Fixed point property on symmetric products of chainable continua

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Abstract. We prove that the third symmetric product of a chainable continuum has the fixed point property.

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1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continuum X and a positive integer n, the nth-*symmetric product of* X is defined as

$$F_n(X) = \{A \subset X : A \text{ is nonempty and } A \text{ has at most } n \text{ points}\}.$$

The hyperspace $F_n(X)$ is considered with the Hausdorff metric H.

Given $\varepsilon > 0$, an ε-*chain* in the continuum X is a finite family of open subsets $U_1, \dots, U_n$ of X such that diameter$(U_i) < \varepsilon$, for each $i \in \{1, \dots, n\}$, and $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A continuum X is said to be *chainable* provided that, for each $\varepsilon > 0$, there exists an ε-chain which covers X.

A *map* is a continuous function. A continuum X has the *fixed point property*, provided that, for each map $f : X \to X$ there exists $p \in X$ such that $f(p) = p$. A map between continua $f : X \to Y$ is said to be *universal*, provided that for each map $g : X \to Y$, there exists a point $p \in X$ such that $g(p) = f(p)$. The *induced map* $f_n : F_n(X) \to F_n(Y)$ is the map defined as $f_n(A) = f(A)$ (the image of A under f).

Symmetric products were introduced by K. Borsuk and S. Ulam in [2], where they asked if every symmetric product of a continuum with the fixed point property must have the fixed point property. J. Oledzki ([8]) constructed a 2-dimensional continuum to answer this question in the negative. On the other hand, the author and G. Higuera have recently constructed a continuum X such that X does not have, but $F_2(X)$ has the fixed point property.

In [6, Exercise 22.25], it is asked to show that the second symmetric product of a chainable continuum has the fixed point property and in [7, p. 77] it is asked if, for each $n \geq 3$, the n-th symmetric product of a chainable continuum has the fixed point property. Some other related questions on this topic can be found in [5] and [7]. A detailed study on the hyperspaces $F_n([0,1])$ can be found in [1].

Let $\mathbb{N}$ be the set of positive integers. Given $n \in \mathbb{N}$, consider the following property $Q(n)$ that may be or may not be true:

$Q(n)$: For every map $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$, the induced map $f_n : F_n([0,1]) \to F_n([0,1])$ is universal.

In this paper we prove the following.

Theorem 3. *Let $n \in \mathbb{N}$. If $Q(n)$ holds, then the n-th symmetric product of every chainable continuum has the fixed point property.*

Theorem 4. *$Q(3)$ holds.*

Corollary 5. *The third symmetric product of each chainable continuum has the fixed point property.*

2. An auxiliary construction

Given $r, n \in \mathbb{N}$, we consider the uniform partition P_r of $[0,1]$ given by

$$P_r = \{\tfrac{k}{r} : k \in \{0, \dots, r\}\}.$$

Define $F_n(P_r) = \{A \in F_n([0,1]) : A \subset P_r\}$. That is, $F_n(P_r)$ is the family of nonempty subsets of P_r with at most n points. Given $A, B \in F_n(P_r)$, notice that the inequality $H(A,B) \leq \frac{1}{r}$ means that, for each element $\frac{k}{r} \in A$ either $\frac{k}{r}$, $\frac{k+1}{r}$ or $\frac{k-1}{r}$ belongs to B and for each element $\frac{j}{r} \in B$ either $\frac{j}{r}$, $\frac{j+1}{r}$ or $\frac{j-1}{r}$ belongs to A.

Let

$$\Delta = \{(A_1, \dots, A_s, t_1, \dots, t_s) : s \in \mathbb{N},\ A_1, \dots, A_s \in F_n(P_r),\ t_1, \dots, t_s \in [0,1],$$
$$t_1 + \dots + t_s = 1 \ \text{ and } \ H(A_i, A_j) \leq \frac{1}{r} \ \text{ for every } \ i, j \in \{1, \dots, s\}\}.$$

Given an element $(A_1, \dots, A_s, t_1, \dots, t_s) \in \Delta$, where $s \geq 2$, and $i \in \{1, \dots, s\}$, we define $A(i) = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_s)$ and $t(i) = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_s)$.

In this section we define a convex structure on the set Δ and we prove some of its properties.

Given a nonempty subset B of P_r, a *block* of B is a nonempty subset D of B such that, if $x, y \in D$ and $x \leq y$, then $[x,y] \cap P_r \subset D$ and D is maximal with this property. We can see the blocks in the following way: let G be the graph in which the points of B are the vertices and the edges are the pairs of adjacent (those at distance $\frac{1}{r}$) points of B. Then a block of B are those vertices that belong to a component of G.

Note that the blocks of B are pairwise disjoint and every point of B belongs to a block of B, so the blocks of B form a partition of B. Given $x \in B$, let $C(x,B)$ be the block of B containing x and let $m(x,B)$ (resp., $M(x,B)$) be the minimum (resp., maximum) of $C(x,B)$. Hence $C(x,B) = [m(x,B), M(x,B)] \cap P_r$ and $B = \bigcup\{C(x,B) : x \in B\}$.

Lemma 1. *Let $s \in \mathbb{N}$ and $A_1, \dots, A_s \in F_n(P_r)$ be such that $H(A_i, A_j) \leq \frac{1}{r}$ for every $i, j \in \{1, \dots, s\}$. Let $A = A_1 \cup \ldots \cup A_s$ and let D be a block of A. Then*

(a) *$D \cap A_i \neq \emptyset$ for each $i \in \{1, \dots, s\}$,*
(b) *diameter$(D) \leq \frac{3n}{r}$,*
(c) *$\{C(a,A) : a \in A_i\} = \{C(a,A) : a \in A_j\}$, for every $i, j \in \{1, \dots, s\}$.*

Proof: (a) Let $i \in \{1, \dots, s\}$. Let $p \in D$. Then there exists $j \in \{1, \dots, s\}$ such that $p \in A_j$. Since $H(A_i, A_j) \leq \frac{1}{r}$, there exists $q \in A_i$ such that $|p - q| \leq \frac{1}{r}$, we may assume that $p \leq q$. Then $q \in \{p, p + \frac{1}{r}\}$. Thus $[p,q] \cap P_r = \{p, q\} \subset A$. Since D is a block of $A, q \in D$. We have shown that $D \cap A_i \neq \emptyset$ and that, for each $p \in D$ there exists $q \in A_i$ such that $|p - q| \leq \frac{1}{r}$.

(b) Let $m = \min D$ and $M = \max D$. Then $D = [m, M] \cap P_r$ and diameter$(D) = M - m$. If $M - m > \frac{3n}{r}$, then we consider the intervals $[m - \frac{1}{r}, m + \frac{1}{r}]$, $[m + \frac{2}{r}, m + \frac{4}{r}]$, $[m + \frac{5}{r}, m + \frac{7}{r}]$,$\dots$,$[m - \frac{3n-1}{r}, m + \frac{3n+1}{r}]$. Since $m + \frac{3n}{r} < M$ and all the elements $m + \frac{3 \cdot 0}{r}, m + \frac{3 \cdot 1}{r}, \dots, m + \frac{3 \cdot n}{r}$ belong to D, by the fact we proved in the paragraph above, each one of these intervals contains an element of A_1. This is a contradiction since A_1 has at most n elements. Therefore, $M - m \leq \frac{3n}{r}$.

