears $(y_j\ y_{j+1})$. In τ_j' we also have the cycle $(\dots\ x_j\ \dots)$ that is equal to the original cycle of τ_j. We have proved:

Lemma 5.4. *Let $\mathbf{x} = x_3x_2x_1$ be a triangle in a trading surface (τ_1,τ_2,τ_3). Then for all $i,j \in \{1,2,3\}$ the permutations $\tau_i^{\mathbf{x}}$ and $(\tau_i^{j,x_j})^{y_{j+1},y_{j-1}}$ coincide* (*assuming that the points y and x_j are identified*).

Corollary 5.5. *A triangle expansion can be replaced by a trapezium expansion followed by a 2-sliding expansion.*

6. The trigonal construction

The triangular construction of Section 4 is a special case of a more general construction we shall now describe.

Suppose that $\mathbf{x} = x_3x_2x_1$ is a triangle in (τ_1, τ_2, τ_3), where τ_i are permutations of X that fulfil (P1) and (P2). Let $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ be another triple of permutations fulfilling (P1) and (P2). Suppose that the latter permutations act on a set Y, where $Y \cap X = \emptyset$. For a fixed $y \in Y$ define permutations $\tau_i' = \tau_i^{\mathbf{x},\Sigma,y}$, $1 \le i \le 3$, on $(X \cup Y) \setminus \{y\}$ in such a way that

$$\begin{aligned}
&\tau_i'(x) = \tau_i(x) \text{ for all } x \in X \setminus \{x_{i-1}\},\\
&\tau_i'(u) = \sigma_i(u) \text{ for all } u \in Y \setminus \{y, \sigma_i^{-1}(y)\},\\
&\tau_i'(x_{i-1}) = \sigma_i(y) \text{ and } \tau_i'(\sigma_i^{-1}(y)) = x_{i+1}.
\end{aligned}$$

We see that the cycles of τ_i and σ_i become cycles of τ_i', with the exception of $(\dots\ x_{i-1}\ x_{i+1}\ \dots)$ and $(y\ \dots\ \sigma_i^{-1}(y))$, which are merged into a common cycle $(\dots\ x_{i-1}\ \dots\ \sigma_i^{-1}(y)\ x_{i+1}\ \dots)$. Denote this cycle by μ_i.

Permutations τ_i' are fixed point free. Because μ_{i-1} and μ_{i+1} meet exactly in x_i, we see that permutations τ_i' fulfil condition (P1). To prove (P2) first note that $\tau_i'(u) = \tau_i(u)$ when both u and $\tau_i'(u)$ belong to X, and that $\tau_i'(u) = \sigma_i(u)$ when both u and $\tau_i'(u)$ belong to Y. Hence it suffices to prove $\tau_{i+1}'\tau_{i-1}'\tau_i'(u) = u$ for $u \in \{x_{i-1}, \sigma_i^{-1}(y)\}$, for all $i \in \{1,2,3\}$.

We have $\tau_i'(x_{i-1}) = \sigma_i(y) \ne \sigma_{i-1}^{-1}(y)$, $\tau_{i-1}'\tau_i'(x) = \sigma_{i-1}\sigma_i(y) = \sigma_{i+1}^{-1}(y)$, which is sent by τ_{i+1}' to x_{i-1}. When i is replaced by $i-1$, then the triangle involves the action of τ_i' on $\sigma_i^{-1}(y)$, and so nothing else is needed.

The permutational order of $(\tau_1', \tau_2', \tau_3')$ is the sum of the incoming orders diminished by three, and the size is the sum of the input sizes diminished by one. We can hence state:

Lemma 6.1. *The permutations* $\tau_i' = \tau_i^{\mathbf{x},\Sigma,y}$, $1 \le i \le 3$, *fulfil conditions* (P1) *and* (P2). *If permutations* τ_i *act on* X *transitively and yield genus* g_X, *and if permutations* σ_i *act transitively on* Y *and yield genus* g_Y, *then permutations* τ_i' *act transitively on* $(X \cup Y) \setminus \{y\}$ *and yield genus* $g_X + g_Y$.

To give conditions for a construction reverse to $\tau_i^{\mathbf{x},\Sigma,y}$ requires a bit of general theory.

Let (τ_1, τ_2, τ_3) be a triple that satisfies (P1) and (P2), with the underlying set X. A (cyclic) sequence $v_3v_2v_1$, where $v_1, v_2, v_3 \in X$, is called a *trigon* if there exists $j \in \{1,2,3\}$ such that

$$\begin{aligned}
&v_3 \text{ and } v_1 \text{ are in the same cycle of } \tau_{j+1} \text{ and } v_3 \notin \{v_1, \tau_{j+1}(v_1)\},\\
&v_2 \text{ and } v_3 \text{ are in the same cycle of } \tau_j \text{ and } v_2 \notin \{v_3, \tau_j(v_3)\}, \text{ and}\\
&v_1 \text{ and } v_2 \text{ are in the same cycle of } \tau_{j-1} \text{ and } v_1 \notin \{v_2, \tau_{j-1}(v_2)\}.
\end{aligned}$$

By default we shall assume $j = 1$. Then for every $i \in \{1,2,3\}$ there exists a cycle ρ_i incident to v_{i-1} and v_{i+1}. Denote by ℓ_i the length of ρ_i. The definition of trigon can be succinctly expressed by saying that $\tau_i^{k_i}(v_{i-1}) = v_{i+1}$ for some k_i, where $1 < k_i < \ell_i$.

Cycles ρ_1, ρ_2 and ρ_3 will be called the *sides* of the trigon $v_3v_2v_1$. Point v_1, v_2 and v_3 are *vertices* of the trigon.

It is useful to realize that if the condition $1 < k_i < \ell_i$ holds for at least one $i \in \{1,2,3\}$, then it holds for all such i:

Lemma 6.2. *Let $v_i \in X$ be such that $\tau_i^{k_i}(v_{i-1}) = v_{i+1}$ for some $k_i \geq 1$, and assume that k_i is the least possible, $1 \leq i \leq 3$. If $k_j = 1$ for at least one $j \in \{1,2,3\}$, then $k_i = 1$ for all $i \in \{1,2,3\}$.*

Proof: Denote by ρ_i the cycle of τ_i that moves both v_{i-1} and v_{i+1}. Let us have $\tau_j(v_{j-1}) = v_{j+1}$ for some $j \in \{1,2,3\}$. The intersection of ρ_{j-1} and ρ_{j+1} contains the point v_j and the point $\tau_{j-1}(v_{j+1}) = \tau_{j+1}^{-1}(v_{j-1})$. These two points coincide, by condition (P1). Thus $v_j = \tau_{j-1}(v_{j+1})$ and $\tau_{j+1}(v_j) = v_{j-1}$. $\square$

Corollary 6.3. *There exists no trigon with a side of length two.*

Corollary 6.4. *Suppose that a point v is incident to two cycles of length two. Then v is a vertex of no trigon.*

Let $v_3v_2v_1$ be a trigon in the trading surface (τ_1, τ_2, τ_3). Associate with it a polygon

$$v_3\,\tau_1(v_3)\,\ldots\,\tau_1^{k_1-1}(v_3)\,v_2\,\tau_3(v_2)\,\ldots\,\tau_3^{k_3-1}(v_2)\,v_1\,\tau_2(v_1)\,\ldots\,\tau_2^{k_2-1}(v_1),$$

where $k_i \geq 1$ is the least possible such that $\tau_i^{k_i}(v_{i-1}) = v_{i+1}$ (we know that $k_i \geq 2$ for all $i \in \{1,2,3\}$).

This polygon can be a separating one, i.e. after its removal the surface is divided into two components. In such a case call the trigon $v_3v_2v_1$ *separating*. Note that all trigons of a trading sphere are separating.

Furthermore, denote by ρ_i the side of $v_3v_2v_1$ that is incident to v_{i-1} and v_{i+1}. Define $\mathrm{In}(v_3v_2v_1)$ as the least set of cycles that fulfils:

(I1) if $\rho \neq \rho_i$ moves $\tau_i^h(v_i)$, $1 \leq h < k_i$, then $\rho \in \mathrm{In}(v_3v_2v_1)$;
(I2) if ρ meets $\rho' \in \mathrm{In}(v_3v_2v_1)$ and $\rho \notin \{\rho_1, \rho_2, \rho_3\}$, then $\rho \in \mathrm{In}(v_3v_2v_1)$.

Let the trigon be separated. The set $\mathrm{In}(v_3v_2v_1)$ is obviously nonempty and yields all cyclic faces in one of the components determined by the polygon associated with $v_3v_2v_1$. The points of this component are called the *points* of trigon $v_3v_2v_1$, and they divide into the vertices v_i and into the *inner points*. The set of inner points will be denoted by $\mathrm{Pnt}(v_3v_2v_1)$.

Lemma 6.5. *Let $v_3v_2v_1$ be a separating trigon. Then $\mathrm{Pnt}(v_3v_2v_1)$ coincides with the set of all points that are moved by some $\rho \in \mathrm{In}(v_3v_2v_1)$.*

The proof does not seem to be needed, since the lemma clearly follows from the geometric interpretation. If a more formal approach is desired, one can apply results of [13], particularly Proposition 3.6. (One of central notions in [13] is the notion of multigon. Trigon $v_3v_2v_1$ is a multigon of profile (ρ_3, ρ_2, ρ_1). Multigons are separating and non-separating, and for each separating multigon P one defines

$\mathrm{In}(P)$ and $\mathrm{Pnt}(P)$ in a way special kind of which are the above definitions of In and Pnt for trigons.)

Let us now turn to the reversal of the trigonal construction described above. Suppose that $\mathbf{x} = x_3x_2x_1$, (τ_1, τ_2, τ_3), $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $y \in Y$ satisfy the initial conditions of this section.

