# A Hajós type result on factoring finite abelian groups by subsets II

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*Abstract.* It is proved that if a finite abelian group is factored into a direct product of lacunary cyclic subsets, then at least one of the factors must be periodic. This result generalizes Hajós's factorization theorem.

*Keywords:* factorization of finite abelian groups, periodic subset, cyclic subset, Hajós's theorem

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## 1. Introduction

Let $G$ be a finite abelian group written multiplicatively with identity element $e$. Let $A_1, \ldots, A_n$ be subsets of $G$. We form the list of elements

$$a_1 \cdots a_n,\ a_1 \in A_1, \ldots, a_n \in A_n. \tag{1}$$

This list contains $|A_1| \cdots |A_n|$ elements. The *product* $A_1 \cdots A_n$ is defined to be the set of all elements on the list (1). If the elements on the list (1) are distinct, that is, if

$$a_1 \cdots a_n = a_1' \cdots a_n',\ a_1, a_1' \in A_1, \ldots, a_n, a_n' \in A_n$$

always implies $a_1 = a_1', \ldots, a_n = a_n'$, then we say that the product $A_1 \cdots A_n$ is *direct*. If the product $A_1 \cdots A_n$ is direct and it is equal to $G$, then we say that $G$ is *factored* into subsets $A_1, \ldots, A_n$. We also express this fact by saying that the equation $G = A_1 \cdots A_n$ is a *factorization* of $G$. Clearly, $G = A_1 \cdots A_n$ is a factorization of $G$ if and only if each element $g$ of $G$ is uniquely expressible in the form

$$g = a_1 \cdots a_n,\ a_1 \in A_1, \ldots, a_n \in A_n.$$

A subset $A$ of $G$ is called *normalized* if $e \in A$. A factorization $G = A_1 \cdots A_n$ is called normalized if each $A_i$ is a normalized subset of $G$. In this paper mainly normalized factorizations will appear.

If $G = A_1 \cdots A_n$ is a normalized factorization of $G$ and $A_n = \{e\}$, then clearly $G = A_1 \cdots A_{n-1}$ is also a normalized factorization of $G$. In other words the factors that are equal to $\{e\}$ can be cancelled from each normalized factorization of $G$ unless $G = \{e\}$. Throughout this note we assume that $G \neq \{e\}$ and that in each normalized factorization of $G$ there is no factor equal to $\{e\}$. Plainly $A_i = \emptyset$ is not possible and so we may assume that $|A_i| \geq 2$.

Let $a$ be an element of $G$ and let $r$ be a nonnegative integer. We form the list of elements

$$e, a, a^2, \ldots, a^{r-1}. \tag{2}$$

We assume that $|a| \geq r$ since otherwise there are repetition among the elements (2). In other words we assume that the elements on the list (2) are distinct. The set of elements on the list (2) of $G$ is called a *cyclic* subset of $G$. We will use the bracket notation $[a, r]$ for this subset of $G$. If $r = 1$, then $[a, r] = \{e\}$ independently of the choice of the element $a$. If $r = 0$, then $[a, r] = \emptyset$ independently of the choice of the element $a$.

In order to solve a long standing geometric conjecture of H. Minkowski, G. Hajós [2] proved the following theorem.

**Theorem 1.** *Let $G$ be a finite abelian group and let $G = A_1 \cdots A_n$ be a factorization of $G$, where each $A_i$ is a cyclic subset of $G$. Then at least one of the factors $A_1, \ldots, A_n$ is a subgroup of $G$.*

This note deals with possible extensions of this theorem. Let $a$, $d$ be elements of $G$ and let $i$, $r$ be positive integers. We form the list of elements

$$e, a, a^2, \ldots, a^{i-1}, a^i d, a^{i+1}, \ldots, a^{r-1}. \tag{3}$$

We assume that $|a| \geq r$ and $1 \leq i \leq r-1$ and further we assume that

$$a^i d \notin \{e, a, a^2, \ldots, a^{i-1}\} \cup \{a^{i+1}, \ldots, a^{r-1}\}.$$

The set of elements on the list (3) is called a *distorted cyclic* subset of $G$. In this note distorted cyclic subsets in the form

$$\{e, a, a^2, \ldots, a^{r-2}, a^{r-1}d\}$$

will mainly appear. The following result has been proved by Sands [3]. In fact he proved a more general result but we cite only a special case.

