On the support of Fourier transform of weighted distributions

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Abstract. We give sufficient conditions for the support of the Fourier transform of a certain class of weighted integrable distributions to lie in the region $x_1 \geq 0$ and $x_2 \geq 0$.

Keywords: S'-convolution, weighted distribution spaces, Fourier transform

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1. Introduction and notation

The goal of this work is to present a partial generalization to the 2-dimensional case of a well known result proved by L. Schwartz in [11]. In that paper, L. Schwartz states for dimension $n = 1$, necessary and sufficient conditions for the Fourier transform of a distribution in an appropriate class to have its support on the half-line $x \geq 0$. To achieve this goal, L. Schwartz introduces a notion of product Tf where T is a distribution that can be represented as a distributional derivative of a continuous function, and f is a locally bounded variation function whose derivative is a measure. The model in mind for f is the one-dimensional Heaviside function H. This definition of product given by L. Schwartz is very natural and it is based on the Leibniz formula for the derivative of a product. However, when we consider the corresponding case for dimension $n > 1$, we need to manage distributions that are derivatives of order at most n of continuous functions, thus, it does not work to follow the scheme set by L. Schwartz because of the existence of terms that might not be possible to define. In this work, we approach our result by appealing to a theorem proved in [6], hence, we turn around the problem of considering a definition of product in the spirit of the one introduced by L. Schwartz in [11].

An important role in this paper is played by the $\mathcal{S}'$-convolution, a commutative operation for tempered distributions developed by Y. Hirata and H. Ogata [6] and R. Shiraishi [9] with the purpose of extending the validity of the Fourier exchange formula $\mathcal{F}(S * T) = \mathcal{F}(S)\mathcal{F}(T)$, where the product on the right-hand side must be understood in an appropriate sense that will be made precise later.

In order to determine sufficient conditions to ensure that the Fourier transform of a distribution T has its support in $x_1 \geq 0, \ldots, x_n \geq 0$, we need to impose some restrictions on T. These restrictions are related to the class of kernels that we need to consider in the setting of our problem. L. Schwartz in [11] considered the classical Hilbert kernel $p.v.\frac{1}{x}$. Thus, for the n-dimensional case we may consider the n-dimensional Hilbert kernel $p.v.\frac{1}{x_1} \otimes \cdots \otimes p.v.\frac{1}{x_n}$. L. Schwartz in [11] and J. Alvarez and C. Carton-Lebrun in [1] have characterized the class of tempered distributions that can be $\mathcal{S}'$-convolved with this kernel. The resulting class is a weighted version of the space of integrable distributions: the space $w_1 \cdots w_n \mathcal{D}'_{L^1}$, where $w_j = (1+x_j^2)^{1/2}$, $j = 1, \dots, n$.

It must be mentioned that there are other approaches to this problem (see, for example, [8]). Basically, the technique employed by them is to define an analytic representation of a distribution $T \in \mathcal{D}'_{L^p}$ on each complexified quadrant in order to characterize the spectrum of T by means of the support of the Fourier transform of their boundary values. In the present work, we consider a larger class of tempered distributions, namely, the family $w_1 \cdots w_n \mathcal{D}'_{L^1}$ (see [5, Proposition 5]) and our tool is the use of the operation of $\mathcal{S}'$-convolution and a Fourier exchange formula. It seems that it is possible to obtain a similar result as in [8, Theorem 6.3] for the larger class of distributions $T \in w_1 \cdots w_n \mathcal{D}'_{L^1}$ using the classical technique of analytic representation, as soon as we state the corresponding results for boundary behavior of the convolutions $T * K$, where K is a kernel in a class $\mathcal{K}$ that contains n-dimensional versions of the Poisson and conjugate Poisson one-dimensional kernels. Some of these boundary behaviors are analyzed in [5].

This paper is organized as follows: in Section 2 we include a brief account of the $\mathcal{S}'$-convolution, as well as several other results related to the space $w_1 \cdots w_n \mathcal{D}'_{L^1}$. In Section 3 we state our main result concerning the support of the Fourier transform of a distribution in this weighted class. For clarity, we will only approach the case $n = 2$.

