Uniqueness and non uniqueness of optimal maps in mass transport problem with not strictly convex cost

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Abstract. In the setting of the optimal transportation problem we provide some conditions which ensure the existence and the uniqueness of the optimal map in the case of cost functions satisfying mild regularity hypothesis and no convexity or concavity assumptions.

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1. Introduction

The wide number of applications of transport problem in several fields as economics, probability, statistics and engineering is one of the reasons of the great interest the problem awakened since its origin due to Monge [28] in 1781. The problem consists in finding a map y which carries one mass distribution into another (described by probability measures μ, ν respectively) and minimizes the total cost

$$\mathcal{C}(y) = \int_{\mathbb{R}^n} c(x - y(x))\, d\mu(x)$$

in the set of transport maps.

As cost function, Monge considered the euclidean distance $c(x) = \|x\|$ but even in this "natural" case the existence of an optimal transport map has been proved only two centuries later by Sudakov [35] (with a gap in the proof fixed by Ambrosio [3]), whereas it was known from the beginning that the solution could not be unique.

The quadratic case, that is $c(x) = \|x\|^2$, of relevant interest in fluid dynamics, was solved by Brenier [8] (see also [17] and [1] for a different approach) who proved the existence and the uniqueness of the optimal transport map. Later Gangbo and McCann ([20], [21]) generalized this result to the case of a cost c which is a strictly convex or strictly concave function of the distance $\|x\|$. If c is not strictly convex, the problem is not yet completely understood as pointed out by Ambrosio, Gigli and Savaré in their recent book ([4, Chapter 6]). A basic tool in Gangbo and McCann's approach is the global invertibility of ∇c as a consequence of the strict convexity or concavity. Later, several authors considered more general assumptions on cost functions $h : U \times V \longrightarrow \mathbb{R}$ where U, V are open subset of $\mathbb{R}^n$. We can summarize them as follows:

(Semi-concavity): the map $x \longrightarrow h(x,y)$ is locally semi-concave, uniformly in y;

(Twist condition): on its domain of definition the map $y \longrightarrow \frac{\partial h}{\partial x}(x,y)$ is injective for every x;

(see Section 2 or [11] for the definition of semi-concavity). These assumptions are satisfied, for example, by cost functions induced by Tonelli Lagrangians [19]. Beyond this case semi-concavity assumption is verified by every C^1-function while twist condition, in general, is not easy to check. Although some results (Theorems 3.1 and 3.2) can be applied to cost functions that are not semi-concave, the aim of this paper is not so much to generalize previous results as to find some analytical assumptions which guarantee the existence and the uniqueness of the optimal map. The cost functions we will consider are related to the special class of mappings

$$\mathcal{A}^+_{p,q}(\Omega) = \{w \in W^{1,p}(\Omega;\mathbb{R}^n) : \operatorname{adj} Dw \in L^q(\Omega;\mathbb{R}^{n\times n}), \det Dw > 0 \quad \text{a.e. on } \Omega\}$$

introduced by Ball [5] in the study of nonlinear elastic phenomena and whose regularity and invertibility properties were finely studied by Šverák [36] and Müller, Qi and Yan [29]. We consider this class of functions since in this setting the global invertibility of ∇c is independent of the convexity or concavity properties of the cost and allows us to consider continuous cost functions c with isolated singular points (i.e. points where ∇c fails to exist) and such that ∇c satisfies an injectivity condition just on the boundary of a suitable set.

The main results are contained in Section 3. We consider the c-subdifferential of a potential of the transport problem (see Section 2 for the definitions) and we look for the optimal transport map as its unique measurable selection. As a first step (Theorems 3.1 and 3.2) we investigate the mass which is carried in a not unique way and relate it to the noninjectivity set of ∇c. The uniqueness is a consequence of the approximate regularity of the potential or of the selection itself and from this we deduce the uniqueness of the optimal map even if the cost is not strictly convex or concave. More precisely we prove the result (Corollary 3.5) in the case of cost function $c \in C^2(\Omega)$ such that ∇c agrees with an homeomorphism on $\partial\Omega$ and the Hessian matrix D^2c has $n-1$ negative eigenvalues in Ω.

In Section 4 we provide an example and an application in an economical setting.

2. Definitions and preliminary results

Notations and definitions.

Throughout the paper n is an integer such that $n \geq 2$.

• We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the Hausdorff k-dimensional measure in $\mathbb{R}^n$. If $N \subset \mathbb{R}^n$, $\dim_H(N)$ will be the Hausdorff dimension of N, that is, $\dim_H(N) := \inf\{s \geq 0 : \mathcal{H}^s(N) = 0\}$.

• $B(x, r)$ is the open ball of center x and radius $r > 0$.

• Let E be a Lebesgue measurable set in $\mathbb{R}^n$. The density of E at $x \in E$ is defined by

$$\theta(E, x) := \lim_{r \to 0} \frac{\mathcal{H}^n(E \cap B(x, r))}{\mathcal{H}^n(B(x, r))}$$

if the limit exists.

• If M is an $n \times n$ real matrix we denote by $\rho(M)$ its rank.

• $D(E)$ will be the derived set of $E \subset \mathbb{R}^n$, that is the set of all accumulation points of E.