**Theorem 2.** *Let $G$ be a finite abelian group and let $G = A_1 \cdots A_n$ be a factorization of $G$, where each $A_i$ is a distorted cyclic subset of $G$. Then at least one of the factors $A_1, \ldots, A_n$ is a subgroup of $G$.*

As a cyclic subset is always a distorted cyclic subset, Theorem 2 can be considered to be an extension of Theorem 1.

Let $g$ be an element of $G$ and let $[a, r]$, $[a, s]$ be cyclic subsets of $G$. A subset of $G$ in the form

$$[a, r] \cup g[a, s]$$

is called a *lacunary cyclic* subset of $G$. If $[a, r] \cap g[a, s] \neq \emptyset$, then $a^i = ga^j$ for some $i$, $j$, $0 \leq i \leq r-1$, $0 \leq j \leq s-1$ and so $g \in \langle a \rangle$. In this case the lacunary cyclic subset $[a, r] \cup g[a, s]$ can be written in the form $[a, r'] \cup a^k[a, s']$ for some integers $r'$, $s'$, $k$. In addition we may choose these integers such that $[a, r'] \cap a^k[a, s'] = \emptyset$. In the rest of this note we will choose the notation such that for the lacunary cyclic subset $[a, r] \cup g[a, s]$, $[a, r] \cap g[a, s] = \emptyset$ holds. As a consequence the number of the elements of the lacunary cyclic subset $[a, r] \cup g[a, s]$ is $r + s$. If $s = 0$, then $[a, r] \cup g[a, s]$ reduces to the cyclic subset $[a, r]$. If $s = 1$, then $[a, r] \cup g[a, s]$ reduces to the distorted cyclic subset

$$\{e, a, a^2, \ldots, a^{r-1}, a^r d\}.$$

Corrádi and Szabó [1] have proved the following theorem.

**Theorem 3.** *Let $G$ be a finite abelian group of odd order and let $G = A_1 \cdots A_n$ be a factorization of $G$, where each $A_i$ is a lacunary cyclic subset of $G$. Then at least one of the factors $A_1, \ldots, A_n$ is a subgroup of $G$.*

From the remarks before Theorem 3 it follows that Theorem 3 is an extension of Theorem 2 for finite abelian groups of odd order. However, examples exhibited in [1] show that Theorem 3 cannot be extended for finite abelian groups of even order.

A subset $A$ of $G$ is defined to be *periodic* if there is an element $g$ of $G$ such that $g \neq e$ and $Ag = A$. Sands [3] established the following about periodic distorted cyclic subsets.

**Theorem 4.** *Let $G$ be a finite abelian group and let $G = AB$ be a factorization of $G$, where $A$ is a distorted cyclic subset of $G$. If $A$ is periodic, then $A$ must be a subgroup of $G$.*

This result implies that the next theorem is an equivalent formulation of Theorem 2.

**Theorem 5.** *Let $G$ be a finite abelian group and let $G = A_1 \cdots A_n$ be a factorization of $G$, where each $A_i$ is a distorted cyclic subset of $G$. Then at least one of the factors $A_1, \ldots, A_n$ is a periodic subset of $G$.*

We will prove the following variant of Theorem 5.

**Theorem 6.** *Let $G$ be a finite abelian group and let $G = A_1 \cdots A_n$ be a factorization of $G$, where each $A_i$ is a lacunary cyclic subset of $G$. Then at least one of the factors $A_1, \ldots, A_n$ is a periodic subset of $G$.*

## 2. The result

In this section we present a proof of Theorem 6.