(c) Given $i \in \{1, \dots, s\}$, by (a) each block of A contains an element of A_i. Then $\{C(a,A) : a \in A_i\}$ coincides with the set of blocks of A. This proves (c). □

Lemma 2 is devoted to define a convex structure on Δ.

Lemma 2. *There exists a function $\sigma : \Delta \to F_n([0,1])$ such that for every $(A_1, \dots, A_s, t_1, \dots, t_s) \in \Delta$, the following properties hold:*

(a) *the function defined by $\sigma(A_1, \dots, A_s, u_1 \dots, u_s)$ from the set $\{(u_1, \dots, u_s) \in [0,1]^s : u_1 + \dots + u_s = 1\}$ into $F_n([0,1])$ is continuous,*
(b) *for each $A \in F_n(P_r)$, $\sigma(A, 1) = A$,*
(c) *if $i \in \{1, \dots, s\}$ and $t_i = 0$, then $\sigma(A_1, \dots, A_s, t_1, \dots, t_s) = \sigma(A(i), t(i))$,*
(d) *if $\alpha : \{1, \dots, s\} \to \{1, \dots, s\}$ is bijective, then $\sigma(A_1, \dots, A_s, t_1, \dots, t_s) = \sigma(A_{\alpha(1)}, \dots, A_{\alpha(s)}, t_{\alpha(1)}, \dots, t_{\alpha(s)})$ (generalized commutativity),*
(e) *if $A = A_1 \cup \ldots \cup A_s$ and $i \in \{1, \dots, s\}$, then $\sigma(A_1, \dots, A_s, t_1, \dots, t_s)$ is contained in the union of, and intersects each one of the intervals of the family $\{[m(a,A), M(a,A)] : a \in A_i\} = \{[m(a,A), M(a,A)] : a \in A\}$,*
(f) *if $i \in \{1, \dots, s\}$, then $H(A_i, \sigma(A_1, \dots, A_s, t_1, \dots, t_s)) \leq \frac{3n}{r}$,*
(g) *if $A_1 = A_2$, then $\sigma(A_1, \dots, A_s, t_1, \dots, t_s) = \sigma(A_2, \dots, A_s, t_1 + t_2, t_3, \dots, t_s)$, that is, if some A_i coincide, then they can be grouped.*

Proof: We define σ by induction on s.

If $(A, 1) \in \Delta$, define

$$\sigma(A, 1) = A. \tag{2.1}$$

Clearly, properties (a)–(g) hold for the case $s = 1$.

If $(A_1, A_2, t_1, t_2) \in \Delta$ and $A_1 = A_2$, let

$$\sigma(A_1, A_2, t_1, t_2) = A_1. \tag{2.2}$$

If $(A_1, A_2, t_1, t_2) \in \Delta$ and $A_1 \neq A_2$, let $A = A_1 \cup A_2$ and

$$\sigma(A_1, A_2, t_1, t_2) = \begin{cases} \{(1-2t_1)a + 2t_1 m(a, A) : a \in A_2\}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } \ t_1 \in [0, \frac{1}{2}], \\ \{(2t_1 - 1)a + (2-2t_1)m(a, A) : a \in A_1\} \\ \qquad\qquad\qquad\qquad\qquad \text{if } \ t_1 \in [\frac{1}{2}, 1]. \end{cases} \tag{2.3}$$

We check that properties (a)–(g) hold for $s = 2$.

In (2.3), if $t_1 = 0$, then $t_2 = 1$ and $\sigma(A_1, A_2, t_1, t_2) = A_2$; if $t_1 = 1$, then $t_2 = 0$ and $\sigma(A_1, A_2, t_1, t_2) = A_1$. These equalities, (2.1) and (2.2) imply property (c). If $t_1 = \frac{1}{2}$, the first line in the definition gives the set $\{m(a, A) : a \in A_2\}$ and the second line gives $\{m(a, A) : a \in A_1\}$. By Lemma 1(c), both sets coincide, so σ is well defined. Clearly, σ depends continuously on (t_1, t_2).

Properties (d) and (g) follow from the equality $t_1 + t_2 = 1$.

Now we prove (e). In the case that $A_1 = A_2$, we have that $A = A_1 = \sigma(A_1, A_2, t_1, t_2)$. Then $\bigcup\{[m(a, A), M(a, A)] : a \in A_1\} \cap P_r = A$. Hence (e) holds. So, we take $(A_1, A_2, t_1, t_2) \in \Delta$ with $A_1 \neq A_2$, let $A = A_1 \cup A_2$ and take $i \in \{1, 2\}$. By Lemma 1(c), we may assume that $i = 1$.

Let $B = \sigma(A_1, A_2, t_1, t_2)$. Take $p \in B$. If $p = (1-2t_1)a + 2t_1 m(a, A)$, for some $a \in A_2$, by Lemma 1(c), there exists a point $x \in A_1$ such that $C(x, A) = C(a, A)$. Thus p belongs to the interval $[m(x, A), M(x, A)]$. In the case that $p = (2t_1 - 1)b + (2-2t_1)m(b, A)$, for some $b \in A_1$, we obtain that $p \in [m(b, A), M(b, A)]$. We have shown that $B \subset \bigcup\{[m(a, A), M(a, A)] : a \in A_1\}$. Now, take $w \in A_1$. By Lemma 1(a), there exists a point $y \in A_2 \cap C(w, A)$. Thus $C(w, A) = C(y, A)$. If $t_1 \in [0, \frac{1}{2}]$, then the point $u = (1-2t_1)y + 2t_1 m(y, A)$ belongs to $B \cap [m(w, A), M(w, A)]$, and if $t_1 \in [\frac{1}{2}, 1]$, then the point $v = (2t_1 - 1)w + (2 - 2t_1)m(w, A)$ belongs to $B \cap [m(w, A), M(w, A)]$. Hence B intersect each one of the intervals of the form $[m(w, A), M(w, A)]$, where $w \in A$. This completes the proof of (e).

Finally, we prove that (e) implies (f). Let $i \in \{1, 2\}$ and $A = A_1 \cup A_2$. Given a point $x \in \sigma(A_1, A_2, t_1, t_2)$, by (e), there exists $a \in A_i$ such that $x \in [m(a, A), M(a, A)]$. By Lemma 1(b), $|x - a| \leq \frac{3n}{r}$. Similarly, for each point $b \in A_i$, there exists $y \in \sigma(A_1, A_2, t_1, t_2)$ such that $|b - y| \leq \frac{3n}{r}$. Therefore, $H(A_i, \sigma(A_1, A_2, t_1, t_2)) \leq \frac{3n}{r}$.

Now, suppose that $s \geq 2$, suppose also that we have defined σ for all the elements in Δ with length at most $2s$ and that properties (a)–(g) are satisfied for these elements. We define σ for elements of Δ with length $2(s+1)$ in the following way. Take $(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) \in \Delta$. Let $A = A_1 \cup \ldots \cup A_{s+1}$. We consider two cases.

Case 1. The set $\{A_1, \ldots, A_{s+1}\}$ has less than $s+1$ elements.

In this case let $\{A_1, \ldots, A_{s+1}\} = \{B_1, \ldots, B_k\}$, where $k \leq s$ and $B_i \neq B_j$, if $i \neq j$. For each $j \in \{1, \ldots, k\}$, let u_j be the sum of all the elements t_i such that $i \in \{1, \ldots, s+1\}$ and $A_i = B_j$. Then define

$$\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) = \sigma(B_1, \ldots, B_k, u_1 \ldots, u_k). \tag{2.4}$$

Notice that $\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1})$ is well defined since we are assuming that the property (d) holds for the integer k.

Case 2. The sets $A_1, \ldots, A_{s+1}$ are pairwise different.

For each $j \in \{1, \ldots, s+1\}$, let $R_j = \bigcup\{A_k : k \in \{1, \ldots, s+1\} - \{j\}\}$. Fix $i \in \{1, \ldots, s+1\}$ such that $t_i = \min\{t_j : j \in \{1, \ldots, s+1\}\}$. Let $u = (s+1)t_i$. Then $0 \leq u \leq 1$.

Subcase 2.1. $u < 1$.

For each $j \in \{1, \ldots, s+1\}$, let $x_j = \frac{1}{1-u}(t_j - t_i)$. Since $1 - t_j = t_1 + \cdots + t_{j-1} + t_{j+1} + \cdots + t_{s+1} \geq st_i$, we have $u - t_i \leq 1 - t_j$ and $t_j - t_i \leq 1 - u$. Hence $0 \leq x_j \leq 1$. Notice that $x_i = 0$ and $x_1 + \cdots + x_{s+1} = \frac{1}{1-u}(1 - (s+1)t_i) = 1$.

Given $w \in \sigma(A(i), x(i))$, by property (e) for the integer s, there exists $a_w \in R_i \subset A$ with the property that $w \in [m(a_w, R_i), M(a_w, R_i)]$. Then define

$$\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) = \{(1-u)w + um(a_w, A) : w \in \sigma(A(i), x(i))\}. \tag{2.5}$$

In order to see that σ is well defined for this case, we need to show that it depends neither on the choice of i nor on the choice of the numbers a_w. So, suppose that $1 \leq i \leq k \leq s+1$ and $t_i = t_k = \min\{t_j : j \in \{1, \ldots, s+1\}\}$. Then $u = (s+1)t_i = (s+1)t_k$ and the points $x_1, \ldots, x_{s+1}$ do not depend on the choice of i or k. Notice that $x_i = x_k = 0$. In the case that $i < k$, we define $W = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{k-1}, A_{k+1}, \ldots, A_{s+1})$ and we define $Y = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{s+1})$, by property (c) for the integer s, $\sigma(A(i), x(i)) = \sigma(W, Y) = \sigma(A(k), x(k))$. And in the case that $i = k$, clearly, $\sigma(A(i), x(i)) = \sigma(A(k), x(k))$. Given $w \in \sigma(A(i), x(i))$, let $a_w \in R_i$ and $b_w \in R_k$ be such that $w \in [m(a_w, R_i), M(a_w, R_i)]$ and $w \in [m(b_w, R_k), M(b_w, R_k)]$. We may assume that $m(a_w, R_i) \leq m(b_w, R_k)$. Then $m(b_w, R_k)$ belongs to both sets $[m(a_w, R_i), M(a_w, R_i)] \cap A$ and $[m(b_w, R_k), M(b_w, R_k)] \cap A$ which are contained in A. Moreover, since $R_i, R_k \subset A$, each one of the sets $[m(a_w, R_i), M(a_w, R_i)] \cap A$ and $[m(b_w, R_k), M(b_w, R_k)] \cap A$ is contained in block of A and they intersect each other. Hence, we have that they are contained in the same block of A. Thus $C(a_w, A) = C(b_w, A)$ and $m(a_w, A) = m(b_w, A)$. This implies that the definition of $\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1})$ ((2.5)) does not depend either on the choice of i nor on the choice of the elements a_w which were taken for each $w \in \sigma(A(i), x(i))$. Thus $\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1})$ is well defined.

Subcase 2.2. $u = 1$.

In this case $t_i = \frac{1}{s+1}$. By the minimality of t_i and the fact that $t_1 + \ldots + t_{s+1} = 1$, we have $(t_1, \ldots, t_{s+1}) = (\frac{1}{s+1}, \ldots, \frac{1}{s+1})$. Then define

$$\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) = \{m(a, A) : a \in A_1\}. \tag{2.6}$$

This completes the definition of σ.

We show that $\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1})$ depends continuously on the variables $(t_1, \ldots, t_{s+1})$. Fix elements $A_1, \ldots, A_{s+1} \in F_n(P_r)$ such that $H(A_i, A_j) \leq \frac{1}{r}$ for every $i, j \in \{1, \ldots, s+1\}$. In the case that $\{A_1, \ldots, A_{s+1}\}$ has less than $s+1$ elements, the continuity follows from the property (a) in the induction hypothesis. Thus suppose that the sets $A_1, \ldots, A_{s+1}$ are pairwise different. Notice that the number $u = (s+1)\min\{t_j : j \in \{1, \ldots, s+1\}\}$ depends continuously on $(t_1, \ldots, t_{s+1})$. Let $\{(t_1^{(k)}, \ldots, t_{s+1}^{(k)})\}_{k=1}^{\infty}$ be a sequence of elements of $[0,1]^{s+1}$ such that $t_1^{(k)} + \cdots + t_{s+1}^{(k)} = 1$ and $\lim(t_1^{(k)}, \ldots, t_{s+1}^{(k)}) = (t_1^{(0)}, \ldots, t_{s+1}^{(0)})$. We may assume that there exists $i \in \{1, \ldots, s+1\}$ such that $t_i^{(k)} = \min\{t_j^{(k)} : j \in \{1, \ldots, s+1\}\}$, for every $k \in \mathbb{N}$. Thus $t_i^{(0)} = \min\{t_j^{(0)} : j \in \{1, \ldots, s+1\}\}$.

First we consider the case that $u_0 = (s+1)t_i^{(0)} < 1$. Since the numbers $u_k = (s+1)\min\{t_j^{(k)} : j \in \{1, \ldots, s+1\}\}$ tend to u_0, we may assume that $u_k < 1$ for every $k \in \mathbb{N}$. Thus we apply definition (2.5) to compute $\sigma(A_1, \ldots, A_{s+1}, t_1^{(k)}, \ldots, t_{s+1}^{(k)})$ and $\sigma(A_1, \ldots, A_{s+1}, t_1^{(0)}, \ldots, t_{s+1}^{(0)})$. For each $k \in \mathbb{N} \cup \{0\}$ and each $j \in \{1, \ldots, s+1\}$, let $x_j^{(k)} = \frac{1}{1-u_k}(t_j^{(k)} - t_i^{(k)})$. Then $\lim x_j^{(k)} = x_j^{(0)}$. By the property (a) for the integer s, we have that $\lim \sigma(A(i), x^{(k)}(i)) = \sigma(A(i), x^{(0)}(i))$. Thus, we assume that $H(\sigma(A(i), x^{(k)}(i)), \sigma(A(i), x^{(0)}(i))) < \frac{1}{r}$, for each $k \in \mathbb{N}$.