Lemma 6.6. *Put $\tau_i' = \tau_i^{\mathbf{x},\Sigma,y}$. Then $x_3x_2x_1$ is a separating trigon in $(\tau_1', \tau_2', \tau_3')$.*

Proof: For all $i \in \{1,2,3\}$ denote by ρ_i' the cycle of τ_i'

$$(\dots\ x_{i-1}\ \dots\ \sigma_i^{-1}(y)\ x_{i+1}\ \dots),$$

and by ρ_i the cycle $(y\ \dots\ \sigma_i^{-1}(y))$ of σ_i. We see that ρ_i' moves both x_{i-1} and x_{i+1}, and that $x_{i+1} \neq \rho_i'(x_{i-1})$. Thus $x_3x_2x_1$ is a trigon in $(\tau_1', \tau_2', \tau_3')$, $\mathrm{In}(x_3x_2x_1) = A \setminus \{\rho_1, \rho_2, \rho_3\}$, where A is the set of all cycles occurring in σ_i, $1 \le i \le 3$, and $\mathrm{Pnt}(x_3x_2x_1) = Y \setminus \{y\}$. We see that $x_3x_2x_1$ is indeed separating. □

Proposition 6.7. *Let μ_i, $1 \le i \le 3$, be permutations of a set $X \cup Y$, $X \cap Y = \emptyset$, that fulfil conditions* (P1) *and* (P2)*, and let $x_3x_2x_1$ be a separating trigon in (μ_1, μ_2, μ_3) such that $\mathrm{Pnt}(x_3x_2x_1) = Y$. Assume $y \notin X \cup Y$, and define a permutation σ_i of $Y \cup \{y\}$ so that it consists of all cycles of $\mathrm{In}(x_3x_2x_1)$ which are induced by μ_i, and of the cycle $(y\ \mu_i(x_{i-1})\ \dots\ \mu_i^{k_i-1}(x_{i-1}))$, where $k_i \ge 1$ is the least integer such that $x_{i+1} = \mu_i^{k_i}(x_{i-1})$. Define also τ_i as the permutation of X which is obtained by removing from the cycle set of μ_i all cycles of $\mathrm{In}(x_3x_2x_1)$, and by replacing the cycle $(\dots\ x_{i-1}\ \mu_i(x_{i-1})\ \dots\ \mu_i^{k_i-1}(x_{i-1})\ x_{i+1}\ \dots)$ with $(\dots\ x_{i-1}\ x_{i+1}\ \dots)$.*

Then both $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) fulfil conditions (P1) *and* (P2)*, $\mathbf{x} = x_3x_2x_1$ is the triangle in the latter triple, and $\mu_i = \tau_i^{\mathbf{x},\Sigma,y}$, for all $i \in \{1,2,3\}$.*

Proof: First note that $k_i \ge 2$ for all $i \in \{1,2,3\}$, by the definition of the trigon. This means that permutations σ_i are fixed point free. For τ_i this is clear immediately. Permutations τ_i satisfy (P1) since they are obtained by retaining or restricting the original cycles. The same reasoning applies to cycles of σ_i, with the exception of the cases when y is a common point of two cycles. However, in such cases there is no other common point since the cycles involved come from the cycles induced by the trigon.

We have $\sigma_i(y) = \mu_i(x_{i-1})$, $\sigma_{i-1}\sigma_i(y) = \mu_{i-1}\mu_i(x_{i-1}) = \mu_{i+1}^{-1}(x_{i-1})$, which is mapped by σ_{i+1} to y. We get thus triangle $y\,\mu_i(x_{i-1})\,\mu_{i+1}^{-1}(x_{i-1})$ in $(\sigma_1, \sigma_2, \sigma_3)$, and for $i = 1,2,3$ these triangles cover all (j, u) such that $j \in \{1,2,3\}$ and either $u = y$ or $\sigma_j(u) \neq \mu_j(u)$, $u \in Y$. It follows that $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ satisfies both (P1) and (P2). In (τ_1, τ_2, τ_3) there arises the triangle $\mathbf{x} = x_3x_2x_1$, and so both (P1) and (P2) are satisfied as well. The equalities $\mu_i = \tau_i^{\mathbf{x},\Sigma,y}$, $i \in \{1,2,3\}$, can be easily derived by comparing the cyclic decompositions. □

Let us remark that both τ_i and σ_i clearly generate a transitive permutation group if mappings μ_i do so. The triple (τ_1, τ_2, τ_3) will be called the *outer trading* *surface* of the trigon $\mathbf{x} = x_3x_2x_1$, while $(\sigma_1, \sigma_2, \sigma_3)$ will be the *inner trading surface*.

7. Trigon-free triangles

Assume that τ_i, $1 \le i \le 3$, are permutations of a set X that fulfil (P1) and (P2). For simplicity, assume also that the group generated by these permutations is transitive on X.

Lemma 7.1. *Every separating trigon $x_3x_2x_1$ is fully determined by the set of inner cycles* $\mathrm{In}(x_3x_2x_1)$.

PROOF: For each $i \in \{1,2,3\}$ there exists exactly one cycle of τ_i that does not belong to $\mathrm{In}(x_3x_2x_1)$, but intersects an element of $\mathrm{In}(x_3x_2x_1)$. The intersections of these three cycles yield the vertices of the trigon. □

Call a triangle $y_3y_2y_1$ *trigon-free* if there exists no trigon $x_3x_2x_1$ such that the set $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\}$ has two elements.

If a triangle $y_3y_2y_1$ is not trigon-free, then there clearly exist a trigon $x_3x_2x_1$ and $i \in \{1,2,3\}$ such that $x_{i-1} = y_{i+1}$ and $x_{i+1} = y_{i-1}$.

Lemma 7.2. *Let $y_3y_2y_1$ be a trigon-free triangle. Denote by k the number of $i \in \{1,2,3\}$ such that $(y_{i-1}\ y_{i+1})$ is a cycle of τ_i.*

(i) *Assume $k = 1$ and choose $j \in \{1,2,3\}$ so that $(y_{j-1}\ y_{j+1})$ is a cycle of τ_j. Set $y = y_j$. There exist permutations σ_i of the set $X \setminus \{y\}$ that fulfil* (P1) *and* (P2) *and satisfy $\tau_i = \sigma_i^{y_{j-1}, y_{j+1}}$, $1 \le i \le 3$. In addition, $\sigma_j(y_{j+1}) = y_{j-1}$.*

(ii) *Assume $k = 2$ and choose $j \in \{1,2,3\}$ so that $(y_{j\pm1}\ y_j)$ is a cycle of $\tau_{j\mp1}$. Then there exist permutations σ_i of the set $X \setminus \{y_{j-1}, y_{j+1}\}$ that fulfil* (P1) *and* (P2) *and satisfy $\tau_i = \sigma_i^{j, y_j}$, $1 \le i \le 3$.*

(iii) *Assume $k = 3$ and suppose that X has more than four elements. Put $x_i = \tau_{i-1}(y_{i-1})$, for $i \in \{1,2,3\}$. Set $\mathbf{x} = x_3x_2x_1$. Then there exist permutations σ_i of the set $X \setminus \{y_1, y_2, y_3\}$ that fulfil* (P1) *and* (P2) *and satisfy $\tau_i = \sigma_i^{\mathbf{x}}$, $1 \le i \le 3$.*

PROOF: The case $k = 3$ is a direct application of Proposition 4.1. If $k = 2$, then $(y_{j-1}\ y_{j+1})$ is not a cycle of τ_j, and hence the length condition of Proposition 4.1 is satisfied. The condition of empty intersection follows from the fact that $y_3y_2y_1$ is trigon-free, and so Proposition 4.1 can be used. It remains to consider the case $k = 1$.

Assume $k = 1$ and put $z = y_{j+1}$ and $x = y_{j-1}$. Let ρ' be the cycle $(x\ z)$, and denote by ρ'' the cycle of τ_j that moves y. Furthermore, let γ' be the cycle of τ_{j-1} that maps z onto y, and let γ'' be the cycle of τ_{j+1} that maps y onto x. If γ is a cycle of τ_{j-1} that connects ρ' and ρ'' and $\gamma \ne \gamma'$, then γ has to move x. Denote by y' the intersection of γ and ρ''. We have $\tau_{j+1}(x) \ne y$, as $k = 1$. This means that xyy' is a trigon, by Lemma 6.2. The triangle zyx, which coincides with $x_3x_2x_1$, is hence not trigon-free, and that contradicts our assumptions. The situation when there exists a cycle $\gamma \neq \gamma''$ of τ_{j+1} that connects ρ' and ρ'' is similar: one gets a trigon yzy', where y' is determined by the intersection of γ and ρ''. We conclude that there exists no cycle different from γ' and γ'' that connects ρ' and ρ''. The assumptions of Proposition 5.3 hence hold, which means that (τ_1, τ_2, τ_3) can be really obtained by the sliding construction. $\square$

Corollary 7.3. *Suppose that τ_i, $1 \leq i \leq 3$, generate on a set X a transitive permutation group, and that they fulfil* (P1) *and* (P2). *If any of permutations τ_i contains a cycle of length two and if X has more than four elements, then (τ_1, τ_2, τ_3) contains a non-separating trigon, or it can be obtained by a trapezium or 2-sliding or trigonal expansion from a triple $(\sigma_1, \sigma_2, \sigma_3)$ that fulfils* (P1) *and* (P2), *and acts on a proper subset of X.*

PROOF: Consider a triangle zyx such that $(x\ y)$ is a cycle of some τ_i. If the triangle is trigon-free, use Lemma 7.2. If it is not, consider the trigon induced by the triangle and use Proposition 6.7. $\square$

Theorem 7.4. *Every trading sphere can be obtained from the trading sphere on four elements by a series of constructions, where each step is a trapezium expansion or a 2-sliding expansion or a trigonal expansion.*

PROOF: This is a direct consequence of Corollary 7.3, Proposition 3.1 and Corollary 5.5. $\square$

8. Inverse mappings

In some proofs the number of needed verifications can be reduced nearly by a half when one considers, in addition to (τ_1, τ_2, τ_3), one of the triples $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$, $(\tau_1^{-1}, \tau_3^{-1}, \tau_2^{-1})$ and $(\tau_3^{-1}, \tau_2^{-1}, \tau_1^{-1})$. In tabular interpretation this corresponds to the exchange of rows and columns, or columns and entry values, or rows and entry values.

Proposition 8.1. *Let (τ_1, τ_2, τ_3) be a triple of permutations that fulfil* (P1) *and* (P2). *Then $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$ fulfils* (P1) *and* (P2) *as well. Furthermore, $x_3x_2x_1$ is a triangle (or a trigon) in (τ_1, τ_2, τ_3) if and only if $x_1x_2x_3$ is a triangle (or a trigon) in $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$.*

PROOF: The cyclic decomposition of τ_i and τ_i^{-1} yield the same sets, and so $(\tau_2^{-1}, \tau_1^{-1}, \tau_3^{-1})$ consists of fixed-point free permutations that fulfil (P1). Since $\tau_2^{-1}\tau_1^{-1}\tau_3^{-1} = \tau_3(\tau_1\tau_2\tau_3)^{-1}\tau_3^{-1}$ is the identity mapping, we see that (P2) is satisfied as well. Now, $\tau_1(x_3) = x_2 \Leftrightarrow \tau_1^{-1}(x_2) = x_3$, $\tau_3(x_2) = x_1 \Leftrightarrow \tau_3^{-1}(x_1) = x_2$ and $\tau_2(x_1) = x_3 \Leftrightarrow \tau_2^{-1}(x_3) = x_1$. This verifies the claim about triangles. The part about trigons is similar. $\square$

Corollary 8.2. *Let π be a permutation of $\{1,2,3\}$, and suppose that (τ_1, τ_2, τ_3) fulfils* (P1) *and* (P2). *If $\operatorname{sgn}(\pi) = 1$, then $(\tau_{\pi(1)}, \tau_{\pi(2)}, \tau_{\pi(3)})$ satisfies these conditions as well. If $\operatorname{sgn}(\pi) = -1$, then they are fulfilled by $(\tau_{\pi(1)}^{-1}, \tau_{\pi(2)}^{-1}, \tau_{\pi(3)}^{-1})$.*

We shall be now describing how the constructions of the earlier sections are influenced by transition to inverse permutations. In principle an expansion of the triple formed by inverse permutations yields the inverses of the expansion from the original permutation triple. However, one has to be careful with details since the added points y_1, y_2 and y_3 depend not only on the permutation structure, but also upon the order of permutations in the structural triple.