A few words about notation: partial derivatives will be denoted as ∂^α, where α is a multi-index $(\alpha_1, \dots, \alpha_n)$. We will use the standard abbreviations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For a function g, we will indicate with $\check{g}$ the function $x \to g(-x)$. Given a distribution T, we will denote with $\check{T}$ the distribution $\varphi \to (T, \check{\varphi})$, where φ is an appropriate test function. The Fourier transform will be denoted as $\mathcal{F}$. The letter C will indicate a positive constant, possibly different at different occurrences.

2. Preliminary results

To introduce the notion of $\mathcal{S}'$-convolution that we use, we give a short review of the spaces of functions and distributions related to this notion (see [12] and [3]).

The space of integrable distributions $\mathcal{D}'_{L^1}$ is, by definition, the strong dual of the space $\dot{B}$ of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that $\partial^\alpha \varphi \to 0$ as $|x| \to \infty$, for each multi-index α. $\dot{B}$ is a closed subspace of the space B consisting of all smooth functions φ with the property that $\partial^\alpha \varphi$ is bounded for every multi-index α, endowed with the topology of uniform convergence in $\mathbb{R}^n$ of each derivative. C_0^∞ is dense in $\dot{B}$ but not in B. According to [12, p. 201], each $T \in \mathcal{D}'_{L^1}$ can be represented as $T = \sum_{\text{finite}} \partial^\alpha f_\alpha$, where $f_\alpha \in L^1$. Thus, we have the strict inclusions $\mathcal{E}' \subset \mathcal{D}'_{L^1} \subset \mathcal{S}'$.

It is also possible to consider $\mathcal{D}'_{L^1}$ as the strong dual of the space B, provided that we endow B with a topology that gives rise to the following notion of sequence convergence: a sequence $\{\varphi_j\}$ converges to φ if, for each multi-index α, one has $\sup_j \|\partial^\alpha \varphi_j\|_\infty < \infty$ and the sequence $\{\partial^\alpha \varphi_j\}$ converges to $\partial^\alpha \varphi$ uniformly on compact sets. If we denote as B_c the resulting topological space, it can be seen that C_0^∞, and so $\dot{B}$, is dense in B_c and hence $\mathcal{D}'_{L^1}$ is the dual of B_c ([12, p. 203]).

We use these spaces to define the notion of $\mathcal{S}'$-convolution.

Definition 1 ([9])**.** Given two tempered distributions T and S, we say that the $\mathcal{S}'$-convolution of T and S exists if $T(\check{S} * \varphi) \in \mathcal{D}'_{L^1}$ for every $\varphi \in \mathcal{S}$. When the $\mathcal{S}'$-convolution exists, the map $\mathcal{S} \to \mathbb{C}$

$$\varphi \longmapsto \left(T\left(\check{S} * \varphi\right), 1\right)_{\mathcal{D}'_{L^1}, B_c}$$

defines a tempered distribution which is denoted by $T * S$.

R. Shiraishi proved in [9] that this operation is commutative. Moreover, Definition 1 coincides with the classical definition in all the cases in which the latter makes sense.

Y. Hirata and H. Ogata in [6] introduced the $\mathcal{S}'$-convolution to extend the validity of the Fourier exchange formula

$$\mathcal{F}(T * S) = \mathcal{F}(T)\mathcal{F}(S) \tag{1}$$

originally proved by L. Schwartz for pairs of distributions in the Cartesian product $\mathcal{O}'_c \times \mathcal{S}'$ ([12]). Later, R. Shiraishi showed in [9] an equivalent definition of $\mathcal{S}'$-convolution, which is the one we are using here.

