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On the Lindelöf property of spaces of continuous functions over a Tychonoff space and its subspaces

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Abstract. We study relations between the Lindelöf property in the spaces of continuous functions with the topology of pointwise convergence over a Tychonoff space and over its subspaces. We prove, in particular, the following: a) if $C_p(X)$ is Lindelöf, $Y = X \cup \{p\}$, and the point p has countable character in Y, then $C_p(Y)$ is Lindelöf; b) if Y is a cozero subspace of a Tychonoff space X, then $l(C_p(Y)^\omega) \leq l(C_p(X)^\omega)$ and $\operatorname{ext}(C_p(Y)^\omega) \leq \operatorname{ext}(C_p(X)^\omega)$.

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All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We use terminology and notation as in [Eng].

Given two spaces X and Z, we denote by $C_p(X, Z)$ the space of all continuous functions from X to Z equipped with the topology of pointwise convergence (that is, the topology of the subspace of the space Z^X of all functions from X to Z endowed with the Tychonoff product topology). The space $C_p(X, \mathbb{R})$ is denoted as $C_p(X)$.

If $p\colon X \to Y$ is a continuous mapping, the *dual mapping* $p^*\colon C_p(Y, Z) \to C_p(X, Z)$ is defined by the rule: $p^*(f) = f \circ p$ for all $f \in C_p(Y)$. The dual mapping is always continuous, is a homeomorphic embedding if p is onto, and is a closed embedding if p is quotient; see [Arh2].

A space X is a $\mathcal{K}_{\sigma\delta}$*-space* if it is an $F_{\sigma\delta}$-set in βX; $\mathcal{K}$*-analytic* spaces are continuous images of $\mathcal{K}_{\sigma\delta}$-spaces.

In [Buz] Buzyakova raised some questions about the behavior of the Lindelöf property of the spaces $C_p(X)$ and $C_p(X, Y)$ for some simple spaces Y under "slight changes" of the spaces X and Y. In this article we give complete or partial answers to a few of these questions.

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1. Adding a point of countable character

Proposition 1.1. *Let X be a non-pseudocompact space. Then $C_p(X) \times \omega^\omega$ is homeomorphic to a closed subspace of $C_p(X)$.*

Proof: Since X is not pseudocompact, there is a discrete family $\{U_n : n \in \omega\}$ of non-empty open sets in X. Choose a point x_n in each U_n; then the set $D = \{x_n : n \in \omega\}$ is closed and discrete in X. For every $n \in \omega$ choose a continuous function from $\phi_n \colon X \to [0,1]$ so that $\phi_n(x_n) = 1$ and $\phi_n(X \setminus U_n) = \{0\}$. For every $f \in \mathbb{R}^D$ put

$$h(f)(x) = \sum_{n=1}^{\infty} f(x_n)\phi_n(x).$$

Note that, by the discreteness of the family $\{U_n : n \in \omega\}$, in a neighborhood of every $x \in X$ at most one term in the sum in the definition of $h(f)$ is distinct from zero; clearly, $h(f)(x_n) = f(x_n)$. It follows that h is a linear extension operator from $C_p(D) = \mathbb{R}^D \to C_p(X)$. Since the value of $h(f)$ at a point $x \in X$ is completely and continuously determined by the value of f at at most one point of D (the one such that $x \in \bar{U}_n$, if there is any), h is continuous.