Proposition 8.3. *Let* (τ_1, τ_2, τ_3) *be a triple of permutations that fulfil* (P1) *and* (P2), *and act on* X, *where* $X \cap \{y_1, y_2, y_3\} = \emptyset$. *Fix* $j \in \{1, 2, 3\}$ *and denote by* $\pi \in S_3$ *the transposition* $(j{-}1\ j{+}1)$. *Put* $\tilde{\tau}_i = \tau_{\pi(i)}^{-1}$, $1 \le i \le 3$. *Transposition* $(y_{j-1}\ y_{j+1})$ *will be denoted (generically) as* g.

(i) *If* $\mathbf{x} = x_3 x_2 x_1$ *is a triangle in* (τ_1, τ_2, τ_3), *then* $\tilde{\mathbf{x}} = x_1 x_2 x_3$ *is a triangle in* $(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3)$, *and*

$$\tilde{\tau}_{\pi(i)}^{\tilde{\mathbf{x}}} = g(\tau_i^{\mathbf{x}})^{-1} g, \quad 1 \le i \le 3.$$

(ii) *For every* $u \in X$

$$\tilde{\tau}_{\pi(i)}^{j,u} = g(\tau_i^{j,u})^{-1} g, \quad 1 \le i \le 3.$$

(iii) *Suppose that* $x, z \in X$ *are distinct points that are incident to the same cycle of* τ_j, $z \ne \tau_j(x)$. *Then*

$$\tilde{\tau}_{\pi(i)}^{z,x} = (\tau_i^{x,z})^{-1}, \quad 1 \le i \le 3.$$

(iv) *Let* σ_i, $1 \le i \le 3$, *be permutations of a set* Y, $X \cap Y = \emptyset$. *Put* $\tilde{\sigma}_i = \sigma_{\pi(i)}^{-1}$, $1 \le i \le 3$, *and assume that* $(\sigma_1, \sigma_2, \sigma_3)$ *fulfils* (P1) *and* (P2). *Let* $\mathbf{x} = x_3 x_2 x_1$ *be a triangle in* (τ_1, τ_2, τ_3) *and let* y *be a point of* Y. *Set* $\tilde{\Sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$ *and* $\tilde{\mathbf{x}} = x_1 x_2 x_3$. *Then* $\tilde{\mathbf{x}}$ *is a triangle in* $(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3)$ *and*

$$\tilde{\tau}_{\pi(i)}^{\tilde{\mathbf{x}}, \tilde{\Sigma}, y} = (\tau_i^{\mathbf{x}, \Sigma, y})^{-1}.$$

Proof: If $\mathbf{x} = x_3 x_2 x_1$ is a triangle in (τ_1, τ_2, τ_3), then $\tilde{\mathbf{x}} = x_1 x_2 x_3$ is a triangle in $(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3)$ by Corollary 8.2. From $\tau_i(x_{i-1}) = x_{i+1}$ we get $x_{i-1} = \tau_i^{-1}(x_{i+1})$, and so

$$\tilde{\tau}_j(x_{j+1}) = x_{j-1},\ \tilde{\tau}_{j-1}(x_{j-1}) = x_j \ \text{ and } \ \tilde{\tau}_{j+1}(x_j) = x_{j+1}.$$

In other words, $\tilde{\tau}_i(x_{\pi(i-1)}) = x_{\pi(i+1)}$, $1 \le i \le 3$, and thus for $\tilde{x}_i = x_{\pi(i)}$ we obtain $\tilde{\mathbf{x}} = \tilde{x}_3 \tilde{x}_2 \tilde{x}_1$ and $\tilde{\tau}_i(\tilde{x}_{i-1}) = \tilde{x}_{i+1}$.

The cycle $(\dots\ \tilde{x}_{i-1}\ \tilde{x}_{i+1}\ \dots)$ of $\tilde{\tau}_i$ is replaced in $\tilde{\tau}_i^{\tilde{\mathbf{x}}}$ by $(\dots\ \tilde{x}_{i-1}\ y_i\ \tilde{x}_{i+1}\ \dots)$. By passing to inverse mappings we see that $(\dots\ x_{\pi(i+1)}\ x_{\pi(i-1)}\ \dots)$ of $\tau_{\pi(i)}$ is replaced by $(\dots \tilde{x}_{\pi(i+1)}\ y_i\ \tilde{x}_{\pi(i-1)}\ \dots)$. Quadruples $(\pi(i), \pi(i+1), \pi(i-1), i)$ are equal, for $i = j, j-1, j+1$, to $(j, j-1, j+1, j)$, $(j+1, j, j-1, j-1)$ and $(j-1, j+1, j, j+1)$, respectively, and so the exchange of y_{j-1} and y_{j+1} makes from $\tilde{\tau}_i^{-1}$ permutation $\tau_{\pi(i)}$, as required by point (i).

Point (iv) can be proved similarly: The mapping $\tilde{\tau}_i^{\tilde{\mathbf{x}},\tilde{\Sigma},y}$ is obtained from $\tilde{\tau}_i$ and $\tilde{\sigma}_i$ by merging the cycles

$$(\ldots\ \tilde{x}_{i-1}\ \tilde{x}_{i+1}\ \ldots) \text{ and } (y\ \tilde{\sigma}_i(y)\ \ldots\ \tilde{\sigma}_i^{-1}(y))$$

into $(\ldots\ \tilde{x}_{i-1}\ \tilde{\sigma}_i(y)\ \ldots\ \tilde{\sigma}_i^{-1}(y)\ \tilde{x}_{i+1}\ \ldots)$ and uniting the other cycles. This means, after turning to the inverses, that the merged cycle equals

$$(\ldots x_{\pi(i+1)}\ \sigma_{\pi(i)}(y)\ \ldots\ \sigma_{\pi(i)}^{-1}(y)\ x_{\pi(i-1)}\ \ldots),$$

and that the other cycles are obtained by uniting the remaining cycles of $\sigma_{\pi(i)}$ and $\tau_{\pi(i)}$.

If i runs through $\{1,2,3\}$, then $(\pi(i+1),\pi(i),\pi(i-1))$ runs through $(i-1,i,i+1)$, and so the constructed permutations are indeed equal to $\tau_i^{x,\Sigma,y}$.

To prove the point (ii) set $u_j = u$, $u_{j-1} = \tau_{j+1}(u)$ and $u_{j+1} = \tau_{j-1}^{-1}(j)$, as in Section 4. Define similarly $\tilde{u}_j = u$, $\tilde{u}_{j-1} = \tilde{\tau}_{j+1}(u)$ and $\tilde{u}_{j+1} = \tilde{\tau}_{j-1}^{-1}(u)$. This clearly means $\tilde{u}_j = u_j$, $\tilde{u}_{j-1} = u_{j+1}$ and $\tilde{u}_{j+1} = u_{j-1}$. In $\tilde{\tau}_{j+1}^{j,u}$ we replace u with y_{j+1} and add the cycle $(y_j\ y_{j+1})$. Thus $\tilde{\tau}_{j+1}^{j,u}$ equals $(\tau_{j-1}^{j,u})^{-1}$ after the exchange of y_{j+1} and y_{j-1}. The case of $\tilde{\tau}_{j-1}^{j,u}$ is very similar. The permutation $(\tilde{\tau}_j^{j,u})^{-1}$ is obtained from τ_j by replacing $(\ldots\ u\ \ldots)$ with $(\ldots\ y_{j+1}\ y_{j-1}\ \ldots)$, and $(\ldots\ u_{j-1}\ u_{j+1}\ \ldots)$ with $(\ldots\ u_{j-1}\ y_j\ u_{j+1}\ \ldots)$. This equals $\tau_j^{j,u}$ when y_{j-1} is exchanged with y_{j+1}.

Let us finally turn to the sliding construction. One obtains $\tilde{\tau}_j^{z,x}$ in such a way that the cycle $(\ldots\ z\ \tau_j^{-1}(z)\ \ldots\ \tau_j(x)\ x\ \ldots)$ is replaced by cycles $(\ldots\ z\ x\ \ldots)$ and $(\tau_j^{-1}(z)\ \ldots\ \tau_j(x)\ y)$. The resulting permutation is clearly equal to $(\tau_j^{x,z})^{-1}$.

Furthermore, to get $\tilde{\tau}_{j-1}^{z,x}$ one uses τ_{j+1}^{-1} in such a way that the cycle $(\ldots\ x\ \ldots)$ is replaced by $(\ldots\ x\ y\ \ldots)$, which is a cycle of $(\tau_{j+1}^{x,z})^{-1}$. For $\tilde{\tau}_{j+1}^{z,x}$ one proceeds similarly. $\square$

9. Overlapping trapezia

A trigon-free triangle $u_3u_2u_1$ such that for some $i \in \{1,2,3\}$ the permutation τ_i contains cycle $(u_{u-1}\ u_{i+1})$ will be called a *seed*. The existence of a seed is a necessary precondition for an application of a local reduction, by Lemma 7.2.

To establish the existence of a seed we define, as an auxiliary notion, an unoriented graph that will be called the 2*-side graph*. The vertices are triangles with a side of length two, and edges connect those triangles that share a side of length two.

Seeds form a subset of vertices of the 2-side graph. To recognize a seed the following criterion is sometimes useful.

Lemma 9.1. *Let $u_3u_2u_1$ be a triangle and $i \in \{1,2,3\}$. There exists no trigon $u_{i+1}u_{i-1}v$ if and only if*

(1) *either $(u_{i-1}\ u_{i+1})$ is a cycle of τ_i,*

(2) *or there is no common point between the cycle of τ_{i-1} that moves u_{i-1} and the cycle of τ_{i+1} that moves u_{i+1}.*

Proof: There is at most one point v in the intersection of cycles that are mentioned in point (2). It exists if and only if $u_{i+1}u_{i-1}v$ is a trigon or a triangle. The latter case corresponds to point (1), by Lemma 6.2. □

Fix now a triple (τ_1, τ_2, τ_3) that determines a trading surface on X. For $u \in X$ and $j \in \{1,2,3\}$ consider a trapezium expansion $\tau_i' = \tau_i^{j,u}$. Set $u_j = u$, $u_{j+1} = \tau_{j-1}^{-1}(u)$ and $u_{j+1} = \tau_{j+1}(u)$. We shall call such an expansion a 2-*trapezium expansion* if $(u_j\ u_{j+1}) = (u\ \tau_{j-1}(u))$ is a cycle of τ_{j-1} or if $(u_j\ u_{j-1}) = (u\ \tau_{j+1}(u))$ is a cycle of τ_{j+1}.

We shall now assume that we are considering a 2-trapezium expansion $(\tau_1', \tau_2', \tau_3')$. We shall choose the variant with $(u_j\ u_{j+1})$ a cycle of τ_{j-1}. This will have no impact on the generality of results obtained since the transition to inverses exchanges $j+1$ with $j-1$, by Proposition 8.3.