Remark 2. Y. Hirata and H. Ogata in [6] proved that if the $\mathcal{S}'$-convolution of two tempered distributions S, T is defined, then the formula (1) holds in the following sense: for any two δ-sequences $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$, the sequences $\{(\mathcal{F}(T) * \varphi_k)\mathcal{F}(S)\}_{k=1}^\infty$ and $\{\mathcal{F}(T)(\mathcal{F}(S) * \psi_k)\}_{k=1}^\infty$ converge in $\mathcal{D}'$ to the same distribution and this common limit is denoted by $\mathcal{F}(T)\mathcal{F}(S)$.

As in [6] one defines a δ-sequence as a sequence $\{\varphi_k\}_{k=1}^\infty$ of non-negative functions in C_0^∞ with the following properties:

1. $\operatorname{Supp} \varphi_k$ converges to 0 when $k \to \infty$;
2. $\int \varphi_k = 1$ for every k.

Mikusiński in [7] proposed another definition for the product of two distributions S and T: ST is the distributional limit (if there exists) of $\{(S * \varphi_k)(T * \psi_k)\}_{k=1}^\infty$ where $\{\varphi_k\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$ are arbitrary δ-sequences. Shiraishi and Itano proved in [10] that both definitions are equivalent.

We return to this point in the following section.

Following [11], we give the next definition. As mentioned before, for clarity we will restrict ourselves to the case $n = 2$.

Definition 3. Let us denote by δ_{x_j} the one-dimensional Dirac measure concentrated at 0 acting on the variable x_j, $j = 1, 2$. We define the following tempered distributions acting on test functions on $\mathbb{R}^2$:

$$K_1 = \frac{1}{2^2}\left[\delta - \frac{1}{(\pi i)} p.v.\frac{1}{x_1} \otimes \delta_{x_2} - \frac{1}{(\pi i)} \delta_{x_1} \otimes p.v.\frac{1}{x_2} + \frac{1}{(\pi i)^2} p.v.\frac{1}{x_1} \otimes p.v.\frac{1}{x_2}\right],$$

$$K_2 = \frac{1}{2^2}\left[\delta + \frac{1}{(\pi i)} p.v.\frac{1}{x_1} \otimes \delta_{x_2} - \frac{1}{(\pi i)} \delta_{x_1} \otimes p.v.\frac{1}{x_2} - \frac{1}{(\pi i)^2} p.v.\frac{1}{x_1} \otimes p.v.\frac{1}{x_2}\right],$$

$$K_3 = \frac{1}{2^2}\left[\delta + \frac{1}{(\pi i)} p.v.\frac{1}{x_1} \otimes \delta_{x_2} + \frac{1}{(\pi i)} \delta_{x_1} \otimes p.v.\frac{1}{x_2} + \frac{1}{(\pi i)^2} p.v.\frac{1}{x_1} \otimes p.v.\frac{1}{x_2}\right],$$

$$K_4 = \frac{1}{2^2}\left[\delta - \frac{1}{(\pi i)} p.v.\frac{1}{x_1} \otimes \delta_{x_2} + \frac{1}{(\pi i)} \delta_{x_1} \otimes p.v.\frac{1}{x_2} - \frac{1}{(\pi i)^2} p.v.\frac{1}{x_1} \otimes p.v.\frac{1}{x_2}\right].$$

A straight computation shows that

$$K_1 + K_2 + K_3 + K_4 = \delta,$$
$$K_1 - K_2 + K_3 - K_4 = \frac{1}{(\pi i)^2} p.v.\frac{1}{x_1} \otimes p.v.\frac{1}{x_2}.$$

From here, it is clear that any tempered distribution is $\mathcal{S}'$-convolvable with $K_1 + K_2 + K_3 + K_4$, however the same is not necessarily true for $K_1 - K_2 + K_3 - K_4$.

