Comment.Math.Univ.Carolin. 51,1 (2010) 85–97

A quasistatic bilateral contact problem with adhesion and friction for viscoelastic materials

Arezki Touzaline

Abstract. We consider a mathematical model which describes a contact problem between a deformable body and a foundation. The contact is bilateral and is modelled with Tresca's friction law in which adhesion is taken into account. The evolution of the bonding field is described by a first order differential equation and the material's behavior is modelled with a nonlinear viscoelastic constitutive law. We derive a variational formulation of the mechanical problem and prove the existence and uniqueness result of the weak solution. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

Keywords: viscoelastic materials, adhesion, Tresca's friction, fixed point, weak solution

Classification: 47J20, 49J40, 74M10, 74M15

1. Introduction

Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. A first study of frictional contact problems within the framework of variational inequalities was made in [7]. The mathematical, mechanical and numerical state of the art can be found in [16]. The bilateral contact problem with Tresca's friction law for viscoelastic materials was studied in [1]. The aim of this paper is to study this model in which moreover the adhesion of contact surfaces is taken into account. Models for dynamic or quasistatic process of frictionless adhesive contact between a deformable body and a foundation have been studied in [4], [5], [10], [17]. In [6] the unilateral quasistatic contact problem with friction and adhesion was studied and an existence result for a friction coefficient small enough was established. As in [9], [10] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$, when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [3], [11], [13], [14], [15], [16], [17], [18]. In this work we derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain a partial regularity result for the solution.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 2.1.

2. Problem statement and variational formulation

Let $\Omega \subset \mathbb{R}^d$; $(d = 2, 3)$, be a domain initially occupied by a viscoelastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ. Γ is partitioned into three measurable parts $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $\operatorname{meas}\Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in bilateral and adhesive contact with Tresca's friction law with a foundation.

Thus, the classical formulation of the mechanical problem is written as follows.

Problem P_1. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to [0, 1]$ such that

$$\operatorname{div} \sigma + \varphi_1 = 0 \text{ in } \Omega \times (0, T), \tag{2.1}$$

$$\sigma = A\varepsilon(\dot{u}) + B\varepsilon(u) \quad \text{in} \quad \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on} \quad \Gamma_2 \times (0, T), \tag{2.4}$$

$$\left.\begin{array}{c} u_\nu = 0 \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)\right| \le g \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)\right| < g \Longrightarrow \dot{u}_\tau = 0 \\ \left|\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)\right| = g \Longrightarrow \\ \exists \lambda \ge 0 \text{ s.t. } \dot{u}_\tau = -\lambda\left(\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)\right) \end{array}\right\} \text{ on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\dot{\beta} = -(c_\tau \beta \left|R_\tau(u_\tau)\right|^2 - \varepsilon_a)_+ \quad \text{on} \quad \Gamma_3 \times (0, T), \tag{2.6}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega, \tag{2.7}$$

$$\beta(0) = \beta_0 \quad \text{on} \quad \Gamma_3. \tag{2.8}$$

We denote by σ the stress field and $\varepsilon(u)$ the strain tensor. Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the viscoelastic constitutive law of the material in which A and B are given nonlinear constitutive functions. Here and below a dot above a variable represents a time derivative. We recall that in linear viscoelasticity the stress tensor $\sigma = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkh}\varepsilon_{kh}\left(\dot{u}\right) + b_{ijkh}\varepsilon_{kh}\left(u\right),$$

where $A = (a_{ijkh})$ is the viscosity tensor and $B = (b_{ijkh})$ is the elasticity tensor, for $i, j, k, h = 1, \ldots, d$. (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Condition (2.5) represents the bilateral contact with Tresca's friction law in which adhesion is taken into account. Here g is a friction bound and the parameters c_τ and ε_a are adhesion coefficients which may depend on $x \in \Gamma_3$. As in [18], R_τ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } |v| \leq L, \\ L\frac{v}{|v|} & \text{if } |v| > L, \end{cases}$$

where $L > 0$ is a characteristic length of the bonds. Equation (2.6) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [18] where $[s]_+ = \max(s, 0)\ \forall\, s \in \mathbb{R}$. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs, bonding cannot be reestablