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# Openly factorizable spaces and compact extensions of topological semigroups

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*Abstract.* We prove that the semigroup operation of a topological semigroup $S$ extends to a continuous semigroup operation on its Stone-Čech compactification $\beta S$ provided $S$ is a pseudocompact openly factorizable space, which means that each map $f : S \to Y$ to a second countable space $Y$ can be written as the composition $f = g \circ p$ of an open map $p : X \to Z$ onto a second countable space $Z$ and a map $g : Z \to Y$. We present a spectral characterization of openly factorizable spaces and establish some properties of such spaces.

*Keywords:* topological semigroup, semigroup compactification, inverse spectrum, pseudocompact space, openly factorizable space, openly generated space, Eberlein compact, Corson compact, Valdivia compact

*Classification:* 22A15, 54B30, 54C20, 54C08, 54D35

This paper was motivated by the problem of detecting topological semigroups that embed into compact topological semigroups. One of the ways to attack this problem is to find conditions on a topological semigroup $S$ guaranteeing that the semigroup operation of $S$ extends to a continuous semigroup operation on the Stone-Čech compactification $\beta S$ of $S$. A crucial step in this direction was made by E. Reznichenko [16] who proved that the semigroup operation on a pseudocompact topological semigroup $S$ extends to a separately continuous semigroup operation on $\beta S$. In this paper we show that the extended operation on $\beta S$ is continuous if the space $S$ is separable and openly factorizable, which means that each continuous map $f : S \to Y$ to a second countable space $Y$ can be written as the composition $f = g \circ p$ of an open continuous map $p : X \to Z$ onto a second countable space $Z$ and a continuous map $g : Z \to Y$. The class of openly factorizable spaces turned out to be interesting by its own so we devote Sections 2–5 to studying such spaces.

We recall that the *Stone-Čech compactification* of a Tychonov space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\bar{f} : \beta X \to Y$.

Replacing in this definition compact Hausdorff spaces by realcompact spaces we obtain the definition of the Hewitt realcompactification $\upsilon X$ of $X$. We recall that a topological space $X$ is *realcompact* if $X$ is homeomorphic to a closed subspace of some power $\mathbb{R}^\kappa$ of the real line. A *Hewitt realcompactification* of a Tychonov space $X$ is a realcompact space $\upsilon X$ containing $X$ as a dense $C$-embedded subspace. A subspace $A$ of a topological space $X$ is *$C$-embedded* if each continuous function $f : A \to \mathbb{R}$ extends to a continuous function $\bar{f} : X \to \mathbb{R}$. By [7, 3.11.16], the Hewitt realcompactification $\upsilon X$ can be identified with the subspace

$$\{x \in \beta X : G \cap X \neq \emptyset \text{ for any } G_\delta\text{-set } G \subset \beta X \text{ with } x \in G\}$$

of the Stone-Čech compactification $\beta X$ of $X$. By [7, 3.11.12], every Lindelöf space $X$ is realcompact and hence $X$ coincides with its Hewitt realcompactification $\upsilon X$.

The Hewitt realcompactification $\upsilon X$ of a Tychonov space $X$ coincides with its Stone-Čech compactification $\beta X$ if and only if the space $X$ is *pseudocompact* in the sense that each continuous real-valued function on $X$ is bounded, see [7, §3.11].

The problem of extending the group operation from a (para)topological group $G$ to its Stone-Čech or Hewitt extensions have been considered in [2], [15], [16], [17]. In this paper we address a similar problem for topological semigroups. All *topological spaces* appearing in this paper *are Tychonov* and all *maps are continuous*.

## 1. Semigroup compactifications of topological semigroups

In this section we recall some information on semigroup compactifications of a given (semi)topological semigroup $S$.

By a *semitopological semigroup* we understand a topological space $S$ endowed with a separately continuous semigroup operation $* : S \times S \to S$. If the operation is jointly continuous, then $S$ is called a *topological semigroup*.

Let $\mathcal{C}$ be a class of compact Hausdorff semitopological semigroups. By a *$\mathcal{C}$-compactification* of a semitopological semigroup $S$ we understand a pair $(\mathcal{C}(S), \eta)$ consisting of a compact semitopological semigroup $\mathcal{C}(S) \in \mathcal{C}$ and a continuous homomorphism $\eta : S \to \mathcal{C}(S)$ (called the *canonical homomorphism*) such that for each continuous homomorphism $h : S \to K$ to a semitopological semigroup $K \in \mathcal{C}$ there is a unique continuous homomorphism $\bar{h} : \mathcal{C}(S) \to K$ such that $h = \bar{h} \circ \eta$. It follows that any two $\mathcal{C}$-compactifications of $S$ are topologically isomorphic.

We shall be interested in $\mathcal{C}$-compactifications for the following classes of semigroups:

- WAP of compact semitopological semigroups;
- AP of compact topological semigroups;
- SAP of compact topological groups.

The corresponding $\mathcal{C}$-compactifications of a semitopological semigroup $S$ will be denoted by WAP($S$), AP($S$), and SAP($S$). The notation came from the abbreviations for weakly almost periodic, almost periodic, and strongly almost periodic function rings that determine those compactifications, see [18, §III.2] or

[5, Ch.IV]. By Theorem 3.4 of [5], any semitopological semigroup $S$ has the $\mathcal{C}$-compactifications $\mathrm{WAP}(S)$, $\mathrm{AP}(S)$ and $\mathrm{SAP}(S)$. Since the closure of a subsemigroup in a semitopological semigroup is a semigroup, we conclude that the WAP- and AP-compactifications $\mathrm{WAP}(S)$ and $\mathrm{AP}(S)$ of $S$ contain the image $\eta(S)$ as a dense subsemigroup. On the other hand, the subsemigroup $\eta(S)$ algebraically generates a dense subgroup of the compact topological group $\mathrm{SAP}(S)$.

The inclusions of the classes SAP $\subset$ AP $\subset$ WAP induce canonical continuous homomorphisms

$$\eta : S \to \mathrm{WAP}(S) \to \mathrm{AP}(S) \to \mathrm{SAP}(S)$$

for each semitopological semigroup $S$. Since the space $\mathrm{WAP}(S)$ is compact, the canonical map $\eta : S \to \mathrm{WAP}(S)$ uniquely extends to a continuous map $\beta\eta : \beta S \to \mathrm{WAP}(S)$ defined on the Stone-Čech compactification of $S$. Since $\eta(S)$ is dense in $\mathrm{WAP}(S)$ and $\mathrm{AP}(S)$, the canonical maps $\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S)$ are surjective.

It should be mentioned that the canonical homomorphism $\eta : S \to \mathrm{WAP}(S)$ need not be injective. For example, for the group $H_+[0,1]$ of orientation-preserving homeomorphisms of the interval the WAP-compactification is a singleton, see [14]. However, for pseudocompact semitopological semigroups the situation is more optimistic. The following two results are due to E. Reznichenko [16]. They allow us to identify the WAP-compactification $\mathrm{WAP}(S)$ of a (countably compact) pseudocompact (semi)topological semigroup $S$ with the Stone-Čech compactification $\beta S$ of $S$. We recall that a topological space $X$ is *countably compact* if each countable open cover of $X$ has a finite subcover.

**Theorem 1.1** (Reznichenko)**.** *For any countably compact semitopological semigroup $S$ the semigroup operation $S \times S \to S$ extends to a separately continuous semigroup operation $\beta S \times \beta S \to \beta S$, which implies that the canonical map $\beta\eta : \beta S \to \mathrm{WAP}(S)$ is a homeomorphism.*

The same conclusion holds for pseudocompact topological semigroups.

**Theorem 1.2** (Reznichenko)**.** *For any pseudocompact topological semigroup $S$ the semigroup operation $S \times S \to S$ extends to a separately continuous semigroup operation $\beta S \times \beta S \to \beta S$, which implies that the canonical map $\beta\eta : \beta S \to \mathrm{WAP}(S)$ is a homeomorphism.*

If a topological semigroup $S$ has pseudocompact square, then its Stone-Čech compactification $\beta S$ coincides with its AP-compactification.

**Theorem 1.3.** *For any topological semigroup $S$ with pseudocompact square $S \times S$ the semigroup operation $S \times S \to S$ extends to a continuous semigroup operation $\beta S \times \beta S \to \beta S$, which implies that the canonical maps $\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S)$ are homeomorphisms.*

Proof: By Theorem 1.2, the semigroup operation $\mu : S \times S \to S$ of $S$ extends to a separately continuous semigroup operation $\overline{\mu} : \beta S \times \beta S \to \beta S$ on $\beta S$. On the other hand, the operation $\mu : S \times S \to S \subset \beta S$ extends to a continuous map $\beta\mu : \beta(S \times S) \to \beta S \times \beta S$. By the Glicksberg Theorem [7, 3.12.20(c)], the pseudocompactness of the square $S \times S$ implies that the Stone-Čech extension $\beta i : \beta(S \times S) \to \beta S \times \beta S$ of the inclusion map $i : S \times S \to \beta S \times \beta S$ is a homeomorphism. Observe that the functions $\beta\mu$ and $\bar{\mu} \circ \beta i$ coincide on the dense subset $S \times S$ of $\beta(S \times S)$. It is an easy exercise to check that those maps coincide everywhere, which implies that the binary operation $\bar{\mu} = \beta\mu \circ (\beta i)^{-1}$ is continuous. This means that $\beta S$ is a compact topological semigroup and hence the canonical map $\beta\eta : \beta S \to \mathrm{AP}(S)$ has continuous inverse. □

It should be mentioned that for a pseudocompact topological semigroup $S$ the canonical map $\eta : S \to \mathrm{AP}(S)$ needs not be a topological embedding. The following counterexample is constructed in [3].

**Example 1.4.** *Under Martin's Axiom there is a countably compact topological semigroup $S$ for which the canonical homomorphism $\eta : S \to \mathrm{AP}(S)$ is not injective.*

Example 1.4 shows that one should impose rather strong restrictions on a topological semigroup $S$ to guarantee that the canonic homomorphism $S \to \mathrm{AP}(S)$ (or $S \to \mathrm{SAP}(S)$) is an embedding.

**Theorem 1.5.** *If a topological semigroup $S$ contains a dense subgroup and has pseudocompact square $S \times S$, then the canonical maps $\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S) \to \mathrm{SAP}(S)$ are homeomorphisms.*

Proof: By Theorem 1.3, the canonical maps $\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S)$ are homeomorphisms. Since the compact topological semigroup $\mathrm{AP}(S)$ contains a dense subgroup (that lies in $S$), we may apply Theorem 1.11 of [6] to conclude that $\mathrm{AP}(S)$ is a group. Being a compact paratopological group, $\mathrm{AP}(S)$ is a topological group according to Theorem 1.13 of [6]. Now the definition of $\mathrm{SAP}(S)$ implies that the canonical map $S \to \mathrm{AP}(S)$ extends to a unique continuous semigroup homomorphism $\mathrm{SAP}(S) \to \mathrm{AP}(S)$, which shows that the canonical map $\mathrm{AP}(S) \to \mathrm{SAP}(S)$ is a homeomorphism. □

We recall that a topological group $G$ is called *totally bounded* if for every non-empty open subset $U \subset G$ there is a finite subset $F \subset G$ such that $G = FU = UF$.

The following important result can be found in [18, III.3.3].

**Theorem 1.6** (Ruppert)**.** *For each totally bounded topological group $G$ the canonical homomorphisms $\mathrm{WAP}(G) \to \mathrm{AP}(G) \to \mathrm{SAP}(G)$ are homeomorphisms and the canonical map $\eta : G \to \mathrm{SAP}(G)$ is a topological embedding.*

The same conclusion holds for Tychonov pseudocompact topological semigroups that contain dense totally bounded topological subgroups.

**Theorem 1.7.** *If a pseudocompact topological semigroup $S$ contains a totally bounded topological group $H$ as a dense subgroup, then the canonical maps*

$$\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S) \to \mathrm{SAP}(S)$$

*are homeomorphisms.*

Proof: The embedding $H \subset S$ induces a continuous homomorphism $h : \mathrm{WAP}(H) \to \mathrm{WAP}(S)$. We claim that this homomorphism is surjective. Indeed, by Theorem 1.6, $\mathrm{WAP}(H)$ is a compact topological group, containing $H$ as a dense subgroup. By Theorem 1.2, the Stone-Čech compactification $\beta S$ of $S$ can be identified with the WAP-compactification $\mathrm{WAP}(S)$ of $S$. Then the image $h(\mathrm{WAP}(H))$ contains the dense subset $H$ of $\beta S = \mathrm{WAP}(S)$ and hence coincides with $\beta S$ being a compact dense subset of $\beta S$. The compact semitopological semigroup $\mathrm{WAP}(S)$, being a continuous homomorphic image of the compact topological group $\mathrm{WAP}(H)$, is a compact topological group. This implies that the canonical homomorphism $\mathrm{WAP}(S) \to \mathrm{SAP}(S)$ is a topological isomorphism. Consequently, the maps $\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S) \to \mathrm{SAP}(P)$ all are homeomorphisms. □

Our final result concerns the AP-compactifications of pseudocompact openly factorizable topological semigroups. Those are pseudocompact topological semigroups whose topological spaces are *openly factorizable*.

We define a topological space $X$ to be *openly factorizable* if for each map $f : X \to Y$ to a second countable space $Y$ there are an open map $p : X \to Z$ onto a second countable space $Z$ and a map $g : Z \to Y$ such that $f = g \circ p$. Openly factorizable spaces will be studied in detail in the next sections. Now we present our main extension result for which we need the notion of a weakly Lindelöf space.

We call a topological space $X$ *weakly Lindelöf* if each open cover $\mathcal{U}$ of $X$ contains a countable subcollection $\mathcal{V} \subset \mathcal{U}$ whose union $\bigcup \mathcal{V}$ is dense in $X$. It is clear that the class of weakly Lindelöf spaces includes all Lindelöf spaces and all countably cellular (in particular, all separable) spaces.

**Theorem 1.8.** *For any openly factorizable topological semigroup $S$ having weakly Lindelöf square $S \times S$, the semigroup operation $S \times S \to S$ extends to a continuous semigroup operation $\upsilon S \times \upsilon S \to \upsilon S$ defined on the Hewitt realcompactification $\upsilon S$ of $S$.*

Proof: By Theorem 3.5 below the semigroup operation $\mu : S \times S \to S$ extends to a continuous map $\bar{\mu} : \upsilon S \times \upsilon S \to \upsilon S$ thought as a continuous binary operation on $\upsilon S$. This operation is associative on $S$ and by the continuity remains associative on $\upsilon S$. □

This theorem implies another one:

**Theorem 1.9.** *For each pseudocompact openly factorizable topological semigroup $S$ with weakly Lindelöf square, the canonical maps $\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S)$ are homeomorphisms.*

Proof: By Theorem 1.8, the semigroup operation $\mu : S \times S \to S$ extends to a continuous semigroup operation $\bar{\mu} : \upsilon S \times \upsilon S \to \upsilon S$ turning the Hewitt realcompactification $\upsilon S$ of $S$ into a topological semigroup that contains $S$ as a dense subsemigroup. Since the space $S$ is pseudocompact, its Hewitt realcompactification coincides with its Stone-Čech compactification $\beta S$ [7, §3.11]. Consequently, $\beta S$ is a compact topological semigroup, which implies that the canonical map

$\beta\eta : \beta S \to \mathrm{AP}(S)$ has a continuous inverse and consequently, the maps

$$\beta S \to \mathrm{WAP}(S) \to \mathrm{AP}(S)$$

are homeomorphisms. $\square$

## 2. Some elementary properties of openly factorizable spaces

In this section we establish some elementary properties of openly factorizable spaces. First we prove a helpful lemma.