L. Schwartz studied in [11] the $\mathcal{S}'$-convolution with the one-dimensional Hilbert kernel $p.v.\frac{1}{x}$. For this purpose he introduced a weighted version of the space $\mathcal{D}'_{L^1}(\mathbb{R})$. Namely,

Definition 4 ([11])**.** Let $w(x) = (1 + x^2)^{1/2}$ for $x \in \mathbb{R}$. Then

$$w\mathcal{D}'_{L^1}(\mathbb{R}) = \left\{T \in \mathcal{D}'(\mathbb{R}) : w^{-1}T \in \mathcal{D}'_{L^1}(\mathbb{R})\right\}$$

with the topology induced by the map

$$\begin{aligned} w\mathcal{D}'_{L^1}(\mathbb{R}) &\longrightarrow \mathcal{D}'_{L^1}(\mathbb{R}) \\ T &\longmapsto w^{-1}T. \end{aligned}$$

L. Schwartz observed in [11] that the condition $T \in w\mathcal{D}'_{L^1}(\mathbb{R})$ should be viewed as the most general condition under which T and $p.v.\frac{1}{x}$ are $\mathcal{S}'$-convolvable.

J. Alvarez and C. Carton-Lebrun extended this result in [1] to the n-dimensional case in two directions: by considering the Riesz kernels $p.v.\frac{x_j}{|x|^{n+1}}$, $j = 1, \ldots, n$, and the n-dimensional Hilbert kernel. The appropriate kernel to consider in this work is the n-dimensional Hilbert kernel $p.v.\frac{1}{x_1} \otimes \cdots \otimes p.v.\frac{1}{x_n}$. The relevant weighted spaces related to this case are given in the following definition.

Definition 5 ([1]). Let $w_j = (1+x_j^2)^{1/2}$, $j = 1, \dots, n$. Then

$$w_1 \cdots w_n \mathcal{D}'_{L^1} = \left\{ T \in \mathcal{D}' : w_1^{-1} \cdots w_n^{-1} T \in \mathcal{D}'_{L^1} \right\}$$

with the topology induced by the map

$$\begin{array}{rcl} w_1 \cdots w_n \mathcal{D}'_{L^1} & \longrightarrow & \mathcal{D}'_{L^1} \\ T & \longmapsto & w_1^{-1} \cdots w_n^{-1} T. \end{array}$$

The space $w_1 \cdots w_n \mathcal{D}'_{L^1}$ is the largest space of tempered distributions for which the $\mathcal{S}'$-convolution with $p.v.\frac{1}{x_1} \otimes \cdots \otimes p.v.\frac{1}{x_n}$ exists. In fact,

Theorem 6 ([1]). *Let $T \in \mathcal{S}'$. Then, the following statements are equivalent:*

(a) $T \in w_1 \cdots w_n \mathcal{D}'_{L^1}$;
(b) *T is $\mathcal{S}'$-convolvable with* $p.v.\frac{1}{x_1} \otimes \cdots \otimes p.v.\frac{1}{x_n}$.

The proof of Theorem 6 is based in the following simple representation formula for distributions in the space $w_1 \cdots w_n \mathcal{D}'_{L^1}$:

Proposition 7 ([1], [2]). *Given $T \in \mathcal{D}'$, the following statements are equivalent:*

(a) $T \in w_1 \cdots w_n \mathcal{D}'_{L^1}$;
(b) $T = T_0 + \sum x_{j_1} \cdots x_{j_k} T_{j_1 \cdots j_k}$, *where* $T_0, T_{j_1 \cdots j_k} \in \mathcal{D}'_{L^1}$ *and the sum is taken over all the different k-tuples $(j_1, \dots, j_k)$ with $1 \le j_1 < \cdots < j_k \le n$, $1 \le k \le n$.*

L. Schwartz observed in [11] that $w\mathcal{D}'_{L^1}(\mathbb{R})$ coincides with the space $\mathcal{D}'_{L^1}(\mathbb{R}) + x\mathcal{D}'_{L^1}(\mathbb{R})$, so Proposition 7 can be considered as an extension of this result.