• Let $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function.

a) The subdifferential of h at x is the set

$$\partial^- h(x) = \{p \in \mathbb{R}^n : \liminf_{y \to x} \frac{h(y) - h(x) - p \cdot (y - x)}{|y - x|} \geq 0\}.$$

b) The set of reachable subgradients of h at x is

$$\nabla_* h(x) = \{\lim_m \nabla h(x_m) : h \text{ is differentiable at } \ x_m, x_m \to x\}.$$

c) The generalized gradient of h at x is the set

$$\partial h(x) = \{p \in \mathbb{R}^n : \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{h(y + tv) - h(y)}{t} \geq p \cdot v \ \text{ for all } \ v \in \mathbb{R}^n\}.$$

d) h is said to be regular at x if

$$\lim_{t \to 0^+} \frac{h(x + tv) - h(x)}{t} \ \text{ exists and}$$
$$\lim_{t \to 0^+} \frac{h(x + tv) - h(x)}{t} = \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{h(y + tv) - h(y)}{t}$$

for every $v \in \mathbb{R}^n$.

A detailed treatise of generalized gradients can be found in the book of Clarke [16].

We now give the definition of a semi-concave function.

Definition 2.1. Let A be an open set in $\mathbb{R}^n$ and $\omega : [0,+\infty) \longrightarrow [0,+\infty)$ a continuous, non-decreasing function such that $\omega(0) = 0$. A function $h : A \longrightarrow \mathbb{R}$ is said to be semi-concave in A with modulus ω if, for each $x \in A$, there exists a linear map $l_x : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$h(y) \leq h(x) + l_x(y - x) + \|y - x\| \, \omega(\|y - x\|)$$

for every $y \in A$.

Besides, $h : A \longrightarrow \mathbb{R}$ is said to be locally semi-concave if, for each $x \in A$, there exists an open neighborhood B_x of x such that h is semi-concave in B_x with a certain modulus.

Further related definitions and properties can be found in [11].

Throughout the paper, we deal with the concepts of approximate continuity and differentiability according to the definitions and properties one can find in the book of Giaquinta, Modica and Souček [22]. Here we recall just the definition of approximate differentiability we frequently use in the following.

Definition 2.2. Let A be a measurable set in $\mathbb{R}^n$ and $u : A \longrightarrow \mathbb{R}$ a measurable function. Suppose that $x \in \mathbb{R}^n$ is such that $\theta(A, x) > 0$. We say that u is approximately differentiable at x if there exists a linear map $l_x : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\operatorname{ap}\limsup_{\substack{y \to x \\ y \in A}} \frac{|u(y) - u(x) - l_x(y - x)|}{\|y - x\|} = 0.$$

We denote by $\operatorname{ap} Du(x)$ the approximate differential of u at x.

Another basic tool is the concept of selection of a set valued map.

Definition 2.3. Let X, Y be sets and $F : X \longrightarrow Y$ a set valued map. A single valued map $f : X \longrightarrow Y$ is called a selection of F if $f(x) \in F(x)$ for every $x \in X$.

We refer to the book of Repovš and Semenov [32] for further definitions and properties.

- We denote by $\mathcal{M}(\mathbb{R}^n)$ the space of non-negative Borel measures on $\mathbb{R}^n$ with finite total mass and compact support. If $\sigma \in \mathcal{M}(\mathbb{R}^n)$ we denote by $\operatorname{spt} \sigma$ the support of σ.

Definition 2.4. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$. We say that the Borel map $v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ pushes μ forward to ν and we write $v_\sharp \mu = \nu$ if $\mu[v^{-1}(B)] = \nu(B)$ for every Borel set $B \subset \mathbb{R}^n$.

- We denote by $\Delta(\mu, \nu)$ the set of all maps that push μ forward to ν.

The transportation problem.

The Monge's problem generalizes for a continuous cost in the following way.

Let $c : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function and $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ such that μ is absolutely continuous with respect to $\mathcal{L}^n$ and $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$. Besides, let $U, V \subset \mathbb{R}^n$ be bounded open sets such that $\operatorname{spt}\mu \subset U$ and $\operatorname{spt}\nu \subset V$.

The variational problem is

$$\inf_{y\in\Delta(\mu,\nu)} \mathcal{C}(y)$$

where

$$\mathcal{C}(y) = \int_{\mathbb{R}^n} c(x - y(x))\, d\mu(x).$$

To overcome the difficulties caused by the nonlinearity of the problem, Kantorovich [23] considered the dual linear problem

$$\sup_{(u,v)\in\mathcal{A}_c} J(u, v)$$

where

$$J(u, v) = \int_{\mathbb{R}^n} u(x)\, d\mu(x) + \int_{\mathbb{R}^n} v(x)\, d\nu(x)$$

and

$$\mathcal{A}_c = \{(u, v) : u, v \in C(\mathbb{R}^n), u(x) + v(y) \le c(x - y) \ \text{ on } \ U \times V\}.$$

It is well-known that the following duality formula holds ([3, Theorem 2.1] or [31, Theorem 4.6.8]).

$$\inf_{y\in\Delta(\mu,\nu)} \mathcal{C}(y) = \sup_{(u,v)\in\mathcal{A}_c} J(u, v) \tag{2.1}$$

and that there exists $(\psi, \phi) \in \mathcal{A}_c$ such that [24]

$$J(\psi, \phi) = \sup\{J(u, v) : (u, v) \in \mathcal{A}_c\}.$$

The potential functions ψ, ϕ have some remarkable properties. First, one may assume (see [20]) that ψ is the c-transform of ϕ and vice-versa, that is, $\psi = \phi^c$ and $\phi = \psi^c$, where

$$\phi^c(x) := \inf_{y\in\overline{V}} \ c(x - y) - \phi(y), \qquad \psi^c(y) := \inf_{x\in\overline{U}} \ c(x - y) - \psi(x).$$

For the definition and properties of c-transforms we refer to the book of Rachev and Rüschendorf [31].