**Lemma 2.1.** *Let $p : X \to Z$ be a map from a Tychonov space to a second countable space and let $\upsilon p : \upsilon X \to \upsilon Z = Z$ be its continuous extension to the Hewitt realcompactification of $X$. The map $\upsilon p$ is surjective* (*open*) *if and only if so is the map $p$.*

Proof: This lemma will follow as soon as we prove that $p(U \cap X) = \upsilon p(U)$ for any open subset $U \subset \upsilon X$. Assuming the opposite, find a point $y \in \upsilon p(U) \setminus p(U \cap X)$. Choose any point $x_0 \in U$ with $\upsilon p(x_0) = y$ and find a continuous function $g : \upsilon X \to [0,1]$ such that $g^{-1}(0)$ is a neighborhood of $x_0$ and $g^{-1}[0,1) \subset U$. Fix a metric $d$ generating the topology of the second countable space $Z$ and consider the continuous function $f : \upsilon X \to [0,\infty)$ defined by $f(x) = g(x) + \mathrm{dist}(\upsilon p(x), y)$. Observe that $f(x_0) = 0$ while $f(x) \in (0,1]$ for all $x \in X$. Indeed, if $x \in X \cap U$, then $f(x) \geq \mathrm{dist}(p(x), y) > 0$ because $y \notin p(U \cap X)$. If $x \in X \setminus U$, then $f(x) \geq g(x) = 1 > 0$. Since $\upsilon X$ is a Hewitt realcompactification of $X$, the function $f|X : X \to (0,+\infty)$ admits a unique continuous extension $\bar{f} : \upsilon X \to (0,+\infty)$. Since $X$ is dense in $\upsilon X$, we get $f = \bar{f}$ and thus $0 = f(x_0) = \bar{f}(x_0) \in (0,\infty)$. This contradiction completes the proof of the equality $\upsilon p(U) = p(U \cap X)$. $\square$

**Proposition 2.2.** *The Hewitt realcompactification $\upsilon X$ of a Tychonov space $X$ is openly factorizable if and only if so is the space $X$.*

Proof: Assume that a Tychonov space $X$ is openly factorizable. To show that the Hewitt realcompactification $\upsilon X$ is openly factorizable, take any map $f : \upsilon X \to Y$ to a second countable space $Y$. Since $X$ is openly factorizable, there are an open surjective map $p : X \to Z$ to a second countable space $Z$ and a map $g : Z \to Y$ such that $f|X = g \circ p$. The space $Z$, being second countable, is realcompact [7, 3.11.12]. Consequently, the map $p$ admits a continuous extension $\upsilon p : \upsilon X \to Z$. It follows that $f = g \circ \upsilon p$. By Lemma 2.1, the map $\upsilon p$ is open and surjective, witnessing that $\upsilon X$ is openly factorizable.

Now assume that $\upsilon X$ is openly factorizable. To show that $X$ is openly factorizable, take any map $f : X \to Y$ to a second countable space $Y$. Since $Y$ is realcompact [7, 3.11.12], the map $f$ extends to a map $\upsilon f : \upsilon X \to Y$. Since $\upsilon X$ is openly factorizable, there are an open surjective map $p : \upsilon X \to Z$ to a second countable space $Z$ and a map $g : Z \to Y$ such that $f = g \circ p$. Then $f|X = g \circ p|X$ and the map $p|X : X \to Z$ is open and surjective by Lemma 2.1. $\square$

**Proposition 2.3.** *The Stone-Čech compactification $\beta X$ of a Tychonov space $X$ is openly factorizable if and only if $X$ is pseudocompact and openly factorizable.*

Proof: If $X$ is pseudocompact and openly factorizable, then the Hewitt realcompactification $\upsilon X$ is openly factorizable by Proposition 2.2. Since $X$ is pseudocompact, its Hewitt realcompactification coincides with the Stone-Čech compactification $\beta X$. So, $\beta X$ is openly factorizable.

Now assume conversely that $\beta X$ is openly factorizable. We claim that $X$ is pseudocompact. In the opposite case, we could find a continuous unbounded function $f : X \to [0,\infty)$. Let $\beta f : \beta X \to [0,\infty]$ be the Stone-Čech extension of the map $f$ to the one-point compactification of the half-line $[0,\infty)$. Since $\beta X$ is openly factorizable, there are an open surjective map $p : \beta X \to Z$ onto a metrizable compact space $Z$ and a map $g : Z \to [0,\infty]$ such that $f = g \circ p$.

Since the function $f$ is unbounded, we can choose a sequence $\{x_n\}_{n\in\omega} \subset X$ such that the sequence $\{f(x_n)\}_{n\in\omega} \subset [0,\infty)$ is strictly increasing and unbounded. Passing to a subsequence, if necessary, we can assume that the sequence $\{p(x_n)\}_{n\in\omega} \subset Z$ converges to some point $z_\infty \in Z$. It follows from $f = g\circ p$ that $g(z_\infty) = \infty$ and the points $z_\infty$, $p(x_n)$, $n \in \omega$, all are distinct. So each point $p(x_n)$ has a neighborhood $U_n \subset Z \setminus \{z_\infty\}$ such that the family $\{U_n : n \in \omega\}$ is disjoint. Moreover, we can assume that the sequence $(U_n)$ converges to $z_\infty$ in the sense that each neighborhood $O(z_\infty)$ contains all but finitely many sets $U_n$. Since the sequence $\{f(x_n)\}_{n\in\omega}$ is closed and discrete in $[0,\infty)$, to each point $f(x_n)$ we can assign an open neighborhood $V_n \subset [0,\infty)$ such that the family $\{V_n : n \in \omega\}$ is discrete in $[0,\infty)$ (in the sense that each point has a neighborhood that meets at most one set $V_n$). Now for every $n \in \omega$ consider the open neighborhood $W_n = f^{-1}(V_n) \cap p^{-1}(U_n)$ of the point $x_n$ in $X$. Since the family $\{V_n\}_{n\in\omega}$ is discrete in $[0,\infty)$, the family $\{W_n\}_{n\in\omega}$ is discrete in $X$. Let $x_\infty \in \beta X$ be any accumulation point of the sequence $\{x_{2n}\}_{n\in\omega}$.

Since the space $X$ is Tychonov and $\{W_{2n}\}_{n\in\omega}$ is discrete, we can construct a continuous function $\varphi : X \to [0,1]$ such that

$$\{x_{2n}\}_{n\in\omega} \subset \varphi^{-1}(1) \subset \varphi^{-1}(0,1] \subset \bigcup_{n\in\omega} W_{2n}.$$

Let $\beta\varphi : \beta X \to [0,1]$ be the Stone-Čech extension of $\varphi$. It follows from the continuity of $\beta\varphi$ that $\beta\varphi(x_\infty) = 1$. Then the set $W = (\beta\varphi)^{-1}(\frac{1}{2},1]$ is an open neighborhood of $x_\infty$ in $\beta X$ with

$$W \cap X \subset \overline{W} \cap X \subset \varphi^{-1}[1/2,1] \subset \bigcup_{n\in\omega} W_{2n}.$$

It follows that $p(W \cap X) \subset V$ where $V = \bigcup_{n\in\omega} V_{2n}$ and consequently,

$$p(W) \subset p(\overline{W}) = p(\overline{W \cap X}) \subset \overline{p(W \cap X)} \subset \overline{V}.$$

Since $\overline{V} \subset X \setminus \bigcup_{n\in\omega} V_{2n+1}$ and $V_{2n+1} \to z_\infty$, the set $\overline{V}$ contains no neighborhood of the point $z_\infty = p(x_\infty)$. Consequently, the set $p(W)$ cannot be open. This contradiction completes the proof of the pseudocompactness of $X$.