Now, concerning the tempered distributions K_j, $j = 1, 2, 3, 4$, considered in Definition 3 above, we obtain the following result:

Proposition 8. *Every $T \in w_1 w_2 \mathcal{D}'_{L^1}$ is $\mathcal{S}'$-convolvable with K_j, $j = 1, 2, 3, 4$.*

Proof: We need to prove that T is $\mathcal{S}'$-convolvable with $\delta_{x_1} \otimes p.v.\frac{1}{x_2}$ and $p.v.\frac{1}{x_1} \otimes \delta_{x_2}$. Since $w_1^{-1} w_2^{-1} T \in \mathcal{D}'_{L^1}$, for the first kernel it suffices to show that $w_1 w_2 (\delta_{x_1} \otimes p.v.\frac{1}{x_2})^{\vee} * \varphi \in B$ for each $\varphi \in \mathcal{S}$ because $\mathcal{D}'_{L^1}$ is closed under multiplication by functions in B.

Indeed

$$\begin{aligned} \left(\left(\delta_{x_1} \otimes p.v.\frac{1}{x_2} \right)^{\vee} * \varphi \right)(\xi_1, \xi_2) &= \lim_{\varepsilon \to 0} \int_{|y_2| > \varepsilon} \frac{\varphi(-\xi_1, y_2 - \xi_2)}{y_2} \, dy_2 \\ &= \lim_{\varepsilon \to 0} \left[\int_{\varepsilon < |y_2| < 1} + \int_{1 < |y_2|} \right] \\ &= \lim_{\varepsilon \to 0} \left[I_1 + I_2 \right]. \end{aligned}$$

Concerning the term I_1 we have

$$\begin{aligned}|I_1| &= \left|\int_{\varepsilon<|y_2|<1} \frac{\varphi(-\xi_1, y_2-\xi_2) - \varphi(-\xi_1,-\xi_2)}{y_2}\, dy_2\right| \\ &\le \int_{\varepsilon<|y_2|<1}\int_0^1 \left|\frac{\partial \varphi}{\partial \xi_2}(-\xi_1, ty_2-\xi_2)\right| dt\, dy_2.\end{aligned}$$

Using the fact that for $0 \le t \le 1$ and $|y_2| < 1$

$$w_2(\xi_2) \le C\left(1+|ty_2-\xi_2|^2\right)^{1/2}$$

we have

$$\begin{aligned}w_1(\xi_1)\, w_2(\xi_2)\, |I_1| &\le C \int_{\varepsilon<|y_2|<1}\int_0^1 \left(1+|\xi_1|^2\right)^{-1/2}\left(1+|\xi_1|^2\right) \\ &\quad\times \left(1+|ty_2-\xi_2|^2\right)^{1/2} \left|\frac{\partial \varphi}{\partial \xi_2}(-\xi_1, ty_2-\xi_2)\right| dt\, dy_2 \\ &\le C.\end{aligned}$$

Also,

$$\begin{aligned}&\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\left[w_1(\xi_1)\, w_2(\xi_2)\int_{\varepsilon<|y_2|<1}\frac{\varphi(-\xi_1, y_2-\xi_2)}{y_2}\, dy_2\right] \\ &= \sum_{0<\beta_1\le\alpha_1}\sum_{0<\beta_2\le\alpha_2} \partial_{\xi_1}^{\beta_1} w_1(\xi_1)\, \partial_{\xi_2}^{\beta_2} w_2(\xi_2) \\ &\qquad\times \int_{\varepsilon<|y_2|<1} \partial_{\xi_1}^{\alpha_1-\beta_1}\partial_{\xi_2}^{\alpha_2-\beta_2}\varphi(-\xi_1, y_2-\xi_2)\,\frac{dy_2}{y_2} \\ &\quad + w_1(\xi_1)\, w_2(\xi_2)\int_{\varepsilon<|y_2|<1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\varphi(-\xi_1, y_2-\xi_2)\,\frac{dy_2}{y_2}\end{aligned} \tag{2}$$

and noticing that

$$\partial_{\xi_1}^{\beta_1} w_1(\xi_1)\, \partial_{\xi_2}^{\beta_2} w_2(\xi_2) = W_{1,\beta_1}(\xi_1)\, W_{2,\beta_2}(\xi_2)\, w_1(\xi_1)\, w_2(\xi_2),$$

where $W_{1,\beta_1}(\xi_1)$, $W_{2,\beta_2}(\xi_2) \in B$, we can proceed in the same way as above to show that both terms in (2) are bounded.