Proof: Assume on the contrary that there is a finite abelian group $G$ and a factorization $G = A_1 \cdots A_n$, where each $A_i$ is a non-periodic lacunary cyclic subset of $G$. Of course we assume that neither $G$ nor any of the factor is equal to $\{e\}$.

Our goal is to prove that if there exists a counter-example $G = A_1 \cdots A_n$, then there exists a counter-example $G = A'_1 \cdots A'_m$ such that each $A'_i$ is a distorted cyclic subset of $G$ that is not periodic. Such an example would contradict Theorem 5, and hence there can be no counter-example at all.

We thus need to demonstrate that each $A_i$ that is not a distorted cyclic subset can be factored into a direct product of such subsets or can be replaced by such a subset. The replacing subsets should not be periodic. To verify that it suffices, by Theorem 4, to verify that they do not form a subgroup. When $n \geq 2$ with no loss of generality we can choose $i = 1$ and set $A = A_1$ and $B = A_2 \cdots A_n$.

Let us turn to the details. If $n = 1$, then $G = A_1$. As $G \neq \{e\}$, $A_1$ is periodic. This contradiction gives that $n \geq 2$.

The factorization $G = A_1 \cdots A_n$ can be written in the form $G = AB$, where $A = A_1$ and $B = A_2 \cdots A_n$. We have assumed that $|A_i| \geq 2$ and so $|A| \geq 2$ and $|B| \geq 2$.

If $|A| = 2$, then $A = A_1$ is a cyclic subset of $G$. In this case we do nothing with $A_1$. For the remaining part of the proof we assume that $|A| \geq 3$.

Let $A = [a, r] \cup g[a, s]$. Multiplying the factorization $G = AB$ by $g^{-1}$ we get the factorization $G = Gg^{-1} = (Ag^{-1})B$. Note that $Ag^{-1} = [a, s] \cup g^{-1}[a, r]$ is also a lacunary cyclic subset of $G$. Clearly, if $Ag^{-1}$ is periodic, then $A$ is periodic too. This shows that the roles of $r$ and $s$ can be reversed in a counter-example. In the remaining part of the proof we assume that $r \geq s$.

In the $r = s$ case the computation

$$\begin{aligned} A &= [a, r] \cup g[a, s] \\ &= [a, r] \cup g[a, r] \\ &= \{e, g\}[a, r] \\ &= [g, 2][a, r] \end{aligned}$$

shows that $A$ is a direct product of two cyclic subsets. If $[g, 2]$ is a subgroup of $G$, then we get the contradiction that $A = A_1$ is periodic. Similarly, if $[a, r]$ is a subgroup of $G$, then we get the contradiction that $A = A_1$ is periodic. Therefore, in the $r = s$ special case the factor $A = A_1$ is a direct product of two non-subgroup cyclic subsets. Thus for the remaining part of the proof we assume that $r > s$.

Let us turn to the $|A| = 3$ case. Now $r + s = 3$, $r > s \geq 0$ and so either $s = 0$ or $s = 1$. If $s = 0$, then $A = A_1$ is a non-periodic cyclic subset of $G$. By Theorem 4, $A = A_1$ is a non-subgroup cyclic subset of $G$. In this case we do nothing with $A_1$. If $s = 1$, then $A = A_1$ is a non-periodic distorted cyclic subset of $G$. By Theorem 4, $A = A_1$ is a non-subgroup distorted cyclic subset of $G$. In this case again we do nothing with $A_1$.

The $|A| = 4$ case is similar. The factor $A = A_1$ is either a non-subgroup cyclic subset of $G$ or a non-subgroup distorted cyclic subset of $G$ and we do nothing with $A_1$. For the remaining part of the proof we assume that $|A| \geq 5$.

If $s = 0$ or $s = 1$, then $A = A_1$ is a non-subgroup cyclic or distorted cyclic subset of $G$ and we do nothing with $A_1$. Suppose $s \geq 2$ and let

$$A' = \{e, a, a^2, \ldots, a^{r+s-2}, a^{s-1}g\}.$$

Define $d \in G$ by $g = a^r d$. Then

$$A' = \{e, a, a^2, \ldots, a^{r+s-2}, a^{r+s-1}d\}.$$

We claim that in the factorization $G = AB$ the factor $A$ can be replaced by $A'$ to get the factorization $G = A'B$.

In order to prove the claim note that the factorization $G = AB$ implies that the sets

$$eB,\ aB, \ldots, a^{r-1}B,\ gB,\ gaB, \ldots, ga^{s-1}B \tag{4}$$

form a partition of $G$. Multiplying the factorization $G = AB$ by $a$ we get the factorization $G = Ga = (Aa)B$. (Here $Aa$ is not a normalized subset of $G$. This is the only factorization in the paper which is not normalized.) Hence the sets

$$aB,\ a^2B, \ldots, a^rB,\ gaB,\ ga^2B, \ldots, ga^sB \tag{5}$$

form a partition of $G$. Comparing partitions (4) and (5) provides that

$$eB \cup gB = a^rB \cup ga^sB. \tag{6}$$

If $gB \cap ga^sB \neq \emptyset$, then $B \cap a^sB \neq \emptyset$. This violates (4) as $2 \leq s < r$. Thus $gB \cap ga^sB = \emptyset$. From (6) it follows that $gB \subset a^rB$. Both sets are of the same size and so $gB = a^rB$. Replacing $gB$ by $a^rB$ in (4) in the following way

$$eB,\ aB, \ldots, a^{r-1}B,\ \underbrace{gB}_{a^rB},\ \underbrace{gaB}_{a^{r+1}B}, \ldots, \underbrace{ga^{s-2}B}_{a^{r+s-2}B}, ga^{s-1}B$$

gives that the sets

$$eB,\ aB, \ldots, a^{r-1}B,\ a^rB,\ a^{r+1}B, \ldots, a^{r+s-2}B,\ ga^{s-1}B$$

form a partition of $G$. This means that $G = A'B$ is a factorization of $G$ as we claimed.

As $G = A'B$ is a factorization of $G$, the elements

$$e, a, a^2, \ldots, a^{r+s-2}, a^{r+s-1}d$$

are distinct. In particular

$$a^{r+s-1}d \notin \{e, a, a^2, \ldots, a^{r+s-2}\}$$

and so $A'$ is a distorted cyclic subset of $G$.

Suppose $s \geq 2$ and let

$$A'' = [a, r+s] = \{e, a, a^2, \ldots, a^{r+s-2}, a^{r+s-1}\}.$$

We claim that in the factorization $G = AB$ the factor $A$ can be replaced by $A''$ to get the factorization $G = A''B$.

In order to prove the claim let us replace $gB$ by $a^r B$ in (4) in the following way

$$eB,\ aB, \ldots, a^{r-1}B,\ \underbrace{gB}_{a^r B},\ \underbrace{gaB}_{a^{r+1}B}, \ldots,\ \underbrace{ga^{s-2}B}_{a^{r+s-2}B},\ \underbrace{ga^{s-1}B}_{a^{r+s-1}B}.$$

Therefore the sets

$$eB,\ aB, \ldots, a^{r-1}B,\ a^r B,\ a^{r+1}B, \ldots, a^{r+s-2}B,\ a^{r+s-1}B$$

form a partition of $G$. This means that $G = A''B$ is a factorization of $G$ as we claimed.

In order to simplify the notations we set $t = r + s$. Now

$$\begin{aligned} A' &= \{e, a, a^2, \ldots, a^{t-2}, a^{s-1}g\} \\ &= \{e, a, a^2, \ldots, a^{t-2}, a^{t-1}d\}. \end{aligned}$$

We claim that $A'$ is not a subgroup of $G$.