In this case the Stone-Čech compactification $\beta X$ coincides with the Hewitt realcompactification $\upsilon X$ of $X$. Applying Proposition 2.2, we conclude that $X$ is openly factorizable. $\square$

**Proposition 2.4.** *The Aleksandrov compactification $\alpha X$ of a locally compact space $X$ is openly factorizable if $X$ is openly factorizable and $\sigma$-compact.*

Proof: Without loss of generality, the space $X$ is not compact. Let $f : \alpha X \to Y$ be any map to a second countable space. Since $X$ is $\sigma$-compact and locally compact, we can find a continuous function $\xi : \alpha X \to [0,1]$ such that $\xi^{-1}(0) = \{\infty_X\}$ where $\infty_X$ is the compactifying point of $\alpha X = \{\infty_X\} \cup X$. Now consider the map $\tilde{f} : X \to Y \times (0,1]$, $\tilde{f} : x \mapsto (f(x), \xi(x))$. Since $X$ is openly factorizable, there is an open map $p : X \to Z$ onto a second countable space $Z$ and a map $\tilde{g} : Z \to Y \times (0,1]$ such that $\tilde{f} = \tilde{g} \circ p$.

The space $Z$ is locally compact as the image of a locally compact space under an open map. So, it is legal to consider its Aleksandrov compactification $\alpha Z = \{\infty_Z\} \cup Z$. Extend the map $p : X \to Z$ to the map $\alpha p : \alpha X \to \alpha Z$ letting $\alpha p|X = p$ and $\alpha p(\infty_X) = \infty_Z$. Let us show that the so-extended map $\alpha p$ is continuous at $\infty_X$. Given an open neighborhood $O(\infty_Z) \subset \alpha Z$ of $\infty_Z$, consider the complement $K = \alpha Z \setminus O(\infty_Z)$ and its image $\tilde{g}(K) \subset Y \times (0,1]$. Being a compact subset of $Y \times (0,1]$, the set $\tilde{g}(K)$ lies in $Y \times [a,1]$ for some $a > 0$. Then $O(\infty_X) = \xi^{-1}([0,a))$ is an open neighborhood of $\infty_X$ such that $\alpha p(O(\infty_X)) \subset O(\infty_Z)$, witnessing that $\alpha p$ is continuous at $\infty_X$. It is equally easy to check that the map $\alpha p : \alpha X \to \alpha Z$ is open.

Denote by $\mathrm{pr}_Y : Y \times (0,1] \to Y$ the projection and define a map $g : \alpha Z \to Y$ letting $g|Z = \mathrm{pr}_Y \circ \tilde{g}$ and $g(\infty_Z) = f(\infty_X)$. It is easy to check that the map $g$ is continuous and $f = g \circ \alpha p$, witnessing the open factorizability of the one-point compactification $\alpha X$ of $X$. $\square$

## 3. Spectral characterization of openly factorizable spaces

In this section we shall present a spectral characterization of openly factorizable topological spaces. First we remind some information related to inverse spectra, see [8, §3.1] and [7, §2.5].

A partially ordered set $(A, \leq)$ is called

- *directed* if for every $a, b \in A$ there exists $c \in A$ with $c \geq a$, $c \geq b$;
- *$\omega$-directed* if for any countable subset $C \subset A$ has an upper bound in $A$ (which is a point $a \in A$ such that $a \geq c$ for every $c \in C$);
- *$\omega$-complete* if $A$ each countable directed subset $C \subset A$ has the smallest upper bound $\sup C$ in $A$.

For example, the ordinal $\omega_1$ endowed with the natural order is a well-ordered $\omega$-complete set.

By a *spectrum* over a directed set $(A, \le)$ we understand a collection $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$ consisting of Tychonov spaces $X_\alpha$, $\alpha \in A$, and continuous surjective maps $\pi^\gamma_\alpha : X_\gamma \to X_\alpha$ for $\alpha \le \gamma$ from $A$ such that $\pi^\gamma_\alpha = \pi^\beta_\alpha \circ \pi^\gamma_\beta$ for every elements $\alpha \le \beta \le \gamma$ of $A$. Let

$$\lim \mathcal{S} = \{(x_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha : \forall\, \alpha, \beta \in A \;\; \alpha \le \beta \Rightarrow x_\alpha = \pi^\beta_\alpha(x_\beta)\} \subset \prod_{\alpha\in A} X_\alpha$$

denote the limit space of the spectrum $\mathcal{S}$.

For a directed subset $B$ of $A$, let $\mathcal{S}|B$ denote the subspectrum $\mathcal{S}|B = \{X_\alpha, \pi^\gamma_\alpha, B\}$ of $\mathcal{S}$, consisting of the spaces $X_\alpha$ and the projections $\pi^\gamma_\alpha$ for which $\alpha, \gamma \in B$. Given a collection $\{f_\alpha : X \to X_\alpha\}_{\alpha\in A}$ of maps from a space $X$ into the spaces of the spectrum $\mathcal{S}$ such that $\pi^\gamma_\alpha \circ f_\gamma = f_\alpha$ for every $\alpha \le \gamma$ in $A$, we denote by $\lim f_\alpha : X \to \lim \mathcal{S}$ the induced map into the limit space of $\mathcal{S}$.

A spectrum $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$ is defined to be

- *continuous* if for every chain $B \subset A$ having supremum $\beta = \sup B$ the map $\lim_{\alpha\in B} \pi^\beta_\alpha : X_\beta \to \lim \mathcal{S}|B$ is a homeomorphism;
- *open* if the projections $\pi^\gamma_\alpha : X_\gamma \to X_\alpha$ are open and surjective for all $\alpha \le \gamma$ in $A$;
- *$\omega$-directed* (resp. *$\omega$-complete*) provided so is its index set $A$;
- an *$\omega$-spectrum* if it is $\omega$-directed and each space $X_\alpha$, $\alpha \in A$, is second countable;
- *factorizing* if every map $f : \lim \mathcal{S} \to \mathbb{R}$ can be written as $f = f_\alpha \circ \pi_\alpha$ for some $\alpha \in A$ and some map $f_\alpha : X_\alpha \to \mathbb{R}$.

According to [8, 3.1.5], a continuous $\omega$-complete spectrum $\mathcal{S}$ with surjective bonding maps is factorizing if and only if every *bounded* map $f : \lim \mathcal{S} \to \mathbb{R}$ can be written as $f = f_\alpha \circ \pi_\alpha$ for some $\alpha \in A$ and some bounded map $f_\alpha : X_\alpha \to \mathbb{R}$. By another result of [8, 3.1.7], a continuous $\omega$-complete open spectrum $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$ is factorizing provided the limit space $\lim \mathcal{S}$ is countably cellular (i.e., contains no uncountable disjoint family of open sets).

In fact, the proof of Proposition 3.1.7 of [8] can be modified to get the following more general statement, cf. [4, 3.2].

**Proposition 3.1.** *Suppose $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$ is an $\omega$-spectrum and $X \subset \lim \mathcal{S}$ is a weakly Lindelöf subspace of its limit such that the restrictions $\pi_\alpha|X : X \to X_\alpha$, $\alpha \in A$, of the limit projections are open and surjective. Then every map $f : X \to Y$ to a second countable space $Y$ can be written as $f = f_\alpha \circ \pi_\alpha|X$ for some $\alpha \in A$ and some map $f_\alpha : X_\alpha \to Y$. In particular, $X$ is $C$-embedded into $\lim \mathcal{S}$ and hence $\lim \mathcal{S}$ is a Hewitt realcompactification of $X$.*

Proof: For every $\alpha \in A$, let $p_\alpha$ denote the restriction $\pi_\alpha|X : X \to X_\alpha$. A subset $C \subset X$ will be called *cylindric* if $C = p_\alpha^{-1}(C_\alpha)$ for some $\alpha \in A$ and some set $C_\alpha \subset X_\alpha$. In this case the set $C$ be called *$\alpha$-cylindric*. Since the projection $p_\alpha : X \to X_\alpha$ is open and surjective, $\mathrm{cl}_X(p_\alpha^{-1}(C_\alpha)) = p_\alpha^{-1}(\mathrm{cl}_{X_\alpha}(C_\alpha))$. Since $X \subset \lim \mathcal{S}$, open cylindric subsets form a base of the topology of $X$. A subset $U \subset X$

is *functionally open* if $U = f^{-1}(V)$ for some continuous function $f : X \to \mathbb{R}$ and some open set $V \subset \mathbb{R}$.