Our notation will follow Lemma 4.4. A cycle of τ_i', $1 \le i \le 3$, that is induced by a cycle ρ of τ_i will be denoted by ρ'. Recall that $\rho' = \rho$ if ρ moves no u_i. Recall also that a cycle of τ_i' is either equal to some ρ', or to one of the cycles $(y_{j+1}\ y_j)$ and $(y_{j-1}\ y_j)$ (then $i = j-1$ and $i = j+1$, respectively).

The cycle $(u_j\ u_{j+1})$ of τ_{j-1} gets transformed into the cycle $(y_{j-1}\ u_{j+1})$ of τ_{j-1}'. Using Corollary 6.4 we can hence state

Lemma 9.2. *Neither y_j, nor y_{j-1} can be a vertex of a trigon.*

For every $x \in X$ put $x' = x$ if $x \neq u$, and $x' = y_{j+1}$ if $x = u$. Under this convention we claim

Lemma 9.3. *If $v_3v_2v_1$ is a trigon in (τ_1, τ_2, τ_3), then $v_3'v_2'v_1'$ is a trigon in $(\tau_1', \tau_2', \tau_3')$ as well. Furthermore, every trigon of $(\tau_1', \tau_2', \tau_3')$ has such a form.*

Proof: Let $v_3v_2v_1$ be a trigon, and let ρ_i be the side of the trigon that is incident both to v_{i+1} and v_{i-1}, $1 \le i \le 3$. Then ρ_i belongs to τ_i, and $\rho_{j-1} \neq (u_j\ u_{j+1})$, by Corollary 6.3. Hence $\rho_{j-1}' \neq (y_{j-1}\ u_{j+1})$, and, of course, $\rho_{j+1}' \neq (y_{j-1}\ y_j)$. Thus at most one of ρ_i', $1 \le i \le 3$, can be incident to y_{j-1}, and therefore ρ_{i-1}' and ρ_{i+1}' cannot meet in y_{j-1}. By Lemma 4.4, ρ_{i-1}' and ρ_{i+1}' meet in some point, for every $i \in \{1,2,3\}$, since $\rho_{j-1} \neq (u_j\ u_{j+1})$. But that means that they meet in v_i', by Lemma 4.4 again.

By Lemma 8.2, every trigon of $(\tau_1', \tau_2', \tau_3')$ has to be of the form $v_3'v_2'v_1'$, with ρ_i' incident to v_{i+1}' and v_{i-1}', ρ_i' a cycle of τ_i, $1 \le i \le 3$. If ρ_{i+1}' meets ρ_{i-1}' in v_i', then ρ_{i+1} meets ρ_{i-1} in v_i. Hence $v_3v_2v_1$ is a triangle or a trigon. If $i \neq j$, then ρ_i is obtained from ρ_i' by replacing every x incident to ρ_i' by x'. That means that $v_3v_2v_1$ cannot be a triangle, by Lemma 6.2. □

Lemma 9.4. *Let $\mathbf{x} = x_3x_2x_1$ be a triangle in (τ_1, τ_2, τ_3). If $\mathbf{x}$ equals neither $u_ju_{j-1}u_{j+1}$, nor $u_ju_{j+1}\tau_j^{-1}(u)$, then $\mathbf{x}' = x_3'x_2'x_1'$ is a triangle of $(\tau_1', \tau_2', \tau_3')$.*

Each triangle in $(\tau_1', \tau_2', \tau_3')$ *is either of this form, or belongs to*

$$\{y_{j+1}u_{j-1}y_j,\ y_{j+1}y_jy_{j-1},\ y_{j-1}y_ju_{j+1},\ y_{j-1}u_{j+1}\tau_j^{-1}(u)\}.$$

Proof: If ρ is a cycle of τ_i, and $x \in X$ is incident to ρ, then ρ' moves x' to $(\rho(x))'$ with the exceptions when $(x, \rho(x))$ is one of

$$(\tau_j^{-1}(u), u_j),\ (u_j, u_{j+1}),\ (u_{j+1}, u_j)\ \text{and}\ (u_{j-1}, u_{j+1}).$$

A triangle $x_3x_2x_1$ of (τ_1, τ_2, τ_3) has one of edges (x_{j-1}, x_{j+1}) equal to one of above four pairs if and only if it equals to $u_ju_{j-1}u_{j+1}$ or to $u_ju_{j+1}\tau_j^{-1}(u)$. Thus if it is not equal to one of these two triangles, then $x_3'x_2'x_1'$ is a triangle of $(\tau_1', \tau_2', \tau_3')$. The number of triangles is the same as the number of points, and hence there are two more triangles in $(\tau_1', \tau_2', \tau_3')$ than in (τ_1, τ_2, τ_3). For the rest it therefore suffices to list the four triangles which are not of the form $x_3'x_2'x_1'$. □

We shall now describe how the 2-side graph of $(\tau_1', \tau_2', \tau_3')$ is obtained from the 2-side graph of (τ_1, τ_2, τ_3) by means of adding edges and deleting edges. Note that every vertex of a 2-side graph has valence at least one. All isolated vertices that will arise from a deletion of an edge will be thus considered as removed from the graph.

Lemma 9.4 defines $\mathbf{x}'$ as $x_3'x_2'x_1'$ for every triangle $\mathbf{x} = x_3x_2x_1 \notin \{u_ju_{j-1}u_{j+1}, u_ju_{j+1}\tau_j^{-1}(u)\}$. We extend the notation by setting $(u_ju_{j-1}u_{j+1})' = y_{j+1}u_{j-1}y_j$ and $(u_ju_{j+1}\tau_j^{-1}(u))' = y_{j-1}u_{j+1}\tau_j^{-1}(u)$.

Lemma 9.5. *The* 2*-side graph of* $(\tau_1', \tau_2', \tau_3')$ *is obtained from the* 2*-side graph of* (τ_1, τ_2, τ_3) *by*

(1) *deleting the edge* $\{u_{j+1}u_{j-1}\tau_{j-1}(u_{j-1}),\ u_{j-1}u_{j+1}u_j\}$ *if* $(u_{j-1}\ u_{j+1})$ *is a cycle of* τ_j*;*
(2) *deleting the edge* $\{\tau_j^{-1}(u)u_j\tau_{j-1}(u),\ u_ju_{j+1}\tau_j^{-1}(u)\}$ *if* $(u_j\ \tau_j^{-1}(u))$ *is a cycle of* τ_j*; and*
(3) *replacing each of the remaining edges* $\{\mathbf{x}, \mathbf{v}\}$ *by* $\{\mathbf{x}', \mathbf{v}'\}$*, with the exception of*

$$\{u_ju_{j-1}u_{j+1},\ u_ju_{j+1}\tau_j^{-1}(u)\},$$

which is replaced by a chain

$$u_{j-1}y_jy_{j+1},\ y_{j+1}y_jy_{j-1},\ y_{j-1}y_ju_{j+1},\ y_{j-1}u_{j+1}\tau_j^{-1}(u).$$

Proof: The cycles $(\ldots\ \tau_j^{-1}(u_j)\ y_{j-1}\ y_{j+1}\ \ldots)$ and $(\ldots\ u_{j-1}\ u_j\ u_{j+1}\ \ldots)$ of τ_j' are the only cycles ρ' that are longer than the corresponding cycle ρ. This is the reason for the deletion described in points (1) and (2).

If $\{\mathbf{x}, \mathbf{v}\}$ is an edge that is induced by a cycle $(x_{i-1}\ x_{i+1})$ which is different from $(u_{j-1}\ u_{j+1})$, $(u_j\ \tau_j^{-1}(u))$ and $(u_j\ u_{j+1})$, then $\mathbf{x}'$ and $\mathbf{v}'$ share the side $(x_{i-1}'\ x_{i+1}')$. The rest is clear. □

Lemma 9.6. *Triangles $y_{j+1}y_jy_{j-1}$ and $y_{j-1}y_ju_{j+1}$ are seeds of $(\tau_1', \tau_2', \tau_3')$. If a triangle $x_3'x_2'x_1'$ of $(\tau_1', \tau_2', \tau_3')$ has a side of length two, then it is a seed if and only if $x_3x_2x_1$ is a seed in (τ_1, τ_2, τ_3). The triangle $u_{j-1}y_jy_{j+1}$ (or the triangle $y_{j-1}y_{j+1}\tau_j^{-1}(u)$) is a seed in $(\tau_1', \tau_2', \tau_3')$ if and only if in (τ_1, τ_2, τ_3) there exists no trigon with vertices u_{j-1} and u_j (or u_{j+1} and $\tau_j^{-1}(u)$, respectively).*

Proof: Points y_{j-1} and y_j are vertices of no trigon, by Lemma 6.5. Hence $y_{j+1}y_jy_{j-1}$ and $y_{j-1}y_ju_{j+1}$ are seeds. Since y_{j-1} and y_j are vertices of no trigon only the side induced by $y_{j+1} = u'$ and u_{j-1} (or by u_{j+1} and $\tau_j^{-1}(u)$) needs to be considered with respect to the triangle $u_{j-1}y_jy_{j+1}$ (or $y_{j-1}y_{j+1}\tau_j^{-1}(u)$). The result then follows from Lemma 9.3, since by this lemma a trigon with vertices v_1' and v_2' exists in $(\tau_1', \tau_2', \tau_3')$ if and only if a trigon with vertices v_1 and v_2 exists in (τ_1, τ_2, τ_3), for all $v_1, v_2 \in X$. □

Lemmas 9.5 and 9.6 make clear how the 2-trapezium expansion influences the 2-side graphs and the seed set. Recall that for a general result on the 2-trapezium construction one has to consider also $(\tilde\tau_1, \tilde\tau_2, \tilde\tau_3)$, where $\tilde\tau_i = \tau_{\pi(i)}^{-1}$ and π is the transposition $(j{-}1\ j{+}1)$. The case when $(u\ \tau_{j+1}^{-1}(u))$ is a cycle of τ_{j+1} then gets transformed to the situation when $(u\ \tilde\tau_{j-1}(u))$ is a cycle of $\tilde\tau_{j-1}$, which corresponds to the situation studied in the lemmas above.

Proposition 9.7. *A trading surface $(\sigma_1, \sigma_2, \sigma_3)$ can be obtained by a 2-trapezium expansion if and only if there exist $j \in \{1, 2, 3\}$ and cycles $(\dots\ x_0\ x_1\ x_2\ \dots)$ and $(\dots\ x_2'\ x_1'\ x_0'\ \dots)$ of σ_j such that*

(1) *either $(x_2\ x_1')$ and $(x_1\ x_0')$ are cycles of σ_{j-1}, and $(x_1\ x_1')$ is a cycle of σ_{j+1},*
(2) *or $(x_2'\ x_1)$ and $(x_1'\ x_0)$ are cycles of σ_{j+1}, and $(x_1'\ x_1)$ is a cycle of σ_{j-1}.*

Proof: Note that cases (1) and (2) induce each other when inverse permutations are considered. Hence we can consider only case (1). Under the conventions of lemmas above the permutation τ_j' contains cycles

$$(\dots\ \tau_j^{-1}(u)\ y_{j-1}\ y_{j+1}\ \dots) \text{ and } (\dots\ u_{j-1}\ y_j\ u_{j+1}\ \dots),$$

while $(y_{j+1}\ y_j)$ and $(y_{j-1}\ u_{j+1})$ are cycles of τ_{j-1} and $(y_j\ y_{j-1})$ is a cycle of τ_{j+1}.