To verify the claim first note that from the factorization $G = A''B$, it follows that the elements $e, a, a^2, \ldots, a^{t-1}$ must be distinct. Next assume on the contrary that $A'$ is a subgroup of $G$. Let us consider the product $a \cdot a^{t-2}$. Either

$$a \cdot a^{t-2} \in \{e, a, \ldots, a^{t-2}\}$$

or

$$a \cdot a^{t-2} \in \{a^{t-1}d\}.$$

In the first case we get $a^{t-1} = a^i$ for some $i$, $0 \leq i \leq t-2$. Therefore $a^{t-1-i} = e$. This is contradiction since the elements $e, a, a^2, \ldots, a^{t-1}$ are distinct.

In the second case $a^{t-1} = a^{t-1}d$ and so $d = e$. This means that $A = A'$. It was assumed that $A$ is not periodic. By Theorem 4, $A$ is not a subgroup. Thus $A'$ is a not a subgroup of $G$ as we claimed.

We may summarize the above argument by saying that in the factorization $G = A_1 A_2 \cdots A_n$ the factor $A_1$ is either a direct product of non-subgroup cyclic subsets of $G$ or can be replaced by a non-subgroup distorted cyclic subset of $G$. It may happen that the distorted cyclic subset is simply a cyclic subset. The essential point is that it is not a subgroup.

We may repeat this replacement in connection with each $A_i$ factor in the factorization and we get a factorization $G = A'_1 \cdots A'_m$, where each $A'_i$ is a non-subgroup distorted cyclic subset of $G$. (The index $m$ is not a typographical error. When we replace the factor $A_i$ it may happen that $A_i$ is replaced by a product of two cyclic subsets. Therefore the number of the factors may change after the replacements.) The new factorization $G = A'_1 \cdots A'_m$ contradicts Theorem 2 and this contradiction completes the proof. $\square$

## 3. An example

In her or his report the anonymous referee writes the following. "For the sake of completeness I suggest to equip the paper with examples of direct products that do not fulfill the hypothesis of Theorem 6 but are close to the hypothesis. For example, can all the factors be lacunary with the exception of one? Can this be achieved for any $n$? Can the non-lacunary factor be of a simple structure, say $[a, r] \cup g[a, s] \cup h[a, t]$?"

We present an example motivated by these questions. Let $G$ be an abelian group with basis elements $x$, $y$, $z$, where $|x| = |y| = |z| = 4$. Set

$$\begin{aligned} A_1 &= \{e, x^2y^2z^2\} \cup xz^2\{e, x^2\} \cup x^2y\{e, y^2\} \cup y^2z\{e, z^2\}, \\ A_2 &= \{e, x\}, \\ A_3 &= \{e, y\}, \\ A_4 &= \{e, z\}. \end{aligned}$$

We claim that the product $A_1A_2A_3A_4$ is direct and it is equal to $G$. Further none of the factors is periodic. The sets $A_2$, $A_3$, $A_4$ are cyclic and so they are lacunary cyclic subsets too. The factor $A_1$ can be written in the form

$$[x^2y^2z^2, 2] \cup xz^2[x^2, 2] \cup x^2y[y^2, 2] \cup y^2z[z^2, 2].$$

This is not exactly the example the referee asks for since the elements $x^2y^2z^2$, $x^2$, $y^2$, $z^2$ are not equal.

We close this section with an open problem. The credit for this problem goes to the referee.

**Problem 1.** *Let $G$ be a finite abelian group and let $G = A_1A_2 \cdots A_n$ be a factorization of $G$ such that $A_1$ is in the form $[a, r] \cup g[a, s] \cup h[a, t]$ and $A_2, \ldots, A_n$ are lacunary cyclic subsets. Does it follow that at least one of the factors is periodic?*

The next problem is a simplified version of Problem 1 and it is motivated by the example above. An answer in the affirmative still would provide a generalization for Hajós's theorem.

**Problem 2.** *Let $G$ be a finite abelian group and let $G = A_1A_2 \cdots A_n$ be a factorization of $G$ such that $A_1$ is in the form $[a, r] \cup g[a, s] \cup h[a, t]$ and $A_2, \ldots, A_n$ are cyclic subsets. Does it follow that at least one of the factors is periodic?*

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