Moreover, since ϕ and c are continuous there exist $x, x' \in \overline{V}$ such that

$$|\phi^c(y) - \phi^c(z)| \le |c(y - x) - c(z - x)| + |c(y - x') - c(z - x')|.$$

Therefore if c is (Lipschitz) continuous, ψ^c and ϕ^c are (Lipschitz) continuous in $\mathbb{R}^n$ as well.

In the following, $(\psi, \phi) = (\psi, \psi^c)$ will denote a maximizer of J on $\mathcal{A}_c$.

Definition 2.5. The c-subdifferential of ψ is the set valued map $\partial^c\psi : U \longrightarrow \overline{V}$ defined by $\partial^c\psi(x) := \{t \in \overline{V} : c(x-t) = \psi(x) + \phi(t)\}$.

By (2.1) every Borel measurable selection of $\partial^c\psi$ that pushes μ forward to ν is an optimal map for the transportation problem. Our aim is to prove that, under suitable assumptions on the cost, such a selection exists and is unique.

In the next theorem we prove that the local invertibility of ∇c implies the approximate differentiability of any measurable selection of $\partial^c\psi$. A similar result can be found in [4] (Theorem 6.2.7) where the thesis is achieved under the assumptions of regularity ($c \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$) and strict convexity of the cost c which guarantees the global invertibility of ∇c. Theorem 2.6 is proved in a slight more regular setting ($c \in C^2$) with an additional assumption ($\det D^2 c \neq 0$) which allows to consider also cost functions which are not convex or concave.

A related result in the setting of Riemannian manifolds can be found in [19] where the authors prove the approximate differentiability in the case of cost $h(x,y) = d^2(x,y)$, d being a Riemannian distance. The result we present here deals with cost functions that satisfy a regularity assumption but they are not necessarily related to the distance in $\mathbb{R}^n$ (as the saddle shaped cost in Example 2).

Theorem 2.6. *Let $c \in C(\mathbb{R}^n)$. Then there exists a Borel measurable selection y of $\partial^c\psi$. If c is locally Lipschitz continuous then $\nabla\psi(x) \in \partial^- c(x - y(x))$ a.e. in U. Moreover if $c \in C^2(\mathbb{R}^n)$ and $\det D^2 c(x) \neq 0$ for every $x \in \mathbb{R}^n$ then y is approximately differentiable a.e. in U.*

Proof: Since c, ψ and ϕ are continuous it follows that $\partial^c\psi$ is a closed valued upper semicontinuous map. Then there exists at least a Borel measurable selection y of $\partial^c\psi$ ([32, Part B, Theorem 6.31]).

If c is locally Lipschitz continuous then ψ is differentiable almost everywhere in U. Let $x \in U$ be such that ψ is differentiable in x and $f(t) := c(t - y(x)) - \psi(t) - \phi(y(x))$. Then $f(t) \geq f(x)$ for every $t \in \mathbb{R}^n$ and $0 \in \partial^- f(x)$, that is, $\nabla\psi(x) \in \partial^- c(x - y(x))$.

Finally let $c \in C^2(\mathbb{R}^n)$ be such that $\det D^2 c(x) \neq 0$ for every $x \in \mathbb{R}^n$ and set $w(x) := x - y(x)$. The regularity of c yields that $\nabla\psi(x) = \nabla c(w(x))$ for a.e. $x \in U$ and the local semi-concavity of ψ in $\mathbb{R}^n$ with modulus $\omega(t) = at$, $a > 0$ ([21, Proposition C.2]). This yields that $\psi(x) - \frac{a}{2}\|x\|^2$ is concave and, as a consequence, $\nabla\psi$ is approximately differentiable a.e. in $\mathbb{R}^n$. Now let $x_0 \in U$ be a point of approximate continuity of y, B an open neighborhood of $w(x_0)$ such that $\nabla c_{|_B}$ is invertible and let $g =: (\nabla c_{|_B})^{-1}$. Since w is approximately continuous in x_0 we have $\theta(w^{-1}(B), x_0) = 1$ and $y(x) = x - g(\nabla\psi(x))$ for a.e. $x \in w^{-1}(B)$. Then for a.e. $x_0 \in U$, y agrees with an approximately differentiable function on a set of density one for x_0. From this fact follows the thesis [22]. $\square$

The set $\mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$.

In the following we consider cost functions belonging to a set related to the classes $\mathcal{A}_{p,q}$ introduced by Ball [5].