By Proposition 2.1 in [Ar1], the space $C_p(X)$ is homeomorphic to $C \times C_p(D) = C \times \mathbb{R}^\omega$ where C is the subset of $C_p(X)$ consisting of all functions equal to 0 on D. Thus, we have homeomorphisms $C_p(X) = C \times \mathbb{R}^\omega = C \times \mathbb{R}^\omega \times \mathbb{R}^\omega = C_p(X) \times \mathbb{R}^\omega$. Since ω^ω is homeomorphic to a closed subspace of $\mathbb{R}^\omega$, we get the statement of the proposition. $\square$

Corollary 1.2. *If X is a non-pseudocompact space, $C_p(X)$ is Lindelöf, and Y is a $\mathcal{K}$-analytic space, then $C_p(X) \times Y$ is Lindelöf.*

Proof: Every $\mathcal{K}$-analytic space is an image of ω^ω under a compact-valued upper semicontinuous mapping (see e.g. [RJ]). Hence, by Proposition 1.1, $C_p(X) \times \mathcal{K}$ is an image under a compact-valued upper semicontinuous mapping of a closed subspace of $C_p(X)$. The statement of the corollary now follows from the well-known fact that compact-valued upper semicontinuous mappings do not raise the Lindelöf number. $\square$

Corollary 1.3. *Let X be a non-pseudocompact space such that $C_p(X)$ is Lindelöf, and P an $F_{\sigma\delta}$-subspace of $C_p(X)$. Then P is Lindelöf.*

Proof: Let $P = \bigcap_{n\in\omega} \bigcup_{m\in\omega} F_{nm}$ where each F_{nm} is a closed set in $C_p(X)$. Then P is the image under the projection onto $C_p(X)$ of the closed subset

$$B = \{(f,\phi) : \forall\, n \in \omega\, f \in F_{n\phi(n)}\}$$

of $C_p(X) \times \omega^\omega$. $\square$

The next theorem provides a positive answer to Question 3.1 in [Buz].

Theorem 1.4. *Let $Y = X \cup \{p\}$ and assume that the point p has countable character in Y. If $C_p(X)$ is Lindelöf, then $C_p(Y)$ is Lindelöf.*

Proof: If p is an isolated point in Y, then $C_p(Y) = C_p(X) \times \mathbb{R}$, and $C_p(Y)$ is Lindelöf. So assume that p is not isolated. Then X is not pseudocompact, by the well-known fact that a pseudocompact space is G_δ-dense in any its extension.

Let $C_0 = \{f \in C_p(Y) : f(p) = 0\}$. Then $C_p(Y)$ is homeomorphic to $C_0 \times \mathbb{R}$ (by virtue of the homeomorphism $f \mapsto (f - f(p), f(p))$ for every $f \in C_p(Y)$). Therefore, it suffices to show that C_0 is Lindelöf. The restriction mapping $r\colon C_p(Y) \to C_p(X)$ embeds C_0 homeomorphically into $C_p(X)$, so we need to show that the subspace $C = r(C_0)$ of $C_p(X)$ is Lindelöf. Clearly, $C = \{f \in C_p(X) : \lim_{x \to p} f(x) = 0\}$.

Let $\{V_n : n \in \omega\}$ be a countable open base for p in Y, and let $U_n = V_n \cap X$, $n \in \omega$. Then

$$C = \{f \in C_p(X) : \forall\, n \in \omega\, \exists\, m \in \omega\ \forall\, x \in U_m \quad |f(x)| \le 1/(n+1)\}.$$

Thus,

$$C = \bigcap_{n\in\omega} \bigcup_{m\in\omega} \bigcap_{x\in U_m} \{f \in C_p(X) : |f(x)| \le 1/(n+1)\}$$

is an $F_{\sigma\delta}$-set in $C_p(X)$, hence is Lindelöf by Corollary 1.3. □

Theorem 1.4 may be slightly generalized:

Theorem 1.5. *Let $Y = X \cup K$ where K is a metrizable compact space, X is dense in Y, $K \cap X = \emptyset$, and $\chi(K, Y) \le \omega$. If $C_p(X)$ is Lindelöf, then $C_p(Y)$ is Lindelöf.*

Proof: Since K is compact metrizable, there is a continuous linear extension operator $h\colon C_p(K) \to C_p(Y)$ [Ar1], so by Proposition 2.1 in [Ar1], $C_p(Y)$ is homeomorphic to $C_0 \times C_p(K)$ where C_0 is the set of all functions in $C_p(Y)$ whose restrictions to K are zero.

Let $Z = Y/K$ be the quotient space, $q\colon Y \to Z$ the natural mapping, and $\{p\} = q(K)$. Since K is compact, q is a perfect mapping, the space Z is Tychonoff, and since the character of K in Y is countable, the character of p in Z is countable. Furthermore, $X = q^{-1}(q(Z \setminus \{p\}))$, so $q|X$ is a perfect bijection from X to $Z \setminus \{p\}$. Thus, $Z \setminus \{p\}$ is homeomorphic to X. By Theorem 1.4, $C_p(Z)$ is Lindelöf.

The dual mapping $q^*\colon C_p(Z) \to C_p(Y)$ is a closed embedding and C_0 is contained in $q^*(C_p(Z))$. Since C_0 is closed in $C_p(Y)$, it is homeomorphic to a closed subspace of $C_p(Z)$. By the density of X in Y, X is not pseudocompact (except the trivial case $K = \emptyset$). The space $C_p(K)$ is $\mathcal{K}$-analytic (in fact, a $\mathcal{K}_{\sigma\delta}$-space, see [Arh2]), so by Corollary 1.2, $C_p(Z) \times C_p(K)$ is Lindelöf. Since C_0 is homeomorphic to a closed set in $C_p(Z)$, $C_0 \times C_p(K)$ is Lindelöf, and $C_p(Y)$ is Lindelöf.

□

Theorem 1.5 does not hold if we only require that K be an Eberlein compact space. Indeed, if Y is the one-point compactification of a Mrówka space, then it is the union of a countable discrete subspace X and the compact space K homeomorphic to the one-point compactification of a discrete space, which is an Eberlein compact space; K has countable character in Y, because its complement is countable (so it is a G_δ-set) and Y is compact. That $C_p(Y)$ for such Y cannot be Lindelöf was proved in [Pol].

On the other hand, a statement similar to Theorem 1.5 holds, with a similar proof, if we require the existence of an extension operator.

Theorem 1.6. *Let $Y = X \cup K$ where K is an Eberlein compact space, X is dense in Y, $K \cap X = \emptyset$, and $\chi(K, Y) \leq \omega$. If $C_p(X)$ is Lindelöf, and there is a continuous extension operator $h\colon C_p(K) \to C_p(Y)$, then $C_p(Y)$ is Lindelöf.*

2. Spaces of functions on cozero sets

In [Buz], Buzyakova proved that *If X is zero-dimensional compact, $C_p(X)$ is Lindelöf, and p is a point of countable character in X, then $C_p(X \setminus \{p\})$ is Lindelöf*, and asks if the same holds for every compact space, or for any space X.

In this section we prove some statements in this direction, which generalize the theorem of Buzyakova.

Theorem 2.1. *Let X be a space such that $C_p(X)^\omega$ is Lindelöf, and Y a cozero set in X. Then $C_p(Y)^\omega$ is Lindelöf.*

Proof: Let $h\colon X \to [0,1]$ be a continuous function such that $Y = h^{-1}((0,1])$.

For each $n \in \omega$ put $F_n = h^{-1}([1/(n+1), 1])$ and $F = X \setminus Y$. Clearly, F and F_n, $n \in \omega$, are zero sets, $F_n \subset \operatorname{Int} F_{n+1}$, and $Y = \bigcup\{F_n : n \in \omega\}$.

Put

$$P = \{G \in C_p(X)^\omega : G(n)|F_n = G(m)|F_n \text{ for all } m, n \in \omega,\ m \geq n\}.$$

Then

$$P = \bigcap_{n\in\omega} \bigcap_{m \geq n} \bigcap_{x \in F_n} \{G \in C_p(X)^\omega : G(m)(x) = G(n)(x)\},$$

so P is closed in $C_p(X)^\omega$, and P^ω is Lindelöf.

Define $T\colon P \to \mathbb{R}^Y$ by the rule:

$$T(G)(x) = G(n)(x) \text{ if } x \in F_n.$$

Obviously, T is well-defined. Let $G \in P$ and $x \in Y$. Then $x \in F_n$ for some n, and $x \in \operatorname{Int} F_{n+1}$. Since $T(G)|F_{n+1} = G(n+1)|F_{n+1}$, $T(G)$ coincides with the continuous function $G(n+1)$ in the neighborhood F_{n+1} of x, and therefore is continuous at x. Thus, G is continuous on Y, and we have proved $T(P) \subset C_p(Y)$.

Let us verify the inverse inclusion. Let $f \in C_p(Y)$. Fix a continuous function $\theta\colon [0,1] \to [0,1]$ so that $\theta(1) = 1$ and $\theta([0,1/2]) = \{0\}$. For every $n \in \omega$ fix a continuous function $h_n\colon X \to [0,1]$ so that $h_n(F_n) \subset \{1\}$ and $h_n(F) \subset \{0\}$, and let $s_n(x) = \theta \circ h_n$. Then $s_n\colon X \to [0,1]$ is continuous, $s_n(F_n) \subset \{1\}$, and s_n is zero in a neighborhood of F. It follows that the function $g_n\colon X \to \mathbb{R}$ defined by the rule

$$g_n(x) = \begin{cases} f(x)s_n(x) & \text{if } x \in Y, \\ 0 & \text{if } x \in F \end{cases}$$

is continuous on X, and coincides with f on F_n. Thus, $f = T(G)$ where $G(n) = g_n$ for all $n \in \omega$. This finishes the proof that $T(P) = C_p(Y)$.

Finally, let us verify that T is continuous. For an open set W in $\mathbb{R}$ and $x \in Y$ denote $O(x,W) = \{f \in C_p(Y) : f(x) \in W\}$. The sets $O(x,W)$ form an open subbase for the topology of $C_p(Y)$, so it suffices to verify that their preimages under T are open in P. So fix x and W; find an $m \in \omega$ so that $x \in F_m$. Then $x \in F_n$ for all $n \geq m$, so $G(n)(x) = G(m)(x)$ for all $G \in P$ and $n \geq m$. We have therefore

$$\begin{aligned} T^{-1}(O(x,W)) &= \{G \in P : G(m)(x) \in W\} \\ &= P \cap \{H \in C_p(X)^\omega : H(m)(x) \in W\}, \end{aligned}$$

an open set in P.

Thus, $C_p(Y)$ is a continuous image of the set P, whence $C_p(Y)^\omega$ is Lindelöf. □

The condition "$C_p(X)^\omega$ is Lindelöf" appears much stronger than "$C_p(X)$ is Lindelöf"; however, as far as the author knows by the moment, whether the two conditions are equivalent is an open problem, both for compact spaces X and in the general case. In some particular cases, however, it is known that the two conditions are equivalent. Thus, R. Pol showed in [Pol] that if X is zero-dimensional compact and $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf. We can slightly improve this statement.

Theorem 2.2. *Let X be a σ-compact zero-dimensional space. If $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf.*

Proof: Since the Cantor cube 2^ω is homeomorphic to a closed subspace of $\mathbb{R}$, the space $C_p(X, 2^\omega)$ is homeomorphic to a closed subspace of $C_p(X)$, and therefore is Lindelöf. We have $C_p(X, 2^\omega) = C_p(X, 2)^\omega$, so $C_p(X, 2)^\omega$ is Lindelöf. Since X is zero-dimensional, $C_p(X, 2)$ separates points and closed sets of X. It follows that the diagonal product $\Phi = \Delta C_p(X, 2)\colon X \to \mathbb{R}^{C_p(X,2)}$ is an embedding; obviously, $\Phi(X) \subset C_p(C_p(X, 2))$. Thus, X is homeomorphic to a σ-compact subspace of $C_p(Y)$ where $Y = C_p(X, 2)$. Then $X \times \omega$ is σ-compact and homeomorphic to a subspace of $C_p(Y^+) = C_p(Y) \times \mathbb{R}$, where Y^+ is the space obtained by adding an isolated point to Y. The space $(Y^+)^\omega$ is Lindelöf: Y^+ is a continuous image of $Y \times 2$, so $(Y^+)^\omega$ is a continuous image of $Y^\omega \times 2^\omega$. By Corollary 2.8 in [Oku], $C_p(X)^\omega = C_p(X \times \omega)$ is Lindelöf. $\square$