For the other direction assume case (1) and set $y_{j-1} = x_1$, $y_{j+1} = x_2$, $y_j = u = x_1'$, $u_{j-1} = x_2'$ and $u_{j+1} = x_0'$. The cycle $(y_{j-1}\ u_{j+1})$ of σ_{j-1} does not meet the cycle of τ_{j+1} that is incident to y_{j+1} since the latter cycle meets $(\dots u_{j-1}\, y_j\, u_{j+1} \dots)$ in $u_{j-1} = \sigma_j^{-1}(y_j) = \sigma_{j-1}^{-1}\sigma_{j+1}^{-1}(y_{j+1}) = \sigma_j(y_{j+1})$, and we assume $u_{j-1} \neq u_{j+1}$. The preconditions of Proposition 4.5 hence hold, and we see that there exists τ_i, $1 \le i \le 3$, such that $\tau_i = \sigma_i^{j,u}$, $1 \le i \le 3$. Furthermore, the cycle $(y_{j-1}\ u_{j+1})$ of σ_{j-1} has to be derived from the cycle $(u\ u_{j+1}) = (u\ \tau_{j-1}^{-1}(u))$ of τ_{j-1}, which means that we are really dealing with a 2-trapezium expansion. □

10. Components of the 2-side graph

A separated latin bi-trade with structural triple (τ_1, τ_2, τ_3) will be called *bicyclic* if there exists $j \in \{1, 2, 3\}$ such that τ_j consists of only two cycles. Such latin bi-trades can be obtained by swapping two consecutive rows in the addition table modulo n (the other rows of the table are ignored). All bicyclic latin bi-trades of size $2n$ are isotopic to such a bi-trade if $j = 1$. This well known fact is reiterated in the following lemma. A trading surface determined by a bicyclic latin bi-trade will be called bicyclic as well.

Lemma 10.1. *Let τ_i, $1 \le i \le 3$, be permutations of a set X that satisfy* (P1) *and* (P2)*. Suppose that τ_j consists for some $j \in \{1, 2, 3\}$ exactly of two cycles. Let one of the cycles be $(x_0 \ \ldots \ x_{n-1})$. Then the other cycle of τ_j can be expressed as $(x'_{n-1} \ \ldots \ x'_0)$ in such a way that τ_{j+1} consists of cycles $(x_i \ x'_i)$ and τ_{j-1} consists of cycle $(x_{i+1} \ x'_i)$, $0 \le i < n$.*

Proof: A cycle of τ_{j+1} that moves x_i, $0 \le i < n$, intersects k different cycles of τ_j, where k is the length of the cycle. Since τ_j has only two cycles we must have $k = 2$. Denote the other point of the cycle by x'_i. From $\tau_{j-1}\tau_j(x_i) = \tau_{j+1}^{-1}(x_i)$ we see that τ_{j-1} consists of cycles $(x_{i+1} \ x'_i)$, $0 \le i < n$, where $x_n = x_0$. Furthermore, $\tau_j(x'_i) = \tau_j\tau_{j+1}(x_i) = \tau_{j-1}(x_i) = x'_{i-1}$. □

Bicyclic bi-trades are without trigons, and each triangle has at least two of the sides induced by a cycle of length two (all three sides are of length two if and only if $n = 2$). For $n \ge 3$ the 2-side graph is a cycle of length $2n$. For $n = 2$ one clearly obtains the complete graph K_4.

Let $\Sigma = (\tau_1, \tau_2, \tau_3)$ be a trading surface on X with a point $x \in X$. Define new permutations τ'_i on $(X \setminus \{x\}) \cup \{y_1, y_2, y_3, y_4\}$ in such a way that the ith permutation is obtained from τ_i by modifying $(\ldots \ x \ \ldots)$ to $(\ldots \ y_{i+1} \ y_{i-1} \ \ldots)$ and by adding cycle $(y_i \ y_4)$. (Indices are computed modulo three.) The cycle $(\ldots \ y_{i+1} \ y_{i-1} \ \ldots)$ can be written as $(y_{i-1} \ \ldots \ \tau_i^{-1}(x) \ y_{i+1})$, and can be regarded as a merging of $(y_{i-1} \ y_{i+1})$ and $(x \ \ldots \ \tau_i^{-1}(x))$. We see that our construction is in fact a special case of the trigonal construction (cf. the beginning of Section 6) such that $\tau'_i = \sigma_i^{\mathbf{y}, \Sigma, x}$, where $(\sigma_1, \sigma_2, \sigma_3)$ is the trading sphere upon $\{y_1, y_2, y_3, y_4\}$ and $\mathbf{y} = y_3y_2y_1$.

Replace now y_4 by x and denote τ'_i simply as τ_i^x, $1 \le i \le 3$. We have thus defined τ_i^x by replacing the occurrence of x in the cycle decomposition of τ_i by the pair $y_{i+1} \ y_{i-1}$, and by adding cycle $(y_i \ x)$. We shall call $(\tau_1^x, \tau_2^x, \tau_3^x)$ the *point expansion* of (τ_1, τ_2, τ_3) at x by $y_3y_2y_1$. Note that $y_3y_2y_1$ is a separating trigon in $(\tau_1^x, \tau_2^x, \tau_3^x)$ and that $\mathrm{Pnt}(y_3y_2y_1) = X \setminus \{x\}$ (this is clear; one can also use Lemma 6.6).

Lemma 10.2. *A trading surface (τ_1, τ_2, τ_3) with more than four points can be obtained by a point expansion if and only if it contains a point x that is moved only by cycles of length two. If $(x \ y_i)$ are cycles of τ_i, $1 \le i \le 3$, then $\tau_i = \sigma_i^x$, where $(\sigma_1, \sigma_2, \sigma_3)$ is a trading surface of the same genus.*

Proof: We have $\tau_i(y_{i+1}) = \tau_i\tau_{i+1}(x) = \tau_{i-1}^{-1}(x) = y_{i-1}$, $1 \le i \le 3$. That makes $y_3y_2y_1$ a trigon, by Lemma 6.2, and we see that $\mathrm{In}(y_3y_2y_1)$ consists of all cycles that move no y_i, $1 \le i \le 3$. To finish one can use Proposition 6.7 or a direct argument. $\square$

Let G be the 2-side graph of a trading surface (τ_1, τ_2, τ_3). By a *j-chain*, $j \in \{1, 2, 3\}$, we shall understand any chain $\mathbf{x_1}, \dots, \mathbf{x_k}$ in G, $k \ge 2$, such that an edge between $\mathbf{x_i}$ and $\mathbf{x_{i+1}}$ is never induced by a cycle of τ_j, $1 \le i < k$, and such that beyond the edges of the chain there exist no other edges of G that would start at $\mathbf{x}_i$, $1 \le i \le k$. Note that in such a chain the connecting edges are alternately induced by cycles of τ_{j-1} and τ_{j+1}. The integer k (which is ≥ 2) is said to be the *length* of the chain.

By a 3-*fan* we shall understand a subgraph of G formed by elements $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$ and $\mathbf{y}$ such that $\mathbf{x_j}$ are of degree 1 and $\{\mathbf{x_j}, \mathbf{y}\}$ is an edge, for each $j \in \{1, 2, 3\}$. We shall always assume that the edge $\{\mathbf{x_j}, \mathbf{y}\}$ is induced by τ_j. The element $\mathbf{y}$ will be called the *centre* of the 3-fan.

In Proposition 10.5 we shall prove that every component of G is a j-chain or a 3-fan or a cycle. However, cycles of length > 3 occur only in the bicyclic bi-trades and they are of even length. Of course, this is true only when the number of points is > 4.

Lemma 10.3. *Suppose that (τ_1, τ_2, τ_3) is a trading surface with more than four points. Let $\mathbf{y} = y_3y_2y_1$ be a vertex of the 2-side graph that is of degree ≥ 3. Then $\mathbf{y}$ is the centre of a 3-fan, and there exist points x_j, $1 \le j \le 3$, such that*

(1) *$y_{j-1}x_jy_{j+1}$ are the other elements of the 3-fan,*
(2) *$\mathbf{x} = x_3x_2x_1$ is a separating trigon,*
(3) *there exists a trading surface $(\sigma_1, \sigma_2, \sigma_3)$ such that $(\tau_1, \tau_2, \tau_3) = (\sigma_1^{\mathbf{x}}, \sigma_2^{\mathbf{x}}, \sigma_3^{\mathbf{x}})$, and*
(4) *all elements of the 3-fan are seeds.*

Proof: We assume, in fact, that $(y_{j-1}\ y_{j+1})$ is a cycle of τ_j for every $j \in \{1, 2, 3\}$. Hence we can use Proposition 4.1. The triangles $y_{j-1}x_jy_{j+1}$ have only one side of length two, and thus they are of degree one in the 2-side graph. By Corollary 6.3 they must be seeds. $\square$

Lemma 10.4. *If $(\tau_1, \tau_2, \tau_3) = (\sigma_1^x, \sigma_2^x, \sigma_3^x)$ is a point expansion, then there exist points y_i, $1 \le i \le 3$, such that $\mathbf{y_i} = y_{i+1}y_{i-1}x$ are triangles that form a cycle of length 3 in the 2-side graph of (τ_1, τ_2, τ_3). Each cycle of length 3 in this graph can be obtained by such a way and none of the triangle $\mathbf{y_i}$ is a seed.*

Proof: The existence of triangles $\mathbf{y_i}$ is clear from the definition of the point expansion. They form a cycle of length three since they share the sides $(x\ y_j)$. None of the cycle elements can have degree ≥ 3 by Lemma 10.3. None of them can be a seed since $y_3y_2y_1$ is a trigon. It remains to show that each cycle of length three can be obtained in this way.

An element of such a cycle is a triangle with two sides of length two. Hence there exist cycles $(y_{j\pm1}\ x)$ of $\tau_{j\pm1}$ that form these sides. There exists a further triangle of the cycle that shares the side $(y_{j+1}\ x)$, and a different triangle that shares the side $(y_{j-1}\ x)$. These two triangles also share a side of length two, and that must belong to τ_j. Therefore it equals $(y_j\ x)$ for some y_j, and the rest follows from Lemma 10.2. □

Proposition 10.5. *Let (τ_1, τ_2, τ_3) be a trading surface that is not bicyclic, and let G be its 2-side graph. Each component of G is either a j-chain for some $j \in \{1, 2, 3\}$, or a 3-fan, or a cycle of length* 3.

Proof: Lemma 10.3 deals with elements of G that are of degree three. It thus suffices to consider the components of G that are chains or cycles of length $k \geq 3$.