**Claim 3.2.** *Each functionally open subset of $X$ is cylindric.*

Proof: A functionally open set $U$ can be written as the countable union $U = \bigcup_{n\in\omega} U_n$ of open sets such that $U_n \subset \mathrm{cl}_X(U_n) \subset U_{n+1}$ for all $n \in \omega$. For every $n \in \omega$ consider the family $\mathcal{U}_n$ of open cylindric subsets of $U_n$. Since $X$ is weakly Lindelöf, for the open cover $\mathcal{U}_n \cup \{X \setminus \mathrm{cl}_X(U_{n-1})\}$ of $X$ there is a countable subfamily $\mathcal{V}_n \subset \mathcal{U}_n$ such that $(\bigcup \mathcal{V}_n) \cup (X \setminus \mathrm{cl}_X(U_{n-1}))$ is dense in $X$ and hence $\mathrm{cl}_X(U_{n-1}) \subset \mathrm{cl}_X(\bigcup \mathcal{V}_n) \subset \mathrm{cl}_X(U_n) \subset U$.

For the countable family $\bigcup_{n\in\omega} \mathcal{V}_n$ of cylindric sets, find an index $\alpha \in A$ such that each set $V \in \mathcal{V}$ is $\alpha$-cylindric. For every $n \in \omega$ consider the closure $F_n$ of the open set $p_\alpha(\bigcup \mathcal{V}_n)$ in $X_\alpha$ and observe that

$$U_{n-1} \subset \mathrm{cl}_X(U_{n-1}) \subset \mathrm{cl}_X(\bigcup \mathcal{V}_n) = p_\alpha^{-1}(F_n) \subset \mathrm{cl}_X(U_n) \subset U.$$

Then for the union $F = \bigcup_{n\in\omega} F_n$ we get

$$U = \bigcup_{n\in\omega} U_n \subset \bigcup_{n\in\omega} p_\alpha^{-1}(F_n) = p_\alpha^{-1}(F) \subset U,$$

which means that the set $U = p_\alpha^{-1}(F)$ is $\alpha$-cylindric. □

Now let $f : X \to Y$ be any map to a second countable space $Y$. Fix a countable base $\mathcal{B}$ of the topology of the space $Y$. Each set $U \in \mathcal{B}$ is functionally open and so is its preimage $f^{-1}(U) \subset X$. By Claim 3.2, the set $f^{-1}(U)$ is cylindric. Since the index set $A$ is $\omega$-directed, there is an index $\alpha \in A$ such that each set $f^{-1}(U)$, $U \in \mathcal{B}$, is $\alpha$-cylindric. Let $s : X_\alpha \to X$ be any (possibly discontinuous) section of the projection $p_\alpha : X_\alpha \to X$ and let $f_\alpha = f \circ s : X_\alpha \to Y$. For each $U \in \mathcal{B}$ the preimage $f_\alpha^{-1}(U) = p_\alpha(f^{-1}(U))$ is open in $X_\alpha$. Consequently, the map $f_\alpha : X_\alpha \to Y$ is continuous. It is clear that $f = f \circ s \circ p_\alpha = f_\alpha \circ p_\alpha = f_\alpha \circ \pi_\alpha|X$.

In order to show that $X$ is $C$-embedded in $\lim \mathcal{S}$, take any continuous function $f : X \to \mathbb{R}$ and find an index $\alpha \in A$ such that $f = f_\alpha \circ \pi_\alpha|X$ for some continuous function $f_\alpha : X_\alpha \to \mathbb{R}$. Then the continuous function $\bar{f} = f_\alpha \circ \pi_\alpha : \lim \mathcal{S} \to \mathbb{R}$ is the required continuous extension of $f$, witnessing that $X$ is $C$-embedded in $\lim \mathcal{S}$. Since the restriction $\pi_\alpha|X : X \to X_\alpha$ of each limit projection $\pi_\alpha : \lim \mathcal{S} \to X_\alpha$, $\alpha \in A$, is surjective, $X$ meets each open cylindric subset of $\lim \mathcal{S}$, which means that $X$ is dense in $\lim \mathcal{S}$.

The limit space $\lim \mathcal{S}$ is realcompact, being a closed subspace of the Tychonov product $\prod_{\alpha\in A} X_\alpha$ of second countable spaces. Since $X$ is $C$-embedded in $\lim \mathcal{S}$, the realcompact space $\lim \mathcal{S}$ is a Hewitt realcompactification of $X$ according to [7, 3.11.16]. □

The following theorem gives a spectral characterization of openly factorizable spaces.

**Theorem 3.3.** *A (weakly Lindelöf) topological space $X$ is openly factorizable (if and) only if $X$ is a dense subspace of the limit space $\lim \mathcal{S}$ of an open $\omega$-spectrum $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$ such that for every $\alpha \in A$ the restriction $\pi_\alpha|X : X \to X_\alpha$ of the limit projection is open and surjective.*

Proof: The "if" part follows immediately from Proposition 3.1. To prove the "only if" part, assume that a Tychonov space $X$ is openly factorizable. Let $A'$ be the set of all open continuous surjective maps $\alpha : X \to X_\alpha$ with $X_\alpha \subset \mathbb{R}^\omega$. The set $A'$ is partially preordered by the relation: $\alpha \le \gamma$ if there is a map $\pi^\gamma_\alpha : X_\gamma \to X_\alpha$ such that $\alpha = \pi^\gamma_\alpha \circ \gamma$. This map $\pi^\gamma_\alpha$ is necessarily open and surjective because the map $\alpha$ is open and surjective while $\gamma$ is continuous. Also the map $\pi^\gamma_\alpha$ is uniquely determined, which implies that $\pi^\gamma_\beta \circ \pi^\beta_\alpha = \pi^\gamma_\alpha$ for any $\alpha \le \beta \le \gamma$ in $A'$. This means that the relation $\le$ on $A'$ is transitive. The preorder $\le$ induces the equivalence relation $\cong$ on $A'$: $\alpha \cong \gamma$ if $\alpha \le \gamma$ and $\gamma \le \alpha$. Let $A$ be a subset of $A'$ intersecting each equivalence class in a single point. Then $A$ becomes a partially ordered set with respect to the order $\le$.

Let us show that the set $(A, \le)$ is $\omega$-directed. Given a countable subset $C \subset A$ consider the diagonal product $f = \Delta_{\gamma\in C}\gamma : X \to \prod_{\gamma\in C} X_\gamma$. Taking into account that $\prod_{\gamma\in C} X_\gamma$ is second countable and $X$ is openly factorizable, find an open surjective map $\alpha : X \to X_\alpha$ onto a second countable space $X_\alpha$ and a map $g : X_\alpha \to \prod_{\gamma\in C} X_\gamma$ such that $g \circ \alpha = f$. We can assume that $X_\alpha \subset \mathbb{R}^\omega$ and thus $\alpha \in A'$. Moreover, we can replace $\alpha$ by an equivalent map and assume that $\alpha \in A$. Let us show that $\alpha \ge \beta$ for each $\beta \in C$. Consider the projection $\mathrm{pr}_\beta : \prod_{\gamma\in C} X_\gamma \to X_\beta$ and observe that the equality $g \circ \alpha = f$ implies $(\mathrm{pr}_\beta \circ g) \circ \alpha = \mathrm{pr}_\beta \circ f = \beta$, which means that $\alpha \ge \beta$.