Given $w \in \sigma(A(i), x^{(0)}(i))$ and $k \in \mathbb{N}$, let w_k be the element of $\sigma(A(i), x^{(k)}(i))$ which is closest to w, then $\lim w_k = w$ and $|w - w_k| < \frac{1}{r}$. Let $a_w, a_{w_k} \in R_i$ be such that $w \in [m(a_w, R_i), M(a_w, R_i)]$ and $w_k \in [m(a_{w_k}, R_i), M(a_{w_k}, R_i)]$. Since the elements $m(a_w, R_i), M(a_w, R_i), m(a_{w_k}, R_i), M(a_{w_k}, R_i)$ belong to P_r, if $[m(a_w, R_i), M(a_w, R_i)] \cap [m(a_{w_k}, R_i), M(a_{w_k}, R_i)] = \emptyset$, the distance from each element of $[m(a_w, R_i), M(a_w, R_i)]$ to each element of $[m(a_{w_k}, R_i), M(a_{w_k}, R_i)]$ is at least $\frac{1}{r}$. This contradicts the fact that $|w - w_k| < \frac{1}{r}$. We have shown that $[m(a_w, R_i), M(a_w, R_i)] \cap [m(a_{w_k}, R_i), M(a_{w_k}, R_i)] \neq \emptyset$. Since both sets $[m(a_w, R_i), M(a_w, R_i)] \cap P_r$ and $[m(a_{w_k}, R_i), M(a_{w_k}, R_i)] \cap P_r$ are blocks of R_i, they must coincide. Thus $C(a_w, R_i) = C(a_{w_k}, R_i)$, $m(a_w, R_i) = m(a_{w_k}, R_i)$, $C(a_w, A) = C(a_{w_k}, A)$ and $m(a_w, A) = m(a_{w_k}, A)$. Thus

$$\begin{aligned} &|(1-u_0)w + u_0 m(a_w, A) - ((1-u_k)w_k + u_k m(a_{w_k}, A))| \\ &\qquad \leq |(1-u_0)w - (1-u_k)w_k| + |u_0 - u_k|. \end{aligned}$$

Similarly, for each $w_k \in \sigma(A(i), x^{(k)}(i))$, there exists $w \in \sigma(A(i), x^{(0)}(i))$ such that

$$|(1-u_0)w + u_0 m(a_w, A) - ((1-u_k)w_k + u_k m(a_{w_k}, A))| \\ \leq |(1-u_0)w - (1-u_k)w_k| + |u_0 - u_k|.$$

Since $\lim |(1-u_0)w - (1-u_k)w_k| + |u_0 - u_k| = 0$, we conclude that

$$\lim \sigma(A_1, \ldots, A_{s+1}, t_1^{(k)}, \ldots, t_{s+1}^{(k)}) = \sigma(A_1, \ldots, A_{s+1}, t_1^{(0)}, \ldots, t_{s+1}^{(0)}).$$

Now consider the case that $u_0 = (s+1)t_i^{(0)} = 1$. In this case $(t_1^{(0)}, \ldots, t_{s+1}^{(0)}) = (\frac{1}{s+1}, \ldots, \frac{1}{s+1})$. Thus $\lim(t_1^{(k)}, \ldots, t_{s+1}^{(k)}) = (\frac{1}{s+1}, \ldots, \frac{1}{s+1})$ and $u_k = (s+1)t_i^{(k)}$ tends to 1. Since the formula (2.6) is clearly continuous in the variables $t_1, \ldots, t_{s+1}$, we may assume that $u_k < 1$ for each $k \in \mathbb{N}$. So we compute $\sigma(A_1, \ldots, A_{s+1}, t_1^{(k)}, \ldots, t_{s+1}^{(k)})$ with (2.5). For each $k \in \mathbb{N}$ and for each $j \in \{1, \ldots, s+1\}$, let $x_j^{(k)} = \frac{1}{1-u_k}(t_j^{(k)} - t_i^{(k)})$. Fix $i_0 \in \{1, \ldots, s+1\} - \{i\}$.

Let $k \in \mathbb{N}$. For each $w \in \sigma(A(i), x^{(k)}(i))$, fix $a_w \in R_i$ such that $w \in [m(a_w, R_i), M(a_w, R_i)]$. We show that

$$(*) \qquad \{m(a_w, A) : w \in \sigma(A(i), x^{(k)}(i))\} = \{m(a, A) : a \in A_{i_0}\}.$$

Given $w \in \sigma(A(i), x^{(k)}(i))$, $a_w \in A_l$ for some $l \in \{1, \ldots, s+1\}$. By Lemma 1(c), there exists $a \in A_{i_0}$ such that $m(a_w, A) = m(a, A)$. On the other hand, given $a \in A_{i_0}$, by property (e) for the integer s, there exists an element $w \in \sigma(A(i), x^{(k)}(i)) \cap [m(a, R_i), M(a, R_i)]$. Since $a \in R_i$ and a and w are in the block $[m(a, R_i), M(a, R_i)] \cap R_i$ of R_i, we obtain that $m(a, R_i) = m(a_w, R_i)$. Since $[m(a, R_i), M(a, R_i)] \cap R_i$ is contained in a block of A, we conclude that $m(a, A) = m(a_w, A)$. This completes the proof of $(*)$.

Notice that $\sigma(A_1, \ldots, A_{s+1}, t_1^{(k)}, \ldots, t_{s+1}^{(k)})$ is computed by using (2.5). So,

$$\begin{aligned}
&\lim \sigma(A_1, \ldots, A_{s+1}, t_1^{(k)}, \ldots, t_{s+1}^{(k)}) \\
&= \lim\{(1-u_k)w + u_k m(a_w, A) : w \in \sigma(A(i), x^{(k)}(i))\} \\
&= \{m(a, A) : a \in A_{i_0}\} \quad \text{(by property } (*)) \\
&= \{m(a, A) : a \in A_1\} \quad \text{(by Lemma 1(c))} \\
&= \sigma(A_1, \ldots, A_{s+1}, t_1^{(0)}, \ldots, t_{s+1}^{(0)}) \quad \text{(by (2.6))}.
\end{aligned}$$

This completes the proof that $\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1})$ depends continuously on $(t_1, \ldots, t_{s+1})$. Therefore, property (a) holds for the integer $s+1$.

Property (b) holds by definition (2.1).

We prove property (c) for $s+1$. Let $(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) \in \Delta$ and $A = A_1 \cup \ldots \cup A_{s+1}$. Suppose that $l \in \{1, \ldots, s+1\}$ is such that $t_l = 0$. We consider two cases.

Case 1. The set $\{A_1, \ldots, A_{s+1}\}$ has less than $s+1$ elements.

Let $\{A_1, \ldots, A_{s+1}\} = \{B_1, \ldots, B_k\}$, where $k \leq s$ and $B_i \neq B_j$, if $i \neq j$. For each $j \in \{1, \ldots, k\}$, let u_j be the sum of all the elements t_i such that $i \in \{1, \ldots, s+1\}$ and $A_i = B_j$. We may assume that $A_l = B_k$. We consider two subcases.

Subcase 1.1. $A_j \neq B_k$ for each $j \neq l$.

In this subcase $u_k = 0$. Using (2.4) and property (c) for k and properties (d) and (g) for s, we obtain

$$\begin{aligned}\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) &= \sigma(B_1, \ldots, B_k, u_1 \ldots, u_k) \\ &= \sigma(B_1, \ldots, B_{k-1}, u_1 \ldots, u_{k-1}) = \sigma(A(l), t(l)).\end{aligned}$$

Subcase 1.2. There exists $j \neq l$ such that $A_j = A_l = B_k$.

We have $\{A_1, \ldots, A_{s+1}\} = \{B_1, \ldots, B_k\} = \{A_1, \ldots, A_{l-1}, A_{l+1}, \ldots, A_{s+1}\}$, u_k is the sum of all the elements t_i such that $i \in \{1, \ldots, s+1\}$ and $A_i = B_k$ and u_k is also the sum of all the elements t_i such that $i \in \{1, \ldots, s+1\} - \{l\}$ and $A_i = B_k$. Using (2.4) and properties (d) and (g) for s, we obtain that

$$\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) = \sigma(B_1, \ldots, B_k, u_1 \ldots, u_k) = \sigma(A(l), t(l)).$$

Case 2. The sets $A_1, \ldots, A_{s+1}$ are pairwise different.

In this case, $t_l = \min\{t_j : j \in \{1, \ldots, s+1\}\}$ and $u = (s+1)t_l = 0 < 1$. For each $j \in \{1, \ldots, s+1\}$, $x_j = \frac{1}{1-u}(t_j - t_l) = t_j$. Applying (2.5), we have $\sigma(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) = \sigma(A(l), x(l)) = \sigma(A(l), t(l))$. This completes the proof of (c).

We prove (d). Let $(A_1, \ldots, A_{s+1}, t_1, \ldots, t_{s+1}) \in \Delta$, let $\alpha : \{1, \ldots, s+1\} \to \{1, \ldots, s+1\}$ be a permutation and $A = A_1 \cup \ldots \cup A_{s+1} = A_{\alpha(1)} \cup \ldots \cup A_{\alpha(s+1)}$. In the case that the set $\{A_1, \ldots, A_{s+1}\}$ has less than $s+1$ elements, property (d) follows easily from property (d) applied to the number s. Thus suppose that the sets $A_1, \ldots, A_{s+1}$ are pairwise different. Let $i \in \{1, \ldots, s+1\}$ be such that $t_{\alpha(i)} = \min\{t_j : j \in \{1, \ldots, s+1\}\} = \min\{t_{\alpha(j)} : j \in \{1, \ldots, s+1\}\}$. Let $u = (s+1)t_{\alpha(i)}$. First, we analyze the case that $u < 1$. Given $j \in \{1, \ldots, s+1\}$, let $x_j = \frac{1}{1-u}(t_j - t_{\alpha(i)})$ and $x'_j = \frac{1}{1-u}(t_{\alpha(j)} - t_{\alpha(i)}) = x_{\alpha(j)}$. Since

$$\{1, \ldots, \alpha(i)-1, \alpha(i)+1, \ldots, s+1\} = \{\alpha(1), \ldots, \alpha(i-1), \alpha(i+1), \ldots, \alpha(s+1)\},$$

by property (d) for s, the set

$$\begin{aligned}&\sigma(A(\alpha(i)), x(\alpha(i))) \\ &= \sigma(A_1, \ldots, A_{\alpha(i)-1}, A_{\alpha(i)+1}, \ldots, A_{s+1}, x_1, \ldots, x_{\alpha(i)-1}, x_{\alpha(i)+1}, \ldots, x_{s+1})\end{aligned}$$

is the set

$$\sigma(A_{\alpha(1)},\dots,A_{\alpha(i-1)},A_{\alpha(i+1)},\dots,A_{\alpha(s+1)},x_{\alpha(1)},\dots,x_{\alpha(i-1)},x_{\alpha(i+1)},\dots,x_{\alpha(s+1)}).$$

Given $w\in\sigma(A(\alpha(i)),x(\alpha(i)))$, let

$$\begin{aligned}a_w\in R_{\alpha(i)}&=A_1\cup\ldots\cup A_{\alpha(i)-1}\cup A_{\alpha(i)+1}\cup\ldots\cup A_{s+1}\\&=A_{\alpha(1)}\cup\ldots\cup A_{\alpha(i-1)}\cup A_{\alpha(i+1)}\cup\ldots\cup A_{\alpha(s+1)}\end{aligned}$$

be such that $w\in[m(a_w,R_{\alpha(i)}),M(a_w,R_{\alpha(i)})]$. By (2.5), we have

$$\sigma(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})=\{(1-u)w+um(a_w,A):w\in\sigma(A(\alpha(i)),x(\alpha(i)))\}$$

and this set is also equal to $\sigma(A_{\alpha(1)},\dots,A_{\alpha(s+1)},t_{\alpha(1)},\dots,t_{\alpha(s+1)})$.

On the other hand, in the case that $u=1$, $t_j=\frac{1}{s+1}=t_{\alpha(j)}$ for each $j\in\{1,\dots,s+1\}$. In this case we apply (2.6) and Lemma 1(c) to obtain that

$$\begin{aligned}&\sigma(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})=\{m(a,A):a\in A_1\}\\&=\{m(a,A):a\in A_{\alpha(1)}\}=\sigma(A_{\alpha(1)},\dots,A_{\alpha(s+1)},t_{\alpha(1)},\dots,t_{\alpha(s+1)}).\end{aligned}$$

This completes the proof of (d).

We prove (e). Let $(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})\in\Delta$, $A=A_1\cup\ldots\cup A_{s+1}$ and $i_0\in\{1,\dots,s+1\}$. In the case that the set $\{A_1,\dots,A_{s+1}\}$ has less than $s+1$ elements, property (e) follows easily from (2.4) and property (e) in the induction hypothesis. Thus suppose that the sets $A_1,\dots,A_{s+1}$ are pairwise different. Let $i\in\{1,\dots,s+1\}$ be such that $t_i=\min\{t_j:j\in\{1,\dots,s+1\}\}$. By Lemma 1(c), the intervals described in property (e) are independent of the choice of i_0, thus we may assume that $i\neq i_0$. Let $u=(s+1)t_i$. In the case that $u=1$, property (e) follows immediatly from (2.6) and Lemma 1(a). So, suppose that $u<1$. For each $j\in\{1,\dots,s+1\}$, let $x_j=\frac{1}{1-u}(t_j-t_i)$. Notice that (see (2.5)) each element p of $\sigma(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})$ is a convex combination of an element $w\in[m(a_w,R_i),M(a_w,R_i)]\subset[m(a_w,A),M(a_w,A)]$ and $m(a_w,A)$. Thus $p\in[m(a_w,A),M(a_w,A)]$. Since $a_w\in R_i\subset A$, this interval is of the form $[m(a,A),M(a,A)]$ for some $a\in A_{i_0}$ (by Lemma 1(c)). Thus $\sigma(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})$ is contained in the union of these intervals. In order to see that $\sigma(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})$ intersects each one of these intervals, let $x\in A_{i_0}\subset R_i$. By property (e) in the induction hypothesis, there exists an element $w\in\sigma(A(i),x(i))\cap[m(x,R_i),M(x,R_i)]$. Then the element $(1-u)w+um(x,A)$ belongs to the set $\sigma(A_1,\dots,A_{s+1},t_1,\dots,t_{s+1})\cap[m(x,A),M(x,A)]$. This completes the proof of (e).

The proof that (e) implies (f) is similar to the proof where we showed the same implication for $s=2$. Thus (f) also holds.