Choose three consecutively adjacent triangles (the triangles are vertices of G) and let $j \in \{1, 2, 3\}$ be such that the sides connecting the middle triangle with the other two belong to τ_{j-1} and τ_{j+1}. These two sides have exactly one common point, say x_1, and we define x'_0 and x'_1 in such a way that $(x'_0\ x_1)$ is a cycle of τ_{j-1} and $(x'_1\ x_1)$ is a cycle of τ_{j+1}. The middle triangle hence equals $x_1x'_1x'_0$, and by setting $x_0 = \tau_j^{-1}(x_1)$ and $x_2 = \tau_j(x_1)$ we see that the triangles adjacent to the middle one are $x_0x_1x'_0$ and $x_1x_2x'_1$. If $x_0 = x_2$, then $(x_0\ x_1)$ is a cycle of τ_j, which makes the latter two triangles adjacent elements of G. We get a cycle of length three, and Lemma 10.4 can be used. Hence we can assume that $x_2 \neq x_0$ and that $k \geq 4$.

In the case of a chain we can also assume that one of $x_0x_1x'_0$ and $x_1x_2x'_1$ is a terminal vertex of the chain. Since $j-1$ gets exchanged with $j+1$ by the transition to inverse permutations, we see that we can choose the former case.

Let the cycle of τ_j that contains x_0, x_1 and x_2 be equal to $(x_0\ x_1\ \dots\ x_n)$, $n \geq 2$. Let $m \leq n$ be the greatest integer such that there exist pairwise distinct elements $x'_1, \dots, x'_m$, for which

(A) $(x_i\ x'_i)$ is a cycle of τ_{j+1}, $1 \leq i \leq m$, and
(B) $(x_i\ x'_{i-1})$ is a cycle of τ_{j-1}, $1 \leq i \leq m$.

We know that $m \geq 1$. From (A) and (B) one gets $\tau_j(x'_i) = \tau_{j-1}^{-1}\tau_{j+1}^{-1}(x'_i) = \tau_{j-1}^{-1}(x_i) = x'_{i-1}$, $1 \leq i \leq m$, and we obtain a chain of adjacent triangles

$$x_0x_1x'_0,\ x_1x'_1x'_0,\ \dots,\ x_{m-1}x_m, x'_{m-1},\ x_mx'_mx'_{m-1}.$$

Since $(x_m\ x'_m)$ is a cycle of τ_{j+1}, we see that $x_mx'_mx'_{m-1}$ is adjacent to $x_mx_{m+1}x'_m$ if $m < n$, and to $x_nx_0x'_n$ if $m = n$.

Assume first $m < n$. Then $x_mx_{m+1}x'_m$ has a side of length two which is equal to $(x_m\ x'_m)$. If it has no other side of length two, then this triangle is the other terminal vertex of a j-chain. Assume the existence of another side of length two. Then this side has to be equal to $(x_{m+1}\ x'_m)$ and belong to τ_{j-1}. Points x'_i, $0 \leq i \leq m$, are images of x_i by τ_{j+1}, and hence none of them can be equal to $x'_{m+1} = \tau_{j+1}(x_{m+1})$. The permutation τ_{j+1} cannot contain a cycle $(x_{m+1}\ x'_{m+1})$ because m is assumed to be maximal, and thus $x_{m+1}x'_{m+1}x'_m$ constitutes the other terminal vertex of the j-chain.

Suppose now that $m = n$. Since $n \geq 2$ we see that $x_n x_0 x'_n$ is a vertex of G that is either the other terminal point of the j-chain, or it has a side $(x_0\ x'_n)$ that belongs to τ_{j-1}. Assume the latter. Then $\tau^2_{j+1}(x'_0) = \tau^2_{j+1}\tau_{j-1}\tau_j(x_0) = \tau_{j+1}(x_0) = \tau_{j+1}\tau_{j-1}(x'_n) = \tau_j^{-1}(x'_n)$. Hence $\tau_j(x'_0) = x'_n$, as the cycle of τ_j that contains both x'_0 and x'_n intersects in exactly one point the cycle of τ_{j+1} that contains x'_0. Thus $(x_0\ x'_0)$ is a cycle of τ_{j+1} We see that conditions (A) and (B) hold modulo $n+1$ for all i, $0 \leq i \leq n$, and that $(x'_n\ x'_{n-1}\ \dots\ x'_1\ x'_0)$ is a cycle of τ'_j. All permutations τ_j thus act upon the set $\{x_i, x'_i;\ 0 \leq i \leq n\}$, and so (τ_1, τ_2, τ_3) has to be bicyclic. □

The explicit description of j-chains that was obtained in the preceding proof is worth recording, and that is what we shall do in the next lemma.

Lemma 10.6. *Let (τ_1, τ_2, τ_3) determine a trading surface. Choose $j \in \{1, 2, 3\}$ and let $m \geq 1$ be an integer. For every j-chain of length $2m$ or $2m+1$ there exist cycles*

$$(\dots\ x_0\ x_1\ \dots\ x_m\ \dots) \text{ and } (\dots\ x'_m\ \dots\ x'_1\ x'_0\ \dots)$$

of τ_j of lengths ℓ and ℓ', respectively, such that for the j-chain of length $2m$

(1) *either $(x_i\ x'_{i-1})$ is a cycle of τ_{j-1}, $1 \leq i \leq m$, and $(x_i\ x'_i)$ is a cycle of τ_{j+1}, $1 \leq i \leq m-1$, with $\tau^2_{j+1}(x'_0) \neq x'_0$ and $\tau^2_{j-1}(x_m) \neq x_m$;*
(2) *or $(x'_i\ x_{i-1})$ is a cycle of τ_{j+1}, $1 \leq i \leq m$, and $(x_i\ x'_i)$ is a cycle of τ_{j-1}, $1 \leq i \leq m-1$, with $\tau^2_{j-1}(x_0) \neq x_0$ and $\tau^2_{j+1}(x'_m) \neq x'_m$.*

For the j-chain of length $2m+1$ the cycles can be chosen in such a way that

(3) *$(x_i\ x'_{i-1})$ is a cycle of τ_{j-1} and $(x_i\ x'_i)$ is a cycle of τ_{j+1}, $1 \leq i \leq m$, with $\tau^2_{j+1}(x'_0) \neq x'_0$ and $\tau^2_{j-1}(x'_m) \neq x'_m$. If $\ell = m+1$, then $\ell' > m+1$ and $x'_m x_0 x'_0$ is a trigon. If $\ell' = m+1$, then $\ell > m+1$ and $x_0 x_{m+1} y$ is a trigon, where $x_{m+1} = \tau_j(x_m)$ and $y = \tau^{-1}_{j+1}(x'_0) = \tau_{j-1}(x'_m)$.*

Proof: The case (1) of length $2m$ corresponds to the situation described in the proof of Proposition 10.5 when m is replaced by $m-1$ in conditions (A) and (B), and when there is assumed that τ_{j-1} contains the cycle $(x_m\ x'_{m-1})$. The case (2) is obtained by the transfer to inverse permutations.

For the case of length $2m+1$ first note that the side of a terminal vertex that is longer than two and does not belong to τ_j belongs to τ_{j-1} for one vertex, and to τ_{j+1} for the other terminal vertex of the j-chain. By choosing the latter alternative we can represent the vertex as $x_0 x_1 x'_1$, where x_0 and x'_1 are incident to the side which belongs to τ_{j+1} and is of length > 2. We can thus assume the situation that is described in the proof of Proposition 10.5 such that $\tau^2_{j-1}(x'_m) \neq x'_m$ and $\tau^2_{j+1}(x_0) \neq x_0$. If $\ell = m+1$, then $\tau_{j-1}(x_0) = \tau^{-1}_{j+1}\tau^{-1}_j(x_0) = \tau^{-1}_j(x_m) = x'_m$. We also have $\tau_{j+1}(x'_0) = x_0$. We see that $x_0 x'_0 x'_m$ constitutes a trigon (the bi-trade is thus a trigonal expansion of a bicyclic bi-trade).

It remains to consider the case $\ell > m+1$ and $\ell' = m+1$. We have $x_{m+1} \neq x_0$, by $\ell > m+1$. The mapping τ_{j-1}^{-1} sends y to x'_m and x'_m to x_{m+1}, while the mapping τ_{j+1}^{-1} sends x_0 to x'_0 and x'_0 to y. The element y belongs to the cycle $(\dots\ x_0\ \dots\ x_{m+1}\ \dots)$ only if $y = x_{m+1} = x_0$. However, that would imply $\ell = m+1$. Hence $x_0 x_{m+1} y$ has to be a trigon. $\square$

By comparing Lemma 10.6 with Proposition 9.7 we see that a j-chain of length $k \geq 4$ describes a situation which can be obtained by a 2-trapezium expansion. In the next lemma we observe that in the case of no trigons this gives us means to describe how the 2-side graph is affected by the expansion if $k \geq 5$.

Lemma 10.7. *Let G be the 2-side graph of a trading surface (τ_1, τ_2, τ_3). Suppose that G contains a j-chain of length $k \geq 5$, $j \in \{1, 2, 3\}$, and that (τ_1, τ_2, τ_3) contains no trigon. Then (τ_1, τ_2, τ_3) can be obtained as a 2-trapezium expansion of a trading surface with no trigon that possesses a 2-side graph isomorphic to the graph obtained from G by replacing a chain of length k by a chain of length $k-2$.*

PROOF: By the remark before the lemma, (τ_1, τ_2, τ_3) is isotopic to a 2-trapezium expansion of a trading surface. This surface is without trigons by Lemma 9.3. We have to show that none of points (1) and (2) of Lemma 9.5 can affect the expansion. Use Lemma 10.6 to describe the considered j-chain. The cycles of lengths ℓ and ℓ' are obtained by expansion of cycles with lengths $\ell - 1$ and $\ell' - 1$. Points (1) or (2) of Lemma 9.5 can apply only when one of the latter values is equal to 2. If this is true, then $3 \in \{\ell, \ell'\}$, $m \leq 2$ and $k \leq 5$. Thus $k = 5$, $m = 2$ and $m+1 \in \{\ell, \ell'\}$. This makes relevant situation (3) of Lemma 10.6, by which (τ_1, τ_2, τ_3) contains a trigon, contrary to our assumptions. $\square$

Theorem 10.8. *Let (τ_1, τ_2, τ_3) determine a trading sphere which is not bicyclic and which contains no trigon. Then it contains at least eight seeds, and its 2-side graph has at least two components. For each of the components there exists $j \in \{1, 2, 3\}$ such that the component is a j-chain.*

PROOF: The 2-side graph G contains no cycle and consists of chains only, by Proposition 10.5. There are at least six edges in G, by Proposition 3.1. This must remain true if a chain of length $k \geq 5$ is replaced by a chain of length $k-2$, by Lemma 10.7. Hence each chain of length $k \geq 5$ can be regarded as if it were contributing at most three edges (which is the contribution of chains of length four). Let r_4 be the number of chains of length $r \geq 4$, and let r_2 and r_3 give the number of chains for lengths 2 and 3. Then $6 \leq \sum r_i(i-1) \leq 3 \sum r_i$ implies $r = \sum r_i \geq 2$, and the number of seeds is at least $\sum i r_i = r + \sum r_i(i-1) \geq 2 + 6 = 8$. $\square$

The structure of trading spheres with exactly two components of the 2-side graph does not seem to be difficult to describe. By doing so one can get the description of all trading spheres with no trigon that have at most eight seeds. However, this would mean introducing additional technical details to this paper.