Let Ω be a bounded open set in $\mathbb{R}^n$ and $p \geq 1$. We consider the classes

$$\mathcal{A}^{\pm}_{p,\frac{n}{n-1}}(\Omega) = \{w \in W^{1,p}(\Omega;\mathbb{R}^n) : \operatorname{adj} Dw \in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^{n\times n}),$$
$$\det Dw \text{ is of one sign a.e. in } \Omega\}$$

and

$$\mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega) = \{v \in W^{2,p}(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega), \nabla v \in \mathcal{A}^{\pm}_{p,\frac{n}{n-1}}(\Omega)\}.$$

Šverák [36] and Müller, Qi and Yan [29] studied some regularity and invertibility properties of functions of the class

$$\mathcal{A}^{+}_{p,\frac{n}{n-1}}(\Omega) = \{w \in W^{1,p}(\Omega;\mathbb{R}^n) : \operatorname{adj} Dw \in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^{n\times n}),$$
$$\det Dw > 0 \quad \text{a.e. on } \Omega\}.$$

The authors proved these properties by using the Brouwer degree; therefore they depend upon the fact that $\det Dw$ does not change its sign and then holds also for $\mathcal{A}^{\pm}_{p,\frac{n}{n-1}}(\Omega)$.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with Lipschitz boundary and $p > n-1$. If $c \in \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}$ then c has a representative $\tilde{c}$ which is locally Lipschitz continuous in Ω and such that $\nabla\tilde{c}$ is continuous in $\Omega \setminus N$ with $\dim_H(N) = n-p$.*

Proof: Let $\tilde{c}$ be a locally Lipschitz continuous representative of the equivalence class of c. We have that

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} \nabla c(z)\,dz = \lim_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} \nabla\tilde{c}(z)\,dz = \nabla\tilde{c}(x)$$

for every $x \in \Omega \setminus E$, where $\dim_H(E) = n-p$ ([40, Corollary 3.3.3]).

Since $\nabla c \in \mathcal{A}^{\pm}_{p,\frac{n}{n-1}}(\Omega)$, we have that $\nabla\tilde{c}$ is continuous outside a set N of Hausdorff dimension $n-p$ (see Lemma 4 and Theorem 4 in [36] and Theorem 5.2 in [29]). $\square$

In the following we call $\tilde{c}$ a regular representative of c.

We recall that if $w \in \mathcal{A}^{\pm}_{p,\frac{n}{n-1}}(\Omega)$ then, for every $x \in \Omega$ there exists a set $N_x \subset (0,r_x)$ [here $r_x = \operatorname{dist}(x,\partial\Omega)$] such that $\mathcal{L}^1(N_x) = 0$ and $w \in \mathcal{A}_{p,\frac{n}{n-1}}(\partial B(x,r))$

for each $r \in (0, r_x) \setminus N_x$ ([36, Proposition 1]). If we consider a continuous representative $\overline{w}$ of $w_{|\partial B(x,r)}$ it is possible to define the degree of $\overline{w}$. Let

$$E(w\,;B(x,r)) = \{y \in \mathbb{R}^n \setminus \overline{w}(\partial B(x,r)) : |\deg(\overline{w}\,;\partial B(x,r)\,;y)| \geq 1\} \cup \\ \cup \{\overline{w}(\partial B(x,r))\}.$$

Among the remarkable properties of this set, we recall here that $E(w\,;B(x,r))$ is a compact set and $E(w\,;B(x,r)) \subset E(w\,;B(x,s))$ if $r, s \in (0, r_x) \setminus N_x$ and $r < s$ ([36, Lemma 3]); finally one defines

$$F(x,w) = \bigcap_{r \in (0,r_x) \setminus N_x} E(w\,;B(x,r)), \quad F(A,w) = \bigcup_{x \in A} F(x,w) \quad \text{if } \ A \subset \Omega,$$

and, if $z \in F(\Omega, w)$,

$$G(z,w) = \{x \in \Omega, z \in F(x,w)\}, \qquad G(B,w) = \bigcup_{z \in B} G(z,w) \quad \text{ if } \ B \subset F(\Omega,w).$$

The set $F(x,w)$ describes the singularity of w at x and if w has a representative $\tilde{w}$ which is continuous at x then $F(x,w) = \tilde{w}(x)$ ([36, Lemma 4]).

3. Main results

In this section we prove the existence and uniqueness results in the case of cost functions $c \in \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded neighborhood of the origin with Lipschitz boundary such that

$$\{x - y : x \in \overline{U}, y \in \overline{V}\} \subset \Omega.$$

Theorem 3.1. *Let $p > n-1$ and let $c \in \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$ be a regular representative of its equivalence class. Besides, let N be the set where ∇c fails to exist and y a measurable selection of $\partial^c \psi$. Suppose that one of the following assumptions holds.*

(a) *$-c$ is regular at every point of Ω.*
(b) *$N \cap D(N) = \emptyset$.*

Then $y(x) \in x - G(\nabla\psi(x), \nabla c)$ a.e. in U.

Remarks. 1) We recall that, by Theorem 2.6, if $c \in C(\Omega)$, then there exists a Borel measurable selection y of $\partial^c \psi$.

2) Assumption (a) is satisfied if c is a semi-concave function ([11, Theorem 3.2.1]).

Proof: We set $w(x) := x - y(x)$ and prove that $\nabla\psi(x) \in \nabla_* c(w(x))$ a.e. in U.

Since c is locally Lipschitz continuous, by Theorem 2.6 we obtain that $\nabla\psi(x) \in \partial^- c(w(x))$ a.e. in U. If $w(x) \notin N$ then $\nabla\psi(x) = \nabla c(w(x))$. In the following we suppose that $w(x) \in N$.