Finally, property (g) follows from definition (2.4) and properties (d) and (g) in the induction hypothesis. This completes the proof of the lemma. □

3. Main results

Proof of Theorem 3: Let $n \in \mathbb{N}$. Suppose that $Q(n)$ holds. Let X be a chainable continuum and suppose that there exists a map $g : F_n(X) \to F_n(X)$ without fixed points. Thus there exists $\varepsilon > 0$ such that $H(A, g(A)) > (3n+4)\varepsilon$ for each $A \in F_n(X)$. Let $\mathfrak{F} = \{U_0, \ldots, U_r\}$ be an ε-chain such that $r > 1$, $X = U_0 \cup \ldots \cup U_r$, there exists a point $p_0 \in U_0 - \mathrm{cl}_X(U_1 \cup \ldots \cup U_r)$, there exists a point $q_0 \in U_r - \mathrm{cl}_X(U_0 \cup \ldots \cup U_{r-1})$ and $\mathrm{cl}_X(U_i) \cap \mathrm{cl}_X(U_j) \neq \emptyset$ if and only if $|i-j| \leq 1$.

Let d be a metric for X. For two nonempty closed subsets A and B of X, let $\mathrm{dist}(A,B) = \min\{d(a,b) : a \in A \text{ and } b \in B\}$. Let $\eta = \min\{\mathrm{dist}(\mathrm{cl}_X(U_i), \mathrm{cl}_X(U_j)) : i,j \in \{0,\ldots,r\} \text{ and } i+1 < j\}$. Since g is uniformly continuous, there is $\delta > 0$ with $\delta < \frac{1}{4}\min\{\mathrm{dist}(\{p_0\}, \mathrm{cl}_X(U_1 \cup \ldots \cup U_r)), \mathrm{dist}(\{q_0\}, \mathrm{cl}_X(U_0 \cup \ldots \cup U_{r-1})), \frac{1}{9r}\}$ and, if $A, B \in F_n(X)$ and $H(A,B) < \delta$, then $H(g(A), g(B)) < \eta$.

Let $\mathfrak{G}$ be a $(\min\{\frac{\delta}{4}, \frac{\delta d(p_0,q_0)}{3}\})$-chain covering X such that $\mathfrak{G}$ refines $\mathfrak{F}$. Let $\mathfrak{H} = \{V_0, \ldots, V_m\}$ be a subchain of $\mathfrak{G}$ such that $p_0 \in V_0 - (V_1 \cup \ldots \cup V_m)$, $q_0 \in V_m - (V_0 \cup \ldots \cup V_{m-1})$. Then $m \geq 3$ and $\frac{1}{m+1} < \delta$. For each $i \in \{0, \ldots, m\}$, choose an element $j(i) \in \{0, \ldots, r\}$ such that $V_i \subset U_{j(i)}$ and choose a point $p_i \in V_i - (\bigcup\{V_k : k \in \{0,\ldots,m\} - \{i\}\})$, where $p_m = q_0$. Notice that, if $i,j \in \{0,\ldots,m\}$ and $|i-j| \leq 1$, then p_i, p_j belong to a set of the form V_k, so $d(p_i,p_j) < \frac{\delta}{4}$. We use the points $p_0, \ldots, p_m$ to define a function $P : F_n(P_m) \to F_n(X)$ as follows.

For each $A = \{\frac{a_1}{m}, \ldots, \frac{a_s}{m}\} \in F_n(P_m)$, where $a_1, \ldots, a_s \in \{0, \ldots, m\}$, let

$$P(A) = \{p_{a_1}, \ldots, p_{a_s}\} \in F_n(X). \tag{3.1}$$

Notice that, if $A, B \in F_n(P_m)$ and $H(A,B) \leq \frac{1}{m}$, then $H(P(A), P(B)) < \frac{\delta}{4}$.

For each $x \in g(P(A))$, choose an index $e(x) \in \{0, \ldots, r\}$ such that

$$x \in U_{e(x)}. \tag{3.2}$$

Define $\varphi_0 : F_n(P_m) \to F_n(P_r)$ by

$$\varphi_0(A) = \{\frac{e(x)}{r} : x \in g(P(A))\}. \tag{3.3}$$

We are going to extend φ_0 to a continuous function φ from $F_n([0,1])$ into itself. It is known that $F_n([0,1])$ is an AR ([3, Korollar 2]). However, we need an extension of φ_0 which will have a property derived from property (f) of Lemma 2, so we use the convex structure defined in the previous section and a Dugundji-type construction.

Define, for each $E \in F_n(P_m)$,

$$\varphi(E) = \varphi_0(E). \tag{3.4}$$

Given $A \in F_n([0,1]) - F_n(P_m)$, define

$$\mathfrak{B}(A) = \Big\{E \in F_n([0,1]) : H(A,E) < \min\Big\{\frac{1}{2}(\min\{H(A,G) : G \in F_n(P_m)\}), \frac{1}{16m}\Big\}\Big\}. \tag{3.5}$$

Let $\mathfrak{W} = \{\mathfrak{W}_\alpha : \alpha \in \Lambda\}$ be a locally finite refinement of the open cover $\{\mathfrak{B}(A) : A \in F_n([0,1]) - F_n(P_m)\}$, of the set $F_n([0,1]) - F_n(P_m)$. Let $\mathfrak{P} = \{\Psi_\alpha : \alpha \in \Lambda\}$ be a partition of the unity subordinated to $\mathfrak{W}$.

For each $\alpha \in \Lambda$, choose an element $C_\alpha \in \mathfrak{W}_\alpha$, also choose an element $A_\alpha \in F_n(P_m)$ such that

$$H(C_\alpha, A_\alpha) = \min\{H(C_\alpha, A) : A \in F_n(P_m)\}. \tag{3.6}$$

Since for each element t of $[0,1]$ there exists an element s of P_m such that $|t-s| \le \frac{1}{2m}$, we have that $H(C_\alpha, A_\alpha) \le \frac{1}{2m}$.

Given $E \in F_n([0,1]) - F_n(P_m)$, let $\alpha_1(E), \dots, \alpha_{k_E}(E)$ be the elements in Λ such that $\Psi_\alpha(E) > 0$. Then define

$$\varphi(E) = \sigma(\varphi_0(A_{\alpha_1(E)}), \dots, \varphi_0(A_{\alpha_{k_E}(E)}), \Psi_{\alpha_1(E)}(E), \dots, \Psi_{\alpha_{k_E}(E)}(E)), \tag{3.7}$$

where φ_0 was previously defined on $F_n(P_m)$ and σ is as in Lemma 2.

We check that φ is well defined. In order to do this, we need to verify that

$$(\varphi_0(A_{\alpha_1(E)}), \dots, \varphi_0(A_{\alpha_{k_E}(E)}), \Psi_{\alpha_1(E)}(E), \dots, \Psi_{\alpha_{k_E}(E)}(E)) \in \Delta,$$

that is, we need to show that, if $i, j \in \{1, \dots, k_E\}$, then $H(\varphi_0(A_{\alpha_i(E)}), \varphi_0(A_{\alpha_j(E)})) \le \frac{1}{r}$. Since $\Psi_{\alpha_i(E)}(E) > 0$, there exists $D \in F_n([0,1]) - F_n(P_m)$ such that $E \in \mathfrak{W}_{\alpha_i(E)} \subset \mathfrak{B}(D)$. Since $C_{\alpha_i(E)} \in \mathfrak{W}_{\alpha_i(E)} \subset \mathfrak{B}(D)$, $H(E, C_{\alpha_i(E)}) < \frac{1}{4m}$ (see (3.5)). Thus $H(E, A_{\alpha_i(E)}) < \frac{3}{4m}$. Similarly, $H(E, A_{\alpha_j(E)}) < \frac{3}{4m}$. Hence, $H(A_{\alpha_i(E)}, A_{\alpha_j(E)}) < \frac{3}{2m}$. Since, for each two points $t, s \in P_m$, the inequality $|t-s| < \frac{3}{2m}$ implies $|t-s| \le \frac{1}{m}$; and $A_{\alpha_i(E)}, A_{\alpha_j(E)} \subset P_m$, we obtain that $H(A_{\alpha_i(E)}, A_{\alpha_j(E)}) \le \frac{1}{m}$. As we noticed after (3.1), this implies that $H(P(A_{\alpha_i(E)}), P(A_{\alpha_j(E)})) < \frac{\delta}{2}$. By the choice of δ, $H(g(P(A_{\alpha_i(E)})), g(P(A_{\alpha_j(E)}))) < \eta$. Let $u = \frac{e(x)}{r} \in \varphi_0(A_{\alpha_i(E)})$, with $x \in g(P(A_{\alpha_i(E)}))$. Then there exists $y \in g(P(A_{\alpha_j(E)}))$ such that $d(x,y) < \eta$. Since $x \in U_{e(x)}$ and $y \in U_{e(y)}$ (see (3.2)), by the choice of η, $|e(x) - e(y)| \le 1$. Thus $v = \frac{e(y)}{r} \in \varphi_0(A_{\alpha_j(E)})$ and $|u - v| \le \frac{1}{r}$. Similarly, for each $v \in \varphi_0(A_{\alpha_j(E)})$, there exists $u \in \varphi_0(A_{\alpha_i(E)})$ such that $|u-v| \le \frac{1}{r}$. Therefore, $H(\varphi_0(A_{\alpha_i(E)}), \varphi_0(A_{\alpha_j(E)})) \le \frac{1}{r}$. We have shown that

$$(\varphi_0(A_{\alpha_1(E)}), \dots, \varphi_0(A_{\alpha_{k_E}(E)}), \Psi_{\alpha_1(E)}(E), \dots, \Psi_{\alpha_{k_E}(E)}(E)) \in \Delta.$$

Combining this with property (d) of Lemma 2, we obtain that φ is well defined and it does not depend on the way we order the indexes $\alpha_1(E), \dots, \alpha_{k_E}(E)$.

We see that φ is continuous. Let $E \in F_n([0,1]) - F_n(P_m)$. Let $\mathfrak{U}$ be an open neighborhood of E in $F_n([0,1])$ such that $\mathfrak{U} \cap F_n(P_m) = \emptyset$ and $\mathfrak{U}$ intersects only finitely many sets, $\mathfrak{W}_{\beta_1}, \ldots, \mathfrak{W}_{\beta_l}$, of the family $\mathfrak{W}$. Notice that for each $D \in \mathfrak{U}$, $\{\alpha_1(D), \ldots, \alpha_{k_D}(D)\} \subset \{\beta_1, \ldots, \beta_l\}$. By properties (c) and (d) of Lemma 2,

$$\begin{aligned}\varphi(D) &= \sigma(\varphi_0(A_{\alpha_1(D)}), \ldots, \varphi_0(A_{\alpha_{k_D}(D)}), \Psi_{\alpha_1(D)}(D), \ldots, \Psi_{\alpha_{k_D}(D)}(D)) \\ &= \sigma(\varphi_0(A_{\beta_1}), \ldots, \varphi_0(A_{\beta_l}), \Psi_{\beta_1}(D), \ldots, \Psi_{\beta_l}(D)).\end{aligned}$$

Hence, property (a) of Lemma 2 implies that φ is continuous on $\mathfrak{U}$. Therefore, φ is continuous at E for each $E \in F_n([0,1]) - F_n(P_m)$.

Now, take $E \in F_n(P_m)$. Let

$$\mathfrak{B} = \left\{ D \in F_n([0,1]) : H(D,E) < \frac{1}{16m} \right\}.$$

Given $D \in \mathfrak{B} - \{E\}$, $D \in F_n([0,1]) - F_n(P_m)$. If $i \in \{1, \ldots, k_D\}$, there exists $G \in F_n([0,1]) - F_n(P_m)$ such that $D \in \mathfrak{W}_{\alpha_i(D)} \subset \mathfrak{B}(G)$. Thus, by (3.5),

$$\begin{aligned}H(D,G) &< \frac{1}{2}(\min\{H(G,L) : L \in F_n(P_m)\}) \leq \frac{1}{2}(H(E,G)) \\ &\leq \frac{1}{2}(H(E,D) + H(D,G)) < \frac{1}{32m} + \frac{1}{2}(H(D,G)).\end{aligned}$$

Hence $H(D,G) < \frac{1}{16m}$, $H(E,G) < \frac{1}{8m}$ and $\min\{H(G,L) : L \in F_n(P_m)\} < \frac{1}{8m}$. Since $C_{\alpha_i(D)} \in \mathfrak{W}_{\alpha_i(D)}$, $H(C_{\alpha_i(D)}, G) < \frac{1}{8m}$. Thus $H(E, C_{\alpha_i(D)}) \leq H(E,G) + H(G, C_{\alpha_i(D)}) < \frac{1}{4m}$. Therefore ((3.6)) $H(A_{\alpha_i(D)}, C_{\alpha_i(D)}) < \frac{1}{4m}$. Hence $H(A_{\alpha_i(D)}, E) < \frac{1}{2m}$. Since $A_{\alpha_i(D)}$ and E belong to $F_n(P_m)$, this implies that $A_{\alpha_i(D)} = E$. From (3.7), for each $D \in \mathfrak{B} - \{E\}$,

$$\varphi(D) = \sigma(\varphi_0(E), \ldots, \varphi_0(E), \Psi_{\alpha_1(D)}(D), \ldots, \Psi_{\alpha_{k_D}(D)}(D)) = \varphi_0(E)$$

(see properties (g) and (b) in Lemma 2). This implies that φ is continuous at E. This completes the proof that φ is continuous.

Define $f : [0,1] \to [0,1]$ as the piecewise linear extension of the function defined on P_m by

$$f(\frac{i}{m}) = \frac{j(i)}{r}. \tag{3.8}$$

Since $p_0 \in U_0 - \mathrm{cl}_X(U_1 \cup \ldots \cup U_r)$ and $q_0 \in U_r - \mathrm{cl}_X(U_0 \cup \ldots \cup U_{r-1})$, $f(0) = 0$ and $f(1) = 1$. Let $f_n : F_n([0,1]) \to F_n([0,1])$ be the induced map. Given $i \in \{0, \ldots, m-1\}$, $V_i \subset U_{j(i)}$ and $V_{i+1} \subset U_{j(i+1)}$. Since $V_i \cap V_{i+1} \neq \emptyset$, $|j(i) - j(i+1)| \leq 1$. This proves that

$$|f(\frac{i}{m}) - f(\frac{i+1}{m})| \leq \frac{1}{r}. \tag{$**$}$$

Since we are assuming that $Q(n)$ is true, there exists an element $D \in F_n([0,1])$ such that $f_n(D) = \varphi(D)$.

We consider two cases.

Case 1. $D \notin F_n(P_m)$.