11. Seeds and trigons

The trigonal construction can combine trading surfaces in a way that destroys some of the seeds in both input trading surfaces. We shall observe that there are cases when the destruction of seeds in a trading sphere takes an extent that permits no application of a 2-sliding reduction or a trapezium reduction. However, we shall prove that if no 2-sliding reduction applies, then there exists a trigon with a bicyclic outer trading sphere.

Let $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ and (τ_1, τ_2, τ_3) determine trading surfaces on Y and X, respectively, and let $\mathbf{x} = x_3x_2x_1$ be a triangle in (τ_1, τ_2, τ_3) and y a point of Y. Denote by $(\tau_1', \tau_2', \tau_3')$ the trigonal expansion, $\tau_i' = \tau_i^{\mathbf{x},\Sigma,y}$.

Lemma 11.1. *The set of triangles of $(\tau_1', \tau_2', \tau_3')$ is obtained from triangles of Σ and (τ_1, τ_2, τ_3) by removing the triangle $\mathbf{x}$ and replacing the triangles $\sigma_{j-1}^{-1}(y)\, y\, \sigma_{j+1}(y)$ by $\sigma_{j-1}^{-1}(y)\, x_j\, \sigma_{j+1}(y)$, for all $j \in \{1, 2, 3\}$. The 2-side graph of $(\tau_1', \tau_2', \tau_3')$ is derived from the disjoint union of the two 2-side graphs by removing $\mathbf{x}$ (if it is a vertex) and by removing each edge induced by a cycle passing through the point y.*

PROOF: The description of triangles follows directly from the definition of $(\tau_1', \tau_2', \tau_3')$. Since there appears no new cycle of length two, the edges in the 2-side graph of $(\tau_1', \tau_2', \tau_3')$ must be inherited from the input trading surface. An edge has to be deleted if the corresponding triangle side becomes a side of a trigon. If such a side is in (τ_1, τ_2, τ_3), then $\mathbf{x}$ is one of its vertices, and hence its deletion is induced by the removal of $\mathbf{x}$. If such a side is in Σ, then it has to move y. □

Lemma 11.2. *A seed $\mathbf{u}$ of (τ_1, τ_2, τ_3) becomes a seed of $(\tau_1', \tau_2', \tau_3')$ if and only if no side of $\mathbf{u}$ is equal to $(x_{i-1}\ x_{i+1})$ for some i, $1 \le i \le 3$.*

PROOF: If ρ is a cycle of τ_i such that ρ does not move x_{i+1}, then set $\rho' = \rho$. If ρ moves x_{i+1}, then ρ equals $(\dots\ x_{i-1}\ x_{i+1}\ \dots)$ and we denote by ρ' the cycle $(\dots\ x_{i-1}\ \sigma_i(y)\ \dots\ \sigma_{i-1}(y)\ x_{i+1}\ \dots)$. Thus each cycle ρ of τ_i is replaced by one cycle ρ' of τ_i'. Let ρ_i be a cycle of τ_i and ρ_j a cycle of τ_j, where $i, j \in \{1, 2, 3\}$ and $i \ne j$. We see immediately that ρ_i meets ρ_j if and only if ρ_i' meets ρ_j'. Hence if ρ_k, $1 \le k \le 3$, are cycle of τ_k such that ρ_k' forms sides of a trigon, then ρ_k' form sides of a trigon or a triangle. By Lemma 11.1 we get a triangle if only if cycles ρ_k are sides of a triangle in (τ_1, τ_2, τ_3) that is different from $\mathbf{x}$, and so if cycles ρ_k' yield a trigon different from $\mathbf{x}$, then cycles ρ_k yield a trigon as well.

If $\mathbf{u} = \mathbf{x}$, then $\mathbf{u}$ does not become a seed in $(\tau_1', \tau_2', \tau_3')$, but one of its sides in (τ_1, τ_2, τ_3) has to equal $(x_{i-1}\ x_{i+1})$, $i \in \{1, 2, 3\}$. Assume $\mathbf{u} \ne \mathbf{x}$ and suppose that $\mathbf{u}$ does not give a seed in $(\tau_1', \tau_2', \tau_3')$. Then one if its sides is a side of a trigon that does not arise from a trigon of (τ_1, τ_2, τ_3). There is only one possibility, and that is $\mathbf{x}$. Hence $\mathbf{x}$ has to be adjacent to $\mathbf{u}$ in (τ_1, τ_2, τ_3). □

Lemma 11.3. *Suppose that (τ_1, τ_2, τ_3) is a trading sphere that contains no trigon. If (τ_1, τ_2, τ_3) is not bicyclic, then $(\tau_1', \tau_2', \tau_3')$ can be obtained by a 2-sliding expansion.*

PROOF: By Theorem 10.8 the 2-side graph of $(\tau_1', \tau_2', \tau_3')$ consists of at least two components and none of the components contains a vertex of degree three or a cycle. Hence there is only one component that involves $\mathbf{x}$ as a vertex. If a seed of (τ_1, τ_2, τ_3) is not a seed of $(\tau_1', \tau_2', \tau_3')$, then it has to be a vertex of that component, by Lemma 11.2. Each terminal point of every other component remains to be a seed that is of degree one in the 2-side graph of $(\tau_1', \tau_2', \tau_3')$, and we can use point (i) of Lemma 7.2. □

Lemma 11.4. *Let (τ_1, τ_2, τ_3) be a bicyclic trading sphere of size $2n$, $n \geq 3$. Then $(\tau_1', \tau_2', \tau_3')$ can be obtained by a 2-trapezium expansion.*

PROOF: There exists exactly one $j \in \{1, 2, 3\}$ such that $(x_j\ x_{j\pm1})$ is a cycle of $\tau_{j\pm1}$. Put $z_i = \tau_j^i(x_{j-1})$, $0 \leq i \leq n-1$, and note that $z_1 = x_{j+1}$. Define points z_{i-1}' so that $(z_i\ z_{i-1}')$ is a cycle of τ_{j-1} (the indices are computed modulo n). If $i = 1, \dots, n-1 \geq 2$, then $(z_i\ z_{i-1}')$ is also a cycle of τ_{j-1}'. In τ_j' there are cycles $(\dots\ z_1\ z_2\ \dots\ z_{n-1}\ z_0\ \dots)$ and $(\tau_{n-1}'\ \dots\ z_2'\ z_1'\ z_0')$, while both τ_{j+1} and τ_{j+1}' contain $(z_1\ z_1')$. Condition (1) of Proposition 9.7 is thus satisfied. □

Lemma 11.5. *For a triangle $\mathbf{u} = u_3u_2u_1$ of Σ put $\mathbf{u}' = \mathbf{u}$ if $y \notin \{u_1, u_2, u_3\}$. If $\mathbf{u} = \sigma_{j+1}^{-1}(y)y\sigma_{j-1}(y)$ for some $u \in \{1, 2, 3\}$, set $\mathbf{u}' = \sigma_{j-1}^{-1}(y)x_j\sigma_{j+1}(y)$. If $\mathbf{u}$ is a seed in Σ, then $\mathbf{u}'$ is a seed in (τ_1, τ_2, τ_3) if and only if $\mathbf{u}$ possesses a side of length two that does not move y.*

PROOF: Every cycle ρ_i of σ_i, $i \in \{1, 2, 3\}$, yields a unique cycle ρ_i' of τ_i'. If ρ_i does not move y, then the lengths of ρ_i and ρ_i' are the same. If $\rho_{i\pm1}$ are cycles of $\sigma_{i\pm1}$, then ρ_{i+1} meets ρ_{i-1} if and only if ρ_{i+1}' meets ρ_{i-1}'. Suppose that every side of length two of the seed $\mathbf{u}$ moves y. Then $\mathbf{u}'$ has no side of length two, and hence it cannot be a seed. Let now $\mathbf{u}$ contain a side $(u_{j-1}\ u_{j+1})$ where $y \neq u_{j\pm1}$. Let ρ_i be the cycle of σ_i that moves u_i. If $\mathbf{u}'$ is not a seed, then $\rho_{i\pm1}'$ and the ith side of $\mathbf{u}'$ form a trigon, for some $i \in \{1, 2, 3\}$. However that cannot be since the preimages of the trigon sides in (τ_1, τ_2, τ_3) meet each other, thus form a triangle, and triangles are transformed only in triangles, by Lemma 11.1. □

Lemma 11.6. *If Σ is trading sphere that contains no trigon, then $(\tau_1', \tau_2', \tau_3')$ can be produced by a 2-sliding expansion.*

PROOF: Suppose first that Σ is a bicyclic trading sphere of size $2n$. Then every triangle of Σ is a seed. From Lemma 11.5 we derive that $\sigma_{j-1}^{-1}(y)x_j\sigma_{j+1}(y)$ is a seed when $(\sigma_{j-1}^{-1}(y)\ \sigma_{j+1}(y))$ is a cycle of Σ. If $n = 2$ this is true for all three j and if $n \geq 3$, then for exactly two of them.

If Σ is not bicyclic, then every component of Σ is a j-chain by Theorem 10.8. There are at least two such components, and only one of them can be affected by the process described in Lemma 11.1. A terminal point of a j-chain that has not been affected is a seed, by Lemma 11.5, and hence we can use point (i) of Lemma 7.2 again. □

Lemma 11.7. *Let $\mathbf{y} = y_3y_2y_1$ be a trigon of Σ. Put $y_i' = y_i$ if $y_i \neq y$, and $y_i' = x_i$ if $y_i = y$. Then $\mathbf{y}' = y_3'y_2'y_1'$ is a trigon in $(\tau_1', \tau_2', \tau_3')$, and this describes* *all trigons in* $(\tau_1', \tau_2', \tau_3')$ *different from* $\mathbf{x} = x_3x_2x_1$ *which have the property that each vertex of the trigon is an inner point or a vertex of* $\mathbf{x}$.

Proof: Let $u, v \in Y$, $u \neq v$, be incident to a cycle ρ of σ_i, and let ℓ be the length of ρ. Assume $\sigma_i^k(u) = v$, where $1 < k < \ell$. Set $u' = u$ if $u \neq y$, and $u' = x_{i-1}$ if $u = y$. Similarly set $v' = v$ if $v \neq y$, and $v' = x_{i+1}$ if $v = y$. Then $(\tau_i')^k(u') = v'$, and k is the least positive integer with this property, by the definition of τ_i'. It follows that $\mathbf{y}'$ is a trigon.

For the converse consider a trigon in $(\tau_1', \tau_2', \tau_3')$ with vertices that are inner points or vertices of $\mathbf{x}$. Such a trigon either equals $\mathbf{x}$, or has at least two vertices outside the set $\{x_1, x_2, x_3\}$. In the latter case we can express the trigon as $\mathbf{y}'$, and it is clear that $\mathbf{y}$ has to be a trigon in Σ. □

Let (μ_1, μ_2, μ_3) be a trading sphere that contains trigons. For a trigon $\mathbf{u}$ denote by $s(\mathbf{u})$ the minimum of the sizes yielded by the outer and inner trading spheres. Say that a trigon $\mathbf{u}$ is *extremal* if $s(\mathbf{u})$ achieves the minimum possible value.