(a) Since $\nabla\psi(x) \in \partial^- c(w(x)), -c$ is regular and by Proposition 2.1.2 in [16], for any $v \in \mathbb{R}^n$ we have

$$v \cdot \nabla\psi(x) \leq \liminf_{\delta\to 0^+} \frac{c(w(x)+\delta v) - c(w(x))}{\delta} = \liminf_{\substack{y\to w(x)\\ \delta\to 0^+}} \frac{c(y+\delta v) - c(y)}{\delta}$$

$$= -\max\{\xi \cdot v : \xi \in \partial(-c)(w(x))\} = \min\{\xi \cdot v : \xi \in \partial c(w(x))\}.$$

Therefore $\partial c(w(x)) = \{\nabla\psi(x)\}$ and this implies that $\nabla c(w(x)) = \nabla\psi(x)$.

(b) We observe that y and $\nabla\psi$ are approximately continuous a.e. in U ([22, Chapter 1.1.5, Proposition 1]). Let $x \in U$ be a point of approximate continuity of y and $\nabla\psi$. This means that there exist measurable sets E_1, E_2 such that $x \in E_1 \cap E_2$, $\theta(E_1, x) = \theta(E_2, x) = 1$ and $y_{|E_1}$, $\nabla\psi_{|E_2}$ are continuous. Therefore there exists a sequence $\{r_n\}_n$ such that $r_n \to 0^+$ and $\mathcal{L}^n((E_1 \cap E_2) \cap B(x, r_n)) > 0$. Let $\{x_n\}_n$ be a sequence convergent to x such that $x_n \in E_1 \cap E_2$. We have that $w(x_n) \to w(x)$ and, since $N \cap D(N) = \emptyset$, we obtain that $w(x_n) \notin N$ and $\nabla\psi(x_n) = \nabla c(w(x_n))$ definitively. Then $\lim_n \nabla c(w(x_n)) = \nabla\psi(x)$, that is, $\nabla\psi(x) \in \nabla_* c(w(x))$.

Now we prove that $\nabla_* c(w(x)) \subset F(w(x), \nabla c)$ a.e. in U.
Let $\{z_n\}_n$ be a sequence convergent to $w(x)$ such that there exist $\nabla c(z_n)$ and $\lim_n \nabla c(z_n)$. Besides for every $r > 0$, let $n_r \in \mathbb{N}$ be such that $z_n \in B(w(x), r)$ if $n > n_r$. We observe that $\deg(\overline{\nabla c}\,; B(w(x), r)\,; \nabla c(z_n)) \neq 0$ for a.e. $r < \operatorname{dist}(w(x), \partial\Omega)$ and $n > n_r$, otherwise we would have

$$\int_{B(w(x),r)} f(\nabla c(y))\, \det D^2 c(y)\, dy = 0$$

for some r and every $f \in C^\infty$ supported in the connected component of $\mathbb{R}^n \setminus \overline{\nabla c}(\partial\Omega)$ containing $\nabla c(z_n)$ ([29, Theorem 5.1]) and this is impossible since $\det D^2 c$ has the same sign a.e. in Ω. Therefore $\nabla c(z_n) \in E(\nabla c\,; B(w(x), r))$ for a.e. $r < \operatorname{dist}(w(x), \partial\Omega)$ if $n > n_r$ and since $E(\nabla c\,; B(w(x), r))$ is compact, also $\lim_n \nabla c(z_n) \in E(\nabla c\,; B(w(x), r))$. Then $\nabla_* c(w(x)) \subset E(\nabla c\,; B(w(x), r))$ for a.e. $r < \operatorname{dist}(w(x), \partial\Omega)$ and this implies that $\nabla_* c(w(x)) \subset F(w(x), \nabla c)$. Finally, $\nabla\psi(x) \in F(w(x), \nabla c)$ a.e. in U and the thesis follows from the definition of G.

□

Now we prove the uniqueness results. At this aim we consider the following assumption on the cost c.

(H) There exists a bounded open set $\Omega_0 \subset \mathbb{R}^n$ such that $\overline{\Omega} \subset \Omega_0$ and a function $g \in \mathcal{A}^{\pm}_{p,\frac{n}{n-1}}(\Omega_0)$ such that g is a homeomorphism onto $g(\Omega_0)$ and $\nabla c_{|\partial\Omega} = g_{|\partial\Omega}$.

Remark. Assumption (H) is satisfied by every nontrivial radial cost $c(x) = f(\|x\|)$ with $f : (0, +\infty) \longrightarrow \mathbb{R}$ derivable a.e. such that $f'(t) \neq 0$ for some t large enough.

In the following we consider the function $h : \Omega_0 \longrightarrow \mathbb{R}^n$ defined by

$$h(x) = \begin{cases} \nabla c(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \Omega_0 \setminus \Omega. \end{cases}$$

Theorem 3.2. *Let $p > n-1$, $c \in \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$ be a regular representative of its equivalence class and y_1, y_2 two measurable selections of $\partial^c\psi$. Suppose that*

(1) *c satisfies* (H),
(2) *one of the assumptions* (a), (b) *of Theorem* 3.1 *holds. Then there exist measurable sets $T \subset \mathbb{R}^n$ and $N_0 \subset U$ such that $\mathcal{H}^{n-1}(T) = 0$, $\mathcal{L}^n(N_0) = 0$ and $\{x \in U : y_1(x) \neq y_2(x)\} \subset \nabla\psi^{-1}(T) \cup N_0$.*

Proof: Let $T = \{y \in \overline{g(\Omega)} : \operatorname{diam} G(y,h) > 0\}$. By Theorem 7(iv) in [36] and Theorem 5.3 in [29] we have $\mathcal{H}^{n-1}(T) = 0$. Besides, if we set $w_k(x) := x - y_k(x)$, $k = 1, 2$, by Theorem 3.1 it follows that $w_k(x) \in G(\nabla\psi(x), \nabla c) \subset G(\nabla\psi(x), h)$ a.e. in U where the inclusion holds since $F(z, \nabla c) \subset F(z, h)$ for every $z \in \Omega$. Therefore there exists a negligible subset N_0 of U such that

$$\{x \in U : y_1(x) \neq y_2(x)\} \subset \{x \in U : \operatorname{diam} G(\nabla\psi(x), h) > 0\} \cup N_0 \subset$$
$$\subset \nabla\psi^{-1}(T) \cup N_0.$$

□

Remarks. 1) If $c \in C^1(\Omega)$ then $F(x,h) = \nabla c(x)$ for every $x \in \Omega$. Since $G(y,h) \subset \overline{\Omega}$ for every $y \in \overline{g(\Omega)}$ ([36, Theorem 7]), one has $G(y,h) = \nabla c^{-1}(y)$ and T is the image set of the "points of noninjectivity" of ∇c. Therefore c satisfies the twist condition if and only if $T = \emptyset$.

2) Assumptions of Theorems 3.1 and 3.2 are satisfied also by cost functions that are not semi-concave.

Theorem 3.3. *Let $p > n-1$, and let $c \in \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$ be a regular representative of its equivalence class. Suppose that*

(1) *c satisfies* (H),
(2) *one of the assumptions* (a), (b) *of Theorem* 3.1 *holds,*
(3) *one of the following assumptions holds:*
   (3a) $\int_{\Omega_0} \|\operatorname{adj} Dh\|^n |\det Dh|^{1-n}\, dx < +\infty$,
   (3b) *ψ is twice approximately differentiable and $\rho(\operatorname{ap} D^2\psi(x)) \geq n-1$ a.e. in U.*

Then there exists a unique (a.e.) measurable selection y of $\partial^c\psi$ with $\partial^c\psi(x) = \{y(x)\}$ a.e. in U and $y \in \Delta(\mu,\nu)$.

Proof: Let y_1, y_2 be two measurable selections of $\partial^c\psi$. By Theorem 3.2 there exist measurable sets $T \subset \mathbb{R}^n$ and $N_0 \subset U$ such that $\mathcal{H}^{n-1}(T) = 0, \mathcal{L}^n(N_0) = 0$ and $\{x \in U : y_1(x) \neq y_2(x)\} \subset \nabla\psi^{-1}(T) \cup N_0$.

If (3a) holds, by Corollary 2 in [36] and Theorem 5.3 in [29] we have that $T = \emptyset$.

Now we suppose that (3b) holds.

Since $\mathcal{H}^{n-1}(T) = 0$ we have that $\rho(\operatorname{ap} D^2\psi(x)) < n-1$ a.e. in $\nabla\psi^{-1}(T)$ ([38, Lemma 3.2]) but this means that $\mathcal{L}^n(\nabla\psi^{-1}(T)) = 0$. Therefore there exists a unique (a.e.) Borel measurable selection y of $\partial^c\psi$ and by Castaing's selection theorem ([32, Part B, Theorem 6.9]) we have $\partial^c\psi(x) = \{y(x)\}$ a.e. in U. Then the c-subdifferential set valued map is a singlevalued map a.e. in U and this implies in standard ways that $y \in \Delta(\mu,\nu)$ (see [13, Section 3, Lemma 2] or [20, Theorem 1, Claim 3]). □

Remarks 3.4. 1) Assumption (3a) was introduced by Ball ([6]) and ensures that $T = \emptyset$, that is, if $c \in C^1$, c satisfies the twist condition.

2) If $c \in C^{1,1}(\Omega)$ then ψ is semi-concave in Ω with modulus $\omega(t) = at$, $a > 0$ and $\psi(x) - \frac{a}{2}\|x\|^2$ is concave ([21, Proposition C.2]). Therefore ψ has the well known regularity properties of convex functions; more precisely ψ has a second order differential at almost every $x \in \Omega$ ([2, Theorem 7.10]), that is, there exists a matrix $D^2\psi(x)$ such that

$$\psi(x+v) = \psi(x) + \langle\nabla\psi(x); v\rangle + \frac{1}{2}\langle D^2\psi(x)v; v\rangle + o(\|v\|^2)$$

for $v \in \mathbb{R}^n$.

The strictly concave functions of the distance are interesting for the economical applications ([21]) or for the relativistic heat equation such as the cost

$$c(x) = \sqrt{1 - \|x\|^2}$$

with $\|x\| < d =: \sup\{\|s-t\| : s \in \operatorname{spt}\mu, t \in \operatorname{spt}\nu\} < 1$ ([9], [27]). In this case Gangbo and McCann have proved the existence and uniqueness of the optimal map. In the next theorem we consider a regular cost ($c \in C^2$) which may be strictly concave or saddle shaped in some direction.

Corollary 3.5. *Let $p > n-1$ and $c \in C^2(\Omega) \cap \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$. Suppose that*

(1) *c satisfies* (H),
(2) *the Hessian matrix $D^2c(x)$ has at least $n-1$ negative eigenvalues for every $x \in \Omega$.*

Then there exists a unique (a.e.) *measurable selection* y *of* $\partial^c\psi$ *with* $\partial^c\psi(x) = \{y(x)\}$ *a.e. in* U *and* $y \in \Delta(\mu,\nu)$.