Now we see that $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$ is an open $\omega$-spectrum. Let $\pi_\alpha : \lim \mathcal{S} \to X_\alpha$, $\alpha \in A$, be the limit projections of this spectrum. The open surjective maps $\alpha \in A$ determine a map

$$E : X \to \lim \mathcal{S},\ E : x \mapsto (\alpha(x))_{\alpha\in A}$$

such that $\pi_\alpha \circ E = \alpha$ for every $\alpha \in A$. The surjectivity of the maps $\alpha \in A$ imply that the map $E : X \to \lim \mathcal{S}$ has dense image $E(X) \subset \lim \mathcal{S}$. Let us show that $E$ is a topological embedding. Given a point $x \in X$ and an open neighborhood $O_x \subset X$ of $x$ we should find an open set $U \subset \lim \mathcal{S}$ such that $E(x) \in U \cap E(X) \subset E(O_x)$. Since $X$ is Tychonov, there is a map $f : X \to [0, 1]$ such that $x \in f^{-1}(0, 1] \subset O_x$. The choice of the set $A$ guarantees that there is a map $\alpha : X \to X_\alpha$ in $A$ and a map $g : X_\alpha \to [0, 1]$ such that $g \circ \alpha = f$. Then the set $V = g^{-1}(0, 1]$ is open in $X_\alpha$ and its preimage $U = \pi^{-1}_\alpha(V)$ is open in $\lim \mathcal{S}$. It is easy to check that this set $U$ has the required property: $E(x) \in U \cap E(X) \subset E(O_x)$. □

Theorem 3.3 implies the following spectral characterization of compact openly factorizable spaces.

**Corollary 3.4.** *A compact Hausdorff space $X$ is openly factorizable if and only if $X$ is homeomorphic to the limit space $\lim \mathcal{S}$ of an open $\omega$-spectrum $\mathcal{S} = \{X_\alpha, \pi^\gamma_\alpha, A\}$.*

**Theorem 3.5.** *Let $X, Y$ be two openly factorizable spaces. If the product $X \times Y$ is weakly Lindelöf, then*

(1) *$X \times Y$ is openly factorizable;*
(2) *each map $f : X \times Y \to Z$ to a Tychonov space $Z$ extends to a map $\bar{f} : \upsilon X \times \upsilon Y \to \upsilon Z$.*

Proof: The space $X$, being an open continuous image of the weakly Lindelöf space $X \times Y$, is weakly Lindelöf. By Theorem 3.3, $X$ is a dense subspace of the limit space $\lim \mathcal{S}_X$ of an open $\omega$-spectrum $\mathcal{S}_X = \{X_\alpha, \pi_\alpha^\gamma, A\}$ such that the restrictions $\pi_\alpha|X : X \to X_\alpha$, $\alpha \in A$, of the limit projections are open and surjective. By Proposition 3.1, the limit space $\lim \mathcal{S}_X$ is a Hewitt realcompactification of $X$.

By the same reason, the Hewitt realcompactification $\upsilon Y$ of $Y$ can be identified with the limit space $\lim \mathcal{S}_Y$ of an open $\omega$-spectrum $\mathcal{S}_Y = \{Y_\alpha, p_\alpha^\gamma, B\}$ such that the restrictions $p_\alpha|Y : Y \to Y_\alpha$, $\alpha \in B$, of the limit projections are open and surjective.

On the product $A \times B$ consider the partial order: $(\alpha, \beta) \le (\alpha', \beta')$ if $\alpha \le \alpha'$ and $\beta \le \beta'$. It is easy to see that the partially ordered set $A \times B$ is $\omega$-directed. It follows that $X \times Y$ is a dense subspace of the limit space $\lim \mathcal{S}_X \times \lim \mathcal{S}_Y$ of the open $\omega$-spectrum

$$\mathcal{S} = \{X_\alpha \times Y_\beta, \pi_\alpha^\gamma \times p_\beta^\delta, A \times B\}$$

such that for every $(\alpha, \beta) \in A \times B$ the restriction $\pi_\alpha \times p_\beta : X \times Y \to X_\alpha \times Y_\beta$ is open and surjective. Since the product $X \times Y$ is weakly Lindelöf, we may apply Proposition 3.1 and Theorem 3.3 and conclude that the product $X \times Y$ is openly factorizable and $\lim \mathcal{S}_X \times \lim \mathcal{S}_Y = \upsilon X \times \upsilon Y$ is a Hewitt realcompactification of $X \times Y$.

Now take any map $f : X \times Y \to Z$ to a second countable space $Z$. By Proposition 3.1, there is an index $(\alpha, \beta) \in A \times B$ and a map $f_{(\alpha,\beta)} : X_\alpha \times Y_\beta \to Z$ such that $f = f_{(\alpha,\beta)} \circ (\pi_\alpha \times p_\beta)|X \times Y$. Then $\bar{f} = f_{(\alpha,\beta)} \circ (\pi_\alpha \times p_\beta)$ is a continuous extension of the map $f$ onto the product $\lim \mathcal{S}_X \times \lim \mathcal{S}_Y = \upsilon X \times \upsilon Y$.

Finally take any map $f : X \times Y \to Z$ to any Tychonov space $Z$. Identify the Hewitt realcompactification $\upsilon Z$ of $Z$ with a closed subspace of $\mathbb{R}^\kappa$ for a suitable cardinal $\kappa$. The preceding case ensures that the map $f$ extends to a map $\bar{f} : \upsilon X \times \upsilon Y \to \mathbb{R}^\kappa$. It follows that

$$\bar{f}(\upsilon X \times \upsilon Y) = \bar{f}(\overline{X \times Y}) \subset \overline{f(X, Y)} \subset \overline{Z} = \upsilon Z \subset \mathbb{R}^\kappa.$$

So $\bar{f}$ is a continuous map into $\upsilon Z$. □

Another operation preserving openly factorizable spaces is the operation of the topological sum.

**Proposition 3.6.** *The topological sum $\bigoplus_{\alpha \in A} X_\alpha$ of non-empty topological spaces $X_\alpha$, $\alpha \in A$, is openly factorizable if and only if each space $X_\alpha$, $\alpha \in A$, is openly factorizable and the index set $A$ is at most countable.*

Proof: To prove the "only if" part, assume that the topological sum $X = \bigoplus_{\alpha\in A} X_\alpha$ is openly factorizable. First we show that each space $X_\alpha$ is openly factorizable. Given any map $f_\alpha : X_\alpha \to Y$ to a second countable space $Y$, pick any point $y_0 \in Y$ and extend the map $f_\alpha$ to a map $f : X \to Y$ letting $f(x) = y_0$ for each $x \notin X_\alpha$. Since $X$ is openly factorizable, there are an open map $p : X \to Z$ onto a second countable space $Z$ and a map $g : Z \to Y$ such that $f = g \circ p$. Let $Z_\alpha = p(X_\alpha) \subset Z$, $p_\alpha = p|X_\alpha : X_\alpha \to Z_\alpha$ and $g_\alpha = g|Z_\alpha$. It is clear that $f_\alpha = g_\alpha \circ p_\alpha$. Since $X_\alpha$ is open in $X$, the restriction $p_\alpha = p|X_\alpha$ is an open map of $X_\alpha$ onto the second countable space $Z_\alpha$. This witnesses that the space $X_\alpha$ is openly factorizable.

Next, we show that $|A| \le \aleph_0$. Assuming the opposite, choose a subset $B \subset A$ of cardinality $|B| = \aleph_1$ and take any function $\xi : A \to \mathbb{R}$, which is injective on the set $B$. Now define a map $f : X \to \mathbb{R}$ letting $f(x) = \xi(\alpha)$ for each $\alpha \in A$ and $x \in X_\alpha$. Since $X$ is openly factorizable, there is an open map $p : X \to Z$ onto a second countable space $Z$ and a map $g : Z \to \mathbb{R}$ such that $f = g \circ p$. For every $\alpha \in A$ the image $Z_\alpha = p(X_\alpha)$ of the open set $X_\alpha \subset X$ is open in $Z$. For every $\alpha, \beta \in B$ the spaces $Z_\alpha$ and $Z_\beta$ are disjoint because $g(Z_\alpha) = \{\xi(\alpha)\} \ne \{\xi(\beta)\} = g(Z_\beta)$. Then the second countable space $Z$ contains an uncountable disjoint family of open non-empty subspaces $Z_\beta$, $\beta \in B$, which is a contradiction.

Now we prove the "if part". Assume that the index set $A$ is at most countable and each space $X_\alpha$, $\alpha \in A$, is openly factorizable. Given any map $f : X \to Y$ from the topological sum $X = \bigoplus_{\alpha\in A} X_\alpha$ to a second countable space $Y$, for every $\alpha \in A$ use the open factorizability of the space $X_\alpha$ to find an open map $p_\alpha : X_\alpha \to Z_\alpha$ onto a second countable space $Z_\alpha$ and a map $g_\alpha : Z_\alpha \to Y$ such that $f|X_\alpha = g_\alpha \circ p_\alpha$. The maps $g_\alpha$, $\alpha \in A$, compose an open map $p = \bigoplus_{\alpha\in A} p_\alpha : X \to Z$ of the topological sum $X = \bigoplus_{\alpha\in A} X_\alpha$ onto the topological sum $Z = \bigoplus_{\alpha\in A} Z_\alpha$, which is second countable because $|A| \le \aleph_0$. On the other hand, the maps $g_\alpha$ compose a continuous map $g = \bigoplus_{\alpha\in A} g_\alpha : Z \to Y$. Since $f = g \circ p$, we conclude that $X$ is openly factorizable. □

Propositions 2.4 and 3.6 imply:

**Corollary 3.7.** *The Aleksandrov compactification $\alpha(\bigoplus_{n\in\omega} X_n)$ of the topological sum $\bigoplus_{n\in\omega} X_n$ of compact openly factorizable spaces $X_n$, $n \in \omega$, is openly factorizable.*

## 4. $G_\delta$-points in openly factorizable spaces

In this section we establish a property of openly factorizable spaces that will help us to recognize spaces which are not openly factorizable.

A point $x$ of a topological space $X$ is called a *$G_\delta$-point* if the singleton $\{x\}$ is a $G_\delta$-subset of $X$. A subset $A$ of a topological space $X$ is called *sequentially closed* if for each sequence $\{a_n\}_{n\in\omega} \subset A$ that converges in $X$ the limit $\lim_{n\to\infty} a_n$ belongs to $A$.

**Theorem 4.1.** *The set of $G_\delta$-points of an openly factorizable space $X$ is sequentially closed in $X$.*

Proof: Let a point $x \in X$ be the limit of a sequence $(x_n)_{n=1}^\infty$ of $G_\delta$-points of $X$. Without loss of generality, the sequence $(x_n)$ consists of pairwise distinct points which are also distinct from $x$. Consider the compact subset

$$K = \{x\} \cup \{x_n : n \in \mathbb{N}\}$$

of $X$ and let $\xi : K \to [0,1]$ be the continuous function defined by $\xi(x) = 0$ and $\xi(x_n) = 2^{-n}$ for $n \in \mathbb{N}$. By the Tietze-Urysohn Theorem and the normality of the Stone-Čech extension $\beta X$, the map $\xi : K \to [0,1]$ admits a continuous extension $\bar{\xi} : \beta X \to [0,1]$. For every $n \in \mathbb{N}$ consider the neighborhood

$$U_n = \{x \in X : |\bar{\xi}(x) - 2^{-n}| < 2^{-n-2}\}$$

of the point $x_n$ in $X$. Using the fact that $x_n$ is a $G_\delta$-point of the Tychonov space $X$, one can construct a continuous function $\eta_n : X \to [0, 2^{-n}]$ such that $\eta_n^{-1}(2^{-n}) = \{x_n\}$ and $\eta_n^{-1}((0, 2^{-n}]) \subset U_n$. It follows from $\sum_{n=1}^\infty \|\eta_n\| < \infty$ that the map $\eta = \sum_{n=1}^\infty \eta_n : X \to [0,1]$ is well-defined and continuous. Moreover, since the sets $U_n$, $n \in \mathbb{N}$, are pairwise disjoint, we conclude that $\eta(x_n) = \eta_n(x_n) = 2^{-n}$ for all $n \in \mathbb{N}$.

Now consider the map $f : X \to [0,1]^2$, $f : x \mapsto (\bar{\xi}(x), \eta(x))$, and observe that $f^{-1}(2^{-n}, 2^{-n}) = \{x_n\}$ for all $n \in \mathbb{N}$. Since the space $X$ is openly factorizable, there is an open surjective map $p : X \to Z$ onto a second-countable space $Z$ and a map $g : Z \to [0,1]^2$ such that $f = g \circ p$. Since the singleton $\{p(x)\}$ is a $G_\delta$-subset of $Z$, the preimage $G = p^{-1}(p(x))$ is a closed $G_\delta$-subset of $X$ containing no point $x_n$. We claim that $G = \{x\}$. Assuming the converse, choose a point $y \in G \setminus \{x\}$ and observe that $X \setminus K$ is an open neighborhood of $y$ in $X$. Since the map $p : X \to Z$ is open, the image $U = p(X \setminus K)$ is an open neighborhood of the point $p(y) = p(x)$. By the continuity of $p$, the sequence $(p(x_n))_{n=1}^\infty$ converges to $p(x)$. Consequently, there is a number $n \in \mathbb{N}$ with $p(x_n) \in U$. Then

$$(2^{-n}, 2^{-n}) = f(x_n) = g \circ p(x_n) \in g(U) = g \circ p(X \setminus K) = f(X \setminus K).$$

Since $f^{-1}(2^{-n}, 2^{-n}) = \{x_n\}$, we conclude that $x_n \in X \setminus K$, which is a contradiction. □

## 5. Scattered openly factorizable spaces

In this section we detect openly factorizable spaces in the class of scattered compacta.

A topological space $X$ is called *scattered* if each subspace of $X$ has an isolated point. A point $x$ of a topological space $X$ is called a *$P$-point* if for any neighborhoods $U_n \subset X$, $n \in \omega$, of $x$ the intersection $\bigcap_{n\in\omega} U_n$ is a neighborhood of $x$ in $X$.

**Theorem 5.1.** *A scattered compact Hausdorff space $X$ is openly factorizable if each point of $X$ is either a $G_\delta$-point or a $P$-point.*

Proof: This theorem will be proved by induction on the scattered height of $X$, defined as follows. For a subset $A \subset X$ let $A^{(1)}$ be the set of non-isolated points of $A$. Let $X^{(0)} = X$ and for every ordinal $\alpha$ let $X^{(\alpha)} = \bigcap_{\beta<\alpha}(X^{(\beta)})^{(1)}$. Since $X$ is scattered, $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. The smallest ordinal $\alpha$ with this property is called the *scattered height* of $X$. The scattered height of the space $X$ is equal to zero if and only if $X$ is empty.

So, the theorem is true for compact scattered spaces $X$ of scattered height 0. Assume that for some ordinal $\alpha$, the theorem is true for scattered compacta of scattered height $< \alpha$. Assume that $X$ has scattered height $\alpha$. This means that $X^{(\alpha)} = \emptyset$ but $X^{(\beta)} \neq \emptyset$ for any ordinal $\beta < \alpha$. Since $X$ is compact, $\alpha = \beta + 1$ is a successor ordinal. In this case $X^{(\beta)}$ is a non-empty finite set. First we consider the case $X^{(\beta)} = \{x\}$ is a singleton.

If $x$ is a $G_\delta$-point, then the compact space $X$ is first countable at $x$. Since scattered compacta are zero-dimensional, we can choose a decreasing sequence $(U_n)_{n\in\omega}$ of closed-and-open neighborhoods of $x$ in $X$ such that $\{x\} = \bigcap_{n\in\omega} U_n$. It follows that $X$ is the Aleksandrov compactification of the topological sum $\bigoplus_{n\in\omega} U_n \setminus U_{n-1}$ of the scattered compacta $U_n \setminus U_{n-1}$ with scattered height $< \alpha$. By the inductive assumption each space $U_n \setminus U_{n-1}$, $n \in \omega$, is openly factorizable. Then $X = \bigoplus_{n\in\omega} U_n \setminus U_{n-1}$ is openly factorizable by Corollary 3.7.