Lemma 11.8. *Suppose that* $\mathbf{x}$ *is the extremal trigon of the trading sphere* $(\tau_1', \tau_2', \tau_3')$. *Then* $(\sigma_1, \sigma_2, \sigma_3)$ *or* (τ_1, τ_2, τ_3) *is a trading sphere without a trigon.*

Proof: Let Σ contain a trigon $\mathbf{y} = y_2y_2y_1$. Define $\mathbf{y}'$ as in Lemma 11.8. If $\operatorname{Pnt}\mathbf{y}' \subseteq \operatorname{Pnt}\mathbf{x}$, then the size of the inner trading sphere of $\mathbf{y}'$ is less than the size of $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$. If $\operatorname{Pnt}\mathbf{y}'$ is not contained in $\operatorname{Pnt}\mathbf{x}$, then there must be $\{x_1, x_2, x_3\} \subseteq \operatorname{Pnt}\mathbf{y}'$. In such a case the outer trading sphere of $\mathbf{y}'$ is smaller than the size of Σ. We conclude that $s(\mathbf{x})$ is not equal to the size of Σ if Σ contains a trigon.

Suppose that $\mathbf{y}$ is a trigon in the outer trading sphere of $\mathbf{x}$, i.e. in (τ_1, τ_2, τ_3). If $\mathbf{x}$ is an inner triangle of $\mathbf{y}$, then the outer trade of $\mathbf{y}$ is less than the size of (τ_1, τ_2, τ_3). If $\mathbf{x}$ is an outer triangle of $\mathbf{y}$, then the inner trade of $\mathbf{y}$ is less than the size of (τ_1, τ_2, τ_3). We conclude that $s(\mathbf{x})$ is equal to the size (τ_1, τ_2, τ_3) if (τ_1, τ_2, τ_3) contains a trigon.

Nothing more is needed since both conclusions cannot hold at the same time. □

Lemma 11.9. *Suppose that a trading sphere cannot be obtained by a* 2*-sliding move. Then it is bicyclic or it contains a trigon with a bicyclic outer trading sphere.*

Proof: If the trading sphere contains no trigon, use Theorem 10.8. Let it contain a trigon and consider an extremal one. By Lemma 11.6 we can deal with the situation when the inside trading sphere of a trigon is without trigons. By Lemma 11.8 we can thus assume that there exists a trigon such that the outside trading sphere is without trigons. If the outside trading sphere is not bicyclic, then the 2-sliding move can be used to produce it, by Lemma 11.3. □

Note that a 2-trapezium expansion yields all bicyclic trades when we start from the trading sphere of the size four. Note also the outer trading sphere is of size four exactly when there can be used point expansion (cf. Lemma 10.2). By combining Lemmas 11.4 and 11.9 we thus obtain

Theorem 11.10. *Every trading sphere can be obtained from the trading sphere of size four by applying* 2*-sliding expansions,* 2*-trapezium expansions and point expansions.*

None of the constructions can be omitted. To see that the 2-sliding expansion cannot be omitted is easy. For the other two cases take any trading sphere $(\sigma_1, \sigma_2, \sigma_3)$ and concurrently replace each point by a trigon in such a way that all outer trading spheres are bicyclic. A special case is the one in which each point of $(\sigma_1, \sigma_2, \sigma_3)$ is subjected to a point expansion. That makes clear its inevitability. In point expansion the outer trading sphere is bicyclic of size 4. If all of the concurrent trigonal expansions use a bicyclic trade of size at least 6, then we obtain a bi-trade that documents the inevitability of the 2-trapezium expansion.

12. Duality

Let (τ_1, τ_2, τ_3) be a trading surface on X. Denote by Y the set of all its triangles $\mathbf{x} = x_3x_2x_1$. Consider a bipartite graph G upon $X \cup Y$ in which the set of edges consists of all pairs $\{x_i, \mathbf{x}\}$, $1 \le i \le 3$. The graph G is cubic and is clearly 3-connected. If the genus is equal to zero, then we get a *planar cubic* 3*-connected bipartite graph*. We have already mentioned in the introduction that all such graphs are face 3-colourable — the faces correspond to the cycles of τ_i, $1 \le i \le 3$, and hence i can be used as the colour.

The rôles of X (the white points) and Y (the black points) are clearly interchangeable. This is not surprising since the points of X represent the triples $a = (a_1, a_2, a_3) \in T^\circ$ of a bi-trade $T = (T^\circ, T^*)$, while the points of Y represent the triples $b = (b_1, b_2, b_3) \in T^*$. An edge occurs if and only if a and b agree in (exactly) two coordinates.

Since the cycles of τ_i represent the faces of G, it is easy to understand how to interpret a trigon $\mathbf{y} = y_3y_2y_1$ in terms of G. The connection is seen better if we identify each y_i with an element of T^*. Let c_i be the element that represents the ith side of $\mathbf{y}$. Then c_i occurs as the ith coordinate of $y_{i\pm1} \in T^*$. To get $\tau_{i-1}(y_i)$ we first change the ith coordinate of y_i and then the $(i+1)$th coordinate. To get $\tau_{i+1}^{-1}(y_i)$ we first change the ith coordinate of y_i and then the $(i-1)$th coordinate. Denote by z_i the triple in T° that is obtained from y_i by the change of the ith coordinate. We see that the $(i\pm1)$th coordinate of both y_i and z_i is equal to $c_{i\pm1}$. The trigon $\mathbf{y}$ can be thus identified with a subgraph of G formed by faces of c_i, $1 \le i \le 3$, in which the faces of c_{i-1} and c_{i+1} always share the edge $\{y_i, z_i\}$. By removing these three edges from the subgraph we get two cycles of the graph G. The cycle that involves vertices z_i will be called the *inner circumference* of G, while the cycle with vertices y_i will be called the *outer circumference*. If $\mathbf{y}$ is separating (which is always the case when G is planar), then the points upon the inner circumference and the points inside the inner circumference determine the inner trading surface (the points of X upon and inside the inner circumference coincide with the set Pnt X). Similarly, the outside trading surface is determined by the outer circumference. The fact that the inner (or outer) trading surface is a bicyclic trading sphere means that there exists $i \in \{1, 2, 3\}$ such that within the inner (outer) circumference one can find only one face of this colour.

By switching between X and Y (between black and white points), we clearly also switch the meaning of the outside and inside of trigons. Lemma 11.9 thus implies that if (T°, T^*) is not bicyclic and cannot be obtained by a 2-sliding move, then (T^*, T°) can be obtained by such a move. This gives us the final theorem of this paper. By applying an expansion to a bi-trade we mean that we consider the induced triple (τ_1, τ_2, τ_3) of structural permutations, expand it to $(\tau'_1, \tau'_2, \tau'_3)$, and convert that back to a bi-trade. To retain the one-to-one correspondence one has to assume that the bi-trades are separated. Call a latin bi-trade *spherical* if it is separated and if its structural permutations form a trading sphere.

Theorem 12.1. *For every spherical latin bi-trade $T = (T^\circ, T^*)$ there exists a sequence $T_i = (T_i^\circ, T_i^*)$ of spherical latin bi-trades, $0 \le i \le k$, such that T_0 is bicyclic, T_k is equal to T and for every i, $0 \le i < k$, one can obtain $(T_{i+1}^\circ, T_{i+1}^*)$ or $(T_{i+1}^*, T_{i+1}^\circ)$ as a 2-sliding expansion of T_i.*

References

[1] Batagelj V., *An improved inductive definition of two restricted classes of triangulations of the plane*, Combinatorics and graph theory (Warsaw 1987), 11–18, Banach Center Publ., 25, PWN, Warsaw, 1989.

[2] Cavenagh N., Donovan D., Drápal A., 3-*homogeneous latin trades*, Discrete Math. **300** (2005), 57–70.

[3] Cavenagh N.J., Donovan D., Drápal A., 4-*homogeneous latin trades*, Australas. J. Combin. **32** (2005), 285–303.

[4] Cavenagh N.J., Hämäläinen C., Drápal A., *Latin bitrades derived from groups*, Discrete Math. **308** (2008), 6189–6202.

[5] Cavenagh N.J., Lisoněk P., *Planar Eulerian triangulations are equivalent to spherical Latin bitrades*, J. Combin. Theory Ser. A **115** (2008), 193–197.

[6] Cavenagh N.J., Wanless I.M., *Latin trades in groups defined on planar triangulations*, J. Algebr. Comb. (in print), DOI 10.1007/s10801-008-0165-9.

[7] Drápal A., Kepka T., *Exchangeable partial groupoids I*, Acta Univ. Carolin. Math. Phys. **24** (1983), 57–72.

[8] Drápal A., Kepka T., *Group modifications of some partial groupoids*, Ann. Discrete Math. **18** (1983), 319–332.

[9] Drápal A., *On a planar construction of quasigroups*, Czechoslovak Math. J. **41** (1991), 538–548.

[10] Drápal A., *Latin Squares and Partial Groupoids* (in Czech), Candidate of Science Thesis, Charles University, Prague, 1988.

[11] Drápal A., *Hamming distances of groups and quasi-groups*, Discrete Math. **235** (2001), 189–197.

[12] Drápal A., *Geometry of Latin Trades*, manuscript circulated at the conference Loops'03, Prague, 2003.

[13] Drápal A., *Geometrical structure and construction of latin trades*, Adv. Geom. **9** (2009), 311–348.

[14] Drápal A., Hämäläinen C., Kala V., *Latin bitrades, dissections of equilateral triangles and abelian groups*, J. Comb. Des. (in print), DOI 10.1002/jcd.20237.

[15] Drápal A., Lisoněk P., *Generating spherical Eulerian triangulations*, Discrete Math., to appear.

[16] Grannell M.J., Griggs T.S., Knor M., *Biembeddings of symmetric configurations and* 3-*homogeneous Latin trades*, Comment. Math. Univ. Carolin. **49** (2008), 411–420.

[17] Hämäläinen C., *Partitioning* 3-*homogeneous latin bitrades*, Geom. Dedicata **133** (2008), 181–193.

[18] Heawood P.J., *On the four colour map theorem*, Quart. J. **29** (1898), 270–285.

[19] Holton D.A., Manvel B., McKay B.D., *Hamiltonian cycles in cubic* 3-*connected bipartite planar graphs*, J. Combin. Theory Ser. B **38** (1985), 279–297.

[20] Keedwell A.D., *Critical sets in latin squares and related matters: an update*, Util. Math. **65** (2004), 97–131.

[21] Lefevre J., Cavenagh N.J., Donovan D., Drápal A., *Minimal and minimum size latin bitrades of each genus*, Comment. Math. Univ. Carolin. **48** (2007), 189–203.

[22] Lefevre J.G., Donovan D., Drápal A., *Permutation representation of* 3 *and* 4-*homogenous latin bitrades*, Fund. Inform. **84** (2008), 99–110.

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(*Received* December 8, 2008, *revised* August 5, 2009)