Corollary 2.3. *Let X be a zero-dimensional σ-compact space such that $C_p(X)$ is Lindelöf. Then for every cozero set Y in X, $C_p(Y)$ is Lindelöf.*

This corollary can also be deduced directly from Theorem 2.2 and Corollary 2.8 in [Oku], using the observation that a cozero set in a σ-compact space is σ-compact.

Corollary 2.4. *Let X be a zero-dimensional σ-compact space such that $C_p(X)$ is Lindelöf. Then for every compact G_δ-set K in X, $C_p(X \setminus K)$ is Lindelöf.*

The proof of Theorem 2.1 actually gives the following statement:

Theorem 2.5. *Let Y be a cozero set in X. Then $C_p(Y)$ is a continuous image of a closed subset of $C_p(X)^\omega$.*

We now can deduce various other corollaries, related to classes of spaces invariant with respect to countable powers, closed subspaces, and continuous images.

Corollary 2.6. *If $C_p(X)$ is a Lindelöf Σ-space, and Y a cozero set in X, then $C_p(Y)$ is a Lindelöf Σ-space.*

Corollary 2.7. *If $C_p(X)$ is a $\mathcal{K}$-analytic space, and Y a cozero set in X, then $C_p(Y)$ is a $\mathcal{K}$-analytic space.*

Corollary 2.8. *If Y is a cozero subspace of X, then $l(C_p(Y)^\omega) \leq l(C_p(X)^\omega)$ and $\operatorname{ext}(C_p(Y)^\omega) \leq \operatorname{ext}(C_p(X)^\omega)$.*

Corollary 2.9. *If $C_p(X)$ is a $L\Sigma(\leq \omega)$-space, and Y a cozero set in X, then $C_p(Y)$ is an $L\Sigma(\leq \omega)$-space.*

(See [KOS] for definition and basic properties of $L\Sigma(\leq \omega)$-spaces.)

And, generally,

Corollary 2.10. *Let $\mathcal{P}$ be a class of spaces invariant with respect to countable powers, closed subspaces and continuous images. If $C_p(X) \in \mathcal{P}$, and Y is a cozero set in X, then $C_p(Y) \in \mathcal{P}$.*

A similar argument applies to the spaces $C_p(X, I)$ where $I = [0, 1]$:

Theorem 2.11. *Let Y be a cozero set in X. Then $C_p(Y, I)$ is a continuous image of a closed subset of $C_p(X, I)^\omega$.*

3. Some open problems

Question 3.1. Let X be a pseudocompact space such that $C_p(X)$ is Lindelöf. Must the product $C_p(X) \times \omega^\omega$ be Lindelöf?

Question 3.2. Let $Y = X \cup K$ where K is a metrizable compact space, X is dense in Y, $\chi(K, Y) \leq \omega$, and $C_p(X)$ is Lindelöf. Must $C_p(Y)$ be Lindelöf?

The question here is if we can omit the condition "$X \cap K = \emptyset$" in Theorem 1.5. If we assume that $C_p(X)^\omega$ is Lindelöf and that $X \setminus K$ is dense in Y, the answer is "yes" by Theorems 1.5 and 2.1.

Question 3.3. Let X be a space such that $C_p(X)^\omega$ is Lindelöf, and Y an open F_σ-subspace of X. Must $C_p(Y)$ be Lindelöf?

Note that for normal spaces X an affirmative answer to this question follows from Theorem 2.1.

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