Let $D_0 = A_{\alpha_1(D)}$. By property (f) of Lemma 2 and (3.7), $H(\varphi_0(D_0), \varphi(D)) \leq \frac{3n}{r}$. For each $x \in D$, choose $k(x) \in \{0, \ldots, m-1\}$ such that $x \in [\frac{k(x)}{m}, \frac{k(x)+1}{m}]$. Let $D_1 = \{\frac{k(x)}{m} \in [0,1] : x \in D\}$ and $D_2 = \{\frac{k(x)+1}{m} \in [0,1] : x \in D\}$. Then $D_1, D_2 \in F_n(P_m)$ and $H(D, D_1)$, $H(D, D_2) \leq \frac{1}{m}$. Let $G \in F_n([0,1]) - F_n(P_m)$ be such that $\mathfrak{W}_{\alpha_1(D)} \subset \mathfrak{B}(G)$. Then $H(D, G)$, $H(C_{\alpha_1(D)}, G) < \frac{1}{16m}$. Thus $H(D_1, G) < \frac{9}{8m}$ and $H(A_{\alpha_1(D)}, C_{\alpha_1(D)}) \leq H(D_1, C_{\alpha_1(D)}) < \frac{10}{8m}$. Hence $H(D_0, D_1) < \frac{20}{8m}$. Since $D_0, D_1 \in F_n(P_m)$, $H(D_0, D_1) \leq \frac{2}{m}$. Given $\frac{i}{m} \in D_0$, there exists $\frac{j}{m} \in D_1$ such that $|\frac{i}{m} - \frac{j}{m}| \leq \frac{2}{m}$. By (**), $|f(\frac{i}{m}) - f(\frac{j}{m})| \leq \frac{2}{r}$. Similarly, Given $\frac{j}{m} \in D_1$, there exists $\frac{i}{m} \in D_0$ such that $|f(\frac{i}{m}) - f(\frac{j}{m})| \leq \frac{2}{r}$. Thus $H(f_n(D_0), f_n(D_1)) \leq \frac{2}{r}$. Given $x \in D$, since $x \in [\frac{k(x)}{m}, \frac{k(x)+1}{m}]$ and $|f(\frac{k(x)}{m}) - f(\frac{k(x)+1}{m})| \leq \frac{1}{r}$, we have that $|f(\frac{k(x)}{m}) - f(x)| \leq \frac{1}{r}$. This implies that $H(f_n(D), f_n(D_1)) \leq \frac{1}{r}$. Since $\varphi(D) = f_n(D)$,

$$\begin{aligned} H(\varphi_0(D_0), f_n(D_0)) \leq H(\varphi_0(D_0), \varphi(D)) + H(\varphi(D), f_n(D_1)) \\ + H(f_n(D_1), f_n(D_0)) \leq \frac{3n+3}{r}. \end{aligned}$$

Thus $H(\varphi_0(D_0), f_n(D_0)) \leq \frac{3n+3}{r}$.

Since $D_0 \in F_n(P_m)$, we can put $D_0 = \{\frac{a_1}{m}, \ldots, \frac{a_s}{m}\}$. Then $f_n(D_0) = \{\frac{j(a_1)}{r}, \ldots, \frac{j(a_s)}{r}\}$ (see (3.8)) and $P(D_0) = \{p_{a_1}, \ldots, p_{a_s}\}$ (see (3.1)). Given $x \in g(P(D_0))$, $\frac{e(x)}{r} \in \varphi_0(D_0)$ ((3.3)). So, there exists $v \in f_n(D_0)$ such that $|\frac{e(x)}{r} - v| \leq \frac{3n+3}{r}$. Then there exists $i \in \{0, \ldots, s\}$ such that $v = f(\frac{a_i}{m}) = \frac{j(a_i)}{r}$ ((3.8)). Thus $|e(x) - j(a_i)| \leq 3n+3$. Recall that $x \in U_{e(x)}$ ((3.2)) and $p_{a_i} \in V_{a_i} \subset U_{j(a_i)}$. Hence $d(x, p_{a_i}) < (3n+4)\varepsilon$. We have shown that, for each $x \in g(P(D_0))$, there exists $p_{a_i} \in P(D_0)$ such that $d(x, p_{a_i}) < (3n+4)\varepsilon$. Similarly, for each $p_{a_i} \in P(D_0)$ there exists $x \in g(P(D_0))$ such that $d(x, p_{a_i}) < (3n+4)\varepsilon$. This proves that $H(P(D_0), g(P(D_0))) < (3n+4)\varepsilon$. Contrary to the choice of ε.

Case 2. $D \in F_n(P_m)$.

In this case, $H(\varphi_0(D), f_n(D)) = 0 \leq \frac{3n+3}{r}$. Thus we can repeat the argument in the paragraph above with D instead D_0 to obtain a contradiction.

We have obtained a contradiction from assuming that $F_n(X)$ does not have the fixed point property. Thus Theorem 3 is proved. $\square$

Proof of Theorem 4: Let $\mathfrak{B} = \{A \in F_3([0,1]) : A \cap \{0,1\} \neq \emptyset\}$ and $\mathfrak{C} = \{A \in F_3([0,1]) : \{0,1\} \subset A\}$. Using Theorem 6 in [2], it is easy to show that there exists a homeomorphism $h : F_3([0,1]) \to D^3$, where D^3 is the unit ball, centered at the origin, in the Euclidean space $\mathbb{R}^3$, such that $h(\mathfrak{B})$ is the unit sphere $S^2 \subset D^3$ and $h(\mathfrak{C})$ is the equator E which results of intersecting S^2 with the plane $z = 0$ in $\mathbb{R}^3$.

Suppose that $Q(3)$ does not hold, then there exists a map $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$, and there exists a map $g : F_3([0,1]) \to F_3([0,1])$ such that $g(A) \neq f_3(A)$ for each $A \in F_3([0,1])$, where f_3 is the induced map of f from $F_3([0,1])$ into itself.

Notice that $f_3(\mathfrak{B}) \subset \mathfrak{B}$ and $f_3(\mathfrak{C}) \subset \mathfrak{C}$. Let $G = h \circ g \circ h^{-1}$ and $F = h \circ f_3 \circ h^{-1}$. Then $G, F : D^3 \to D^3$, $F|S^2 : S^2 \to S^2$, and $G(p) \neq F(p)$ for each $p \in D^3$. Define $\varphi : D^3 \to S^2$ by $\varphi(p)$ is the only point in the intersection of S^2 and the convex ray which starts in $G(p)$ and passes through $F(p)$. Then φ is continuous and $\varphi(p) = F(p)$ for each $p \in S^2$.

Consider the map $K : S^2 \times [0,1] \to S^2$ given by $K(p,t) = \varphi(tp)$. Then, for each $p \in S^2$, $K(p,1) = F(p)$ and $K(p,0) = \varphi(0)$. Thus $F|S^2$ is homotopic to a constant map.

Let $\lambda : [0,1] \times [0,1] \to [0,1]$ be given by $\lambda(x,t) = tx + (1-t)f(x)$. Then λ is continuous, $\lambda(0,t) = 0$ and $\lambda(1,t) = 1$ for each $t \in [0,1]$. Let $\Lambda : S^2 \times [0,1] \to S^2$ be given by $\Lambda(p,t) = h(\lambda(h^{-1}(p) \times \{t\}))$. Then Λ is continuous, $\Lambda(p,0) = F(p)$ and $\lambda(p,1) = p$, for each $p \in S^2$. Thus $F|S^2$ is homotopic to the identity map defined on S^2. This is impossible since S^2 is not contractible. Hence $Q(3)$ holds and Theorem 4 is proved. $\square$

Question 6. Does $Q(n)$ hold for each $n \geq 4$?

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