For the term I_2 we can use the estimate

$$\begin{aligned}w_2(\xi_2) &\le C\left(1+|\xi_2-y_2|^2\right)^{1/2}\left(1+|y_2|^2\right)^{1/2} \\ &\le C\left(1+|\xi_2-y_2|^2\right)^{1/2}|y_2|\end{aligned}$$

if $|y_2| > 1$. Thus

$$\begin{aligned} & w_1(\xi_1)\, w_2(\xi_2)\, |I_2| \\ & \leq \int_{1<|y_2|} \left(1+|\xi_1|^2\right)^{1/2} \left(1+|\xi_2-y_2|^2\right)^{1/2} |\varphi(-\xi_1, y_2-\xi_2)|\, dy_2 \\ & \leq C. \end{aligned}$$

For the derivatives we proceed as before.

To manage the other kernel we use exactly the same techniques.

This completes the proof. □

3. Fourier transform of distributions in $w_1 w_2 \mathcal{D}'_{L^1}$

Let us denote by H_1 the classical Heaviside function in $\mathbb{R}^2$, that is,

$$H_1(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 > 0 \text{ and } x_2 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and let H_2, H_3 and H_4 the following modified versions of H_1:

$$H_2(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 < 0 \text{ and } x_2 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_3(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 < 0 \text{ and } x_2 < 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_4(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 > 0 \text{ and } x_2 < 0, \\ 0 & \text{otherwise.} \end{cases}$$

A straight computation shows that for $j = 1, 2, 3, 4$

$$\mathcal{F}(K_j) = H_j.$$

In the following proposition we consider the Fourier transform of distributions in $w_1 w_2 \mathcal{D}'_{L^1}$.

Proposition 9. *Given $T \in w_1 w_2 \mathcal{D}'_{L^1}$, $\mathcal{F}(T)$ can be represented (in many forms) as a distributional derivative*

$$\mathcal{F}(T) = \frac{\partial^2}{\partial x_1 \partial x_2} f,$$

where f is a continuous function, slowly increasing at infinity.

Proof: By Proposition 7 we can write

$$T = T_0 + x_1 T_1 + x_2 T_2 + x_1 x_2 T_{12},$$

where T_0, T_1, T_2 and T_{12} belong to $\mathcal{D}'_{L^1}$. Thus

$$\mathcal{F}(T) = \mathcal{F}(T_0) - \frac{1}{2\pi i}\frac{\partial}{\partial x_1}\mathcal{F}(T_1) - \frac{1}{2\pi i}\frac{\partial}{\partial x_2}\mathcal{F}(T_2) + \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial x_1 \partial x_2}\mathcal{F}(T_{12}).$$

Since the Fourier transform sends $\mathcal{D}'_{L^1}$ into the space of continuous functions slowly increasing at infinity, we have that $\mathcal{F}(T_0)$, $\mathcal{F}(T_1)$, $\mathcal{F}(T_2)$ and $\mathcal{F}(T_{12})$ satisfy these conditions.

Now, a standard procedure (see [3, p. 180]) allows us to conclude the desired result. □

We remark the fact that we can also obtain a similar result for the space $w_1 \cdots w_n \mathcal{D}'_{L^1}$ using Proposition 7.

According to Proposition 8 every $T \in w_1 w_2 \mathcal{D}'_{L^1}$ is $\mathcal{S}'$-convolvable with the kernels K_j, $j = 1, 2, 3, 4$. Since the Fourier exchange formula (1) is valid for any two tempered distributions that are $\mathcal{S}'$-convolvable, we have that

$$\begin{aligned} \mathcal{F}(T * K_j) &= \mathcal{F}(T)\mathcal{F}(K_j) \\ &= \mathcal{F}(T)\, H_j, \end{aligned} \tag{3}$$

$j = 1, 2, 3, 4$. This shows that the product $\mathcal{F}(T)H_j$ is defined in the sense described in Remark 2.