Proof: By Remark 3.4, ψ is locally semi-concave and has a second order differential a.e. in $x \in \Omega$. We prove that at every point x_0 where ψ has a second order differential we have $\rho(D^2\psi(x_0)) \geq n-1$. Suppose for the contradiction that $\rho(D^2\psi(x_0)) \leq n-2$ and let y be a Borel measurable selection of $\partial^c\psi(x_0)$. As in Theorem 2.6 we consider $f(t) := c(t - y(x_0)) - \psi(t) - \phi(y(x_0))$. Clearly f has a second order differential at x_0 and since x_0 is a global minimum for f, we have

$$\begin{aligned} f(x_0+v) &= \frac{1}{2}\langle D^2 f(x_0)v; v\rangle + o(\|v\|^2) \\ &= \frac{1}{2}[\langle D^2 c(x_0 - y(x_0))v; v\rangle - \langle D^2\psi(x_0)v; v\rangle] + o(\|v\|^2) \geq 0 \end{aligned} \tag{3.1}$$

for every $v \in \mathbb{R}^n$.

Since $\rho(D^2\psi(x_0)) \leq n-2$ there exists a subspace V of $\mathbb{R}^n$ such that $\dim V \geq 2$ and $\langle D^2\psi(x_0)v; v\rangle = 0$ for every $v \in V$. Now let $\lambda_1, \dots, \lambda_q$ be the distinct negative eingevalues of $D^2c(x_0 - y(x_0))$ and $W_{\lambda_1}, \dots, W_{\lambda_q}$ the relative eigenspaces. By assumption we have that $\dim \bigoplus_{i=1}^q W_{\lambda_i} \geq n-1$ and if we set $W =: V \cap \bigoplus_{i=1}^q W_{\lambda_i}$ we get $\dim W \geq 1$. Therefore $\langle D^2c(x_0 - y(x_0))w; w\rangle < 0$ for every $w \in W \setminus \{0\}$ and this is a contradiction with (3.1).

Thus $\rho(D^2\psi(x)) \geq n-1$ for a.e. $x \in \Omega$, assumption (3b) of Theorem 3.4 is satisfied and the thesis follows. □

Theorem 3.6. *Let* $p > n-1$, *and let* $c \in \mathcal{A}^{2,\pm}_{p,\frac{n}{n-1}}(\Omega)$ *be a regular representative of its equivalence class. Suppose that*

1. *c satisfies* (H),
2. *one of the assumptions* (a), (b) *of Theorem* 3.1 *holds,*
3. *there exists a measurable selection* y *of* $\partial^c\psi$ *such that* y *is approximate differentiable a.e. in* U *and* $\|\operatorname{ap} Dy(x)\| < 1$ *a.e. in* U.

Then y *is the unique measurable selection of* $\partial^c\psi(x)$, $\partial^c\psi(x) = \{y(x)\}$ *a.e. in* U *and* $y \in \Delta(\mu,\nu)$.

Proof: We set $w(x) := x - y(x)$ and

$$N_0 = \{x \in \Omega : \nabla c \text{ is not differentiable in } x\} \cup \{x \in \Omega : \det D^2c(x) = 0\}.$$

By Theorem 3.1 we have that $\nabla\psi(x) = \nabla c(w(x))$ a.e. in $\Omega \setminus w^{-1}(N_0)$. Since $\|\operatorname{ap} Dy(x)\| < 1$ a.e. in U we have that $\operatorname{ap} Dw$ is invertible a.e. ([10, Proposition VI.7]) and $\det \operatorname{ap} Dw(x) \neq 0$ a.e. Therefore $\mathcal{L}^n(w^{-1}(N_0)) = 0$ ([38, Lemma 3.2]) and ψ is twice approximately differentiable a.e. in U. Moreover

$$\det \operatorname{ap} D^2\psi(x) = \det D^2c(w(x)) \det \operatorname{ap} Dw(x) \neq 0$$

a.e in U and the thesis follows from Theorem 3.3. □

4. Examples

1) *An economic application.*

We give an economic application of the proved results to multidimensional incentive problem in a situation of *Adverse Selection* (we refer to [7] or [25] for a survey on this subject). In this setting the aim of the principal is to contract with an agent concerning a service (commonly called *action*) and a monetary compensatory transfer, the principal being not informed about the individual characteristics of the agent.

We assume that the agent is characterized by the quasi linear utility function

$$V(x, y, t) = f(x, y) + t,$$

where $x \in A \subset \mathbb{R}^n$ is the agent's characteristics, unobservable by the principal, $y \in B \subset \mathbb{R}^n$ is the action or choice of the agent, $t \in \mathbb{R}$ is the compensatory transfer and A, B bounded open sets. Besides we recall that a *contract* is a pair of functions $(h, t) : A \longrightarrow B \times \mathbb{R}$ and the aim of the principal is to look for a incentive-compatible contract, that is a contract (h, t) such that

$$f(x, h(x)) + t(x) \geq f(x, h(z)) + t(z) \quad \text{for all} \quad (x, z) \in A \times A.$$

Finally we say that a function $h : A \longrightarrow B$ is implementable (or rationalizable) if there exists a function $t : A \longrightarrow \mathbb{R}$ such that (h, t) is incentive-compatible.