If $x$ is a $P$-point, then for each continuous function $f : X \to Y$ to a second countable space $Y$ the preimage $G = f^{-1}f(x)$ is a closed neighborhood of $x$. Since $X$ is zero-dimensional, we can choose a closed-and-open neighborhood $U \subset G$ of $x$ in the space $X$ and consider the quotient space $X/U$ which is a scattered compact space of scattered height $< \alpha$. By the inductive assumption, $X/U$ is openly factorizable. Consequently, there is an open surjective map $p : X/U \to Z$ onto a second countable space $Z$ and a map $g : Z \to Y$ such that $f = g\circ p\circ q$ where $q : X \to X/U$ is the quotient map. Then the open surjective map $p \circ q : X \to Z$ and the function $g : Z \to Y$ witness the open factorizability of the space $X$.

Now we consider the general case of finite set $X^{(\beta)}$. The scattered space $X$, being zero-dimensional, can be written as the disjoint sum $X = X_1 \cup \cdots \cup X_n$ of open-and-closed sets $X_i$ that meet the finite set $X^{(\beta)}$ in a unique point. By the preceding case, each space $X_i$ is openly factorizable and so is their disjoint union $X$. □

Theorems 4.1 and 5.1 imply the following characterization of openly factorizable scattered linearly ordered compacta.

**Corollary 5.2.** *A scattered linearly ordered compact space $X$ is openly factorizable if and only if each point $x \in X$ is either a $G_\delta$-point or a $P$-point.*

This corollary has another:

**Corollary 5.3.** *Any closed segment of ordinals $[0, \alpha]$ endowed with the order topology is openly factorizable.*

## 6. Some comments and open problems

In this section we discuss the relation of the class of openly factorizable compact spaces to other known classes of compact spaces and pose some open problems. The survey [19] provides the necessary information on various classes of compact spaces.

We recall that a compact space $X$ is called

- *Dugundji compact* if for each embedding $X \to Y$ to another compact space $Y$ there is a linear positive norm one operator $u : C(X) \to C(Y)$ extending continuous functions from $X$ to $Y$;
- *AE(0)-space* if each map $f : B \to X$ defined on a closed subspace $B$ of a zero-dimensional compact space $A$ can be extended to a continuous map $\bar{f} : A \to X$;
- *openly generated* if $X$ is homeomorphic to the limit $\lim \mathcal{S}$ of an open continuous $\omega$-complete $\omega$-spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\gamma, A\}$;
- *dyadic compact* if $X$ is a continuous image of the Cantor cube $\{0,1\}^\kappa$ for some cardinal $\kappa$;
- *$\kappa$-adic* if $X$ is a continuous image of some $\kappa$-metrizable compact space;
- *$\kappa$-metrizable* if $X$ admits a $\kappa$-metric.

We recall that a *$\kappa$-metric* on $X$ is a function assigning to each point $x \in X$ and a regular closed set $F \subset X$ a non-negative number $\rho(x, F)$ so that the following axioms hold:

(1) $\rho(x, F) = 0$ if and only if $x \in F$;
(2) $\rho(x, F) \geq \rho(x, F')$ for any regular closed sets $F \subset F'$ of $X$;
(3) for any regular closed set $F$ the function $\rho(\cdot, F) : x \mapsto \rho(x, F)$ is continuous with respect to the first argument;
(4) for any point $x \in X$ and a linearly ordered family $\mathcal{F}$ of regular closed subsets of $X$, we get $\rho(x, \overline{\bigcup \mathcal{F}}) = \inf_{F \in \mathcal{F}} \rho(x, F)$.

By the classical result of Haydon [10], the classes of Dugundji and AE(0)-compacta coincide. By [21], the classes of openly generated and $\kappa$-metrizable compacta coincide. It is well-known that each compact topological group is Dugundji compact. Each Dugundji compact is openly generated and each openly generated compact space of weight $\leq \aleph_1$ is Dugundji [21]. Each $\kappa$-adic compact space has countable cellularity [21]. The hyperspace $\exp(\{0,1\}^{\aleph_2})$ is openly generated but not Dugundji, see [21], [20].

The spectral characterization of openly factorizable spaces from Corollary 3.4 implies that each openly generated compact space is openly factorizable. The simplest example of an openly factorizable compact space which is not openly generated is the ordinal space $[0, \omega_1]$. It is not openly generated because has uncountable cellularity. By the same reason, $[0, \omega_1]$ is not $\kappa$-adic.

Thus we have the following chain of implications:

$$\begin{array}{ccccccccccc}
 & & & & \text{dyadic} & \Rightarrow & \kappa\text{-adic} & & & & \\
 & & & & \Uparrow & & \Uparrow & & & & \\
\begin{matrix}\text{compact}\\ \text{topological}\\ \text{group}\end{matrix} & \Rightarrow & \begin{matrix}\text{Dugundji}\\ \text{compact}\end{matrix} & \Leftrightarrow & \text{AE(0)-compact} & \Rightarrow & \kappa\text{-metrizable} & \Leftrightarrow & \begin{matrix}\text{openly}\\ \text{generated}\end{matrix} & \Rightarrow & \begin{matrix}\text{openly}\\ \text{factorizable}\end{matrix}
\end{array}$$

Let us observe that the classes of openly generated and openly factorizable compact spaces are preserved by open normal functors in the sense of Shchepin [21], see also [22]. This allows us to construct many openly factorizable compacta failing to be Dugundji compact.

There is another chain of important classes of compact spaces, that is "orthogonal" to the chain of classes considered above.

We recall that a compact space $X$ of weight $\kappa$ is

(1) *Corson compact* if $X$ embeds into the $\Sigma$-product of lines

$$\Sigma = \{(x_\alpha) \in \mathbb{R}^\kappa : |\{\alpha \in \kappa : x_\alpha \neq 0\}| \leq \aleph_0\} \subset \mathbb{R}^\kappa;$$

(2) *Eberlein compact* if $X$ embeds into the subspace

$$\Sigma_0 = \{(x_\alpha) \in \mathbb{R}^\kappa : \forall\, \varepsilon > 0\ |\{\alpha \in \kappa : |x_\alpha| < \varepsilon\}| < \aleph_0\} \subset \mathbb{R}^\kappa;$$

(3) *Valdivia compact* if $X$ embeds into $\mathbb{R}^\kappa$ so that $X \cap \Sigma$ is dense in $X$.

Those properties relate as follows:

$$\text{Eberlein compact} \ \Rightarrow\ \text{Corson compact} \ \Rightarrow\ \text{Valdivia compact}.$$

Each Eberlein compact with countable cellularity is metrizable [1]. So the classes of Eberlein compacta and $\kappa$-adic compacta intersect by the class of metrizable compacta.

**Problem 6.1.** *Is each openly factorizable Eberlein* (*or Corson*) *compact space metrizable?*

According to [12] or [13], a scattered linearly ordered compact space $X$ is Valdivia compact if and only if $X$ has weight $\leq \aleph_1$, each non-$G_\delta$-point of $X$ is isolated from one side, and each closed first-countable subset of $X$ is metrizable. This characterization combined with Corollary 5.2 implies:

**Corollary 6.2.** *If a scattered linearly ordered compactum is Valdivia compact, then it is openly factorizable.*

On the other hand,

- the compactified long line of weight $\aleph_1$ is Valdivia compact but is not openly factorizable;
- the one-point compactification $\alpha\aleph_1$ of a discrete space of cardinality $\aleph_1$ is scattered Eberlein (and thus Valdivia) compact but it is not openly factorizable (by Theorem 4.1);

- the space made from $[0, \omega_1] \oplus [0, \omega]$ by collating $\omega_1$ and $\omega$ is neither Valdivia compact nor openly factorizable;
- the space made from $[0, \omega_1] \oplus [0, \omega_1]$ by collating the points $\omega_1$ is openly factorizable (by Theorem 5.1) but is not Valdivia compact.

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