We are looking for sufficient conditions for the Fourier transform of $T \in w_1 w_2 \mathcal{D}'_{L^1}$ to have its support in the set $x_1 \geq 0$ and $x_2 \geq 0$. Thus, it seems natural to give an important role to the function $H_2 + H_3 + H_4$, for which the product $\mathcal{F}(T)[H_2 + H_3 + H_4]$ is defined.

Indeed, since T is $\mathcal{S}'$-convolvable with $K_2 + K_3 + K_4$ we have

$$\begin{aligned} \mathcal{F}(T * (K_2 + K_3 + K_4)) &= \mathcal{F}(T)\, \mathcal{F}(K_2 + K_3 + K_4) \\ &= \mathcal{F}(T)\, [\mathcal{F}(K_2) + \mathcal{F}(K_3) + \mathcal{F}(K_4)] \\ &= \mathcal{F}(T)\, [H_2 + H_3 + H_4]. \end{aligned} \tag{4}$$

Now, we prove the following result:

Theorem 10. *Let S be a distribution that can be written as $S = \frac{\partial^2}{\partial x_1 \partial x_2} f$, where f is a continuous function, slowly increasing at infinity, and the derivative is taken in $\mathcal{D}'$. Let us assume that the product $S[H_2 + H_3 + H_4]$ is defined in the sense described in Remark 2. Then, S has its support in the set $x_1 \geq 0$ and $x_2 \geq 0$ if $S[H_2 + H_3 + H_4] = 0$.*

Proof: Let us suppose that $S[H_2 + H_3 + H_4] = 0$. Thus, for every δ-sequence $\{\varphi_k\}_{k=1}^{\infty}$ we have

$$\lim_{k \to \infty} (S * \varphi_k)\,(H_2 + H_3 + H_4) = 0 \quad \text{in } \mathcal{D}'.$$

This implies that for every $\psi \in C_0^\infty$

$$\lim_{k\to\infty} \langle (S * \varphi_k)(H_2 + H_3 + H_4), \psi\rangle = 0.$$

Denote $C = \mathbb{R}^2 \setminus \{(x_1, x_2) : x_1 \geq 0,\, x_2 \geq 0\}$. Take a smooth and compactly supported function θ such that $\operatorname{supp}\theta \subset C$. Then, using the fact that $S * \varphi_k \to S$ in $\mathcal{D}'$ and that $\theta = \theta(H_2 + H_3 + H_4)$ we have

$$\begin{aligned}\langle S, \theta\rangle &= \lim_{k\to\infty} \langle S * \varphi_k, \theta\rangle \\ &= \lim_{k\to\infty} \langle S * \varphi_k, \theta (H_2 + H_3 + H_4)\rangle \\ &= \lim_{k\to\infty} \langle (S * \varphi_k)(H_2 + H_3 + H_4), \theta\rangle \\ &= 0.\end{aligned}$$

Therefore, S has its support in $\{(x_1, x_2) : x_1 \geq 0,\, x_2 \geq 0\}$. □

As an immediate consequence we obtain:

Corollary 11. *Let $T \in w_1 w_2 \mathcal{D}'_{L^1}$. Then, $\mathcal{F}(T)$ has its support in the set $x_1 \geq 0$ and $x_2 \geq 0$ if $T * (K_2 + K_3 + K_4) = 0$.*

Proof: If $T * (K_2 + K_3 + K_4) = 0$, then we can apply the Fourier exchange formula (1) to obtain

$$\mathcal{F}(T)\mathcal{F}(K_2 + K_3 + K_4) = 0$$

or

$$\mathcal{F}(T)\,[H_2 + H_3 + H_4] = 0$$

and by Theorem 10, $\mathcal{F}(T)$ has its support in the set $x_1 \geq 0$ and $x_2 \geq 0$. □

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