Here we consider the case $f(x, y) = -c(x - y)$ (considered also in [12] and [14] under the assumption that c is strictly convex) and a Borel map $h_0 : A \longrightarrow B$ as a referred action profile. Finally we define $\nu(C) = \mu(h_0^{-1}(C))$ for every Borel set $C \subset B$ and we denote $(\psi, \phi) = (\psi, \psi^c)$ a maximizer of J on $\mathcal{A}_c$.

Proposition 4.1. *Let c satisfy the assumptions of Corollary 3.5. Then there exists a unique (a.e.) incentive-compatible contract $(h, \phi \circ h)$ such that $h(x) \in \partial^c \psi(x)$ for every $x \in A$ and $h \in \Delta(\mu, \nu)$.*

Proof: By Corollary 3.5 there exists a unique (a.e.) Borel measurable selection h of $\partial^c \psi$. We have that $h \in \Delta(\mu, \nu)$ and $\psi(x) + \phi(h(x)) = c(x - h(x))$ for every $x \in A$. Thus

$$\phi(h(x)) - c(x - h(x)) = -\psi(x) \geq \phi(h(z)) - c(x - h(z))$$

for every $(x, z) \in A \times A$ and this implies that the contract $(h, \phi \circ h)$ is incentive-compatible. □

Remark. We recall that in the one dimensional case (that is, A, B are intervals), if $f \in C^2$ and satisfies the Spence-Mirless condition $\frac{\partial^2 f}{\partial x \partial y} > 0$ then h is implementable if and only if h is non-decreasing [33]. In the case $f(x, y) = c(x - y)$

the Spence-Mirless condition yields that $c'' < 0$, that is, c is strictly concave. The economic interpretation of this fact is that the agent is risk averse meanwhile in general the strict convexity of c means that the agent is risk prone. Proposition 4.1 allows to consider, in a multidimensional setting, saddle shaped utility functions where different attitude to risk of the agent can be represented.

2) *A saddle-shaped cost.*

We consider the cost function $c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$c(x, y) = \sqrt{x^4 + (1 + y^2)^{-1}}.$$

Clearly $c \in C^\infty(\mathbb{R}^2)$ and

$$\det D^2 c(x, y) = 2x^2[x^4(1+y^2)(3y^2-1)+3(2y^2-1)][x^2(1+y^2)+1]^{-2}(1+y^2)^{-2}.$$

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^4 + y^2 < \frac{1}{2}\}$ and suppose that $\{P - Q : P \in \operatorname{spt}\mu, Q \in \operatorname{spt}\nu\} \subset \Omega$. A straightforward computation shows that $\det D^2 c \leq 0$ in Ω and $\det D^2 c(x, y) = 0$ if and only if $x = 0$; besides we have that $c_{xx}(0, y) = c_{xy}(0, y) = 0$ and $c_{yy}(0, y) = (2y^2 - 1)(y^2 + 1)^{-\frac{5}{2}} < 0$ if $y^2 < \frac{1}{2}$. Therefore $D^2 c$ is indefinite or negative semi-definite in Ω and c is neither locally convex nor concave in Ω.

Now let $\Omega_0 = \{(x, y) \in \mathbb{R}^2 : x^4 + y^2 < \frac{3}{4}\}$ and $g : \Omega_0 :\longrightarrow \mathbb{R}$ be defined by

$$g(x, y) = (p(x), q(y)),$$

where

$$p(x) = \frac{2x^3}{\sqrt{x^4 + 2(3 - 2x^4)^{-1}}} \qquad q(y) = -\frac{\sqrt{2}y}{(1 + y^2)\sqrt{(1 - y^4)(3 + 2y^2)}}.$$

It is easy to show that p and q are strictly monotone on $\left(-\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{3}{4}}\right)$, $\det Dg(x, y) = p'(x)q'(y) \leq 0$ in Ω_0 and $\det Dg(x, y) = 0$ if and only if $x = 0$. Hence g is an homeomorphism of Ω_0 onto $g(\Omega_0)$, $g \in \mathcal{A}^{\pm}_{p,2}(\Omega_0)$ for any $p > 1$ and one verify that $g = \nabla c$ on $\partial\Omega$. Then c satisfies assumption (H) and, by Corollary 3.5, there exists a unique optimal map for the transportation problem.
Finally we observe that

$$h(P) := \begin{cases} \nabla c(P) & \text{if } P \in \Omega \\ g(P) & \text{if } P \in \Omega_0 \setminus \Omega \end{cases}$$

does not satisfy assumption (3a) of Theorem 3.3; in fact we have that there exist $a, b > 0$ such that

$$\int_{\Omega_0} \frac{\|\operatorname{adj} Dh\|^2}{|\det Dh|}\, dx\, dy \geq \int_{\Omega} \frac{\|\operatorname{adj} D^2 c\|^2}{|\det D^2 c|}\, dx\, dy \geq \int_{\Omega} \frac{c_{yy}^2}{|\det D^2 c|}\, dx\, dy$$

$$= \int_{\Omega} \frac{[x^4(1+y^2)(1-3y^2) + (1-2y^2)]^2}{2x^2(1+y^2)^3[x^4(1+y^2)+1][x^4(1+y^2)(1-3y^2)+3(1-2y^2)]}\, dx\, dy$$

$$> \int_{\Omega^*} \frac{a}{bx^2}\, dx\, dy = +\infty\,,$$

where $\Omega^* = \{(x,y) \in \Omega : y^2 < \frac{1}{4}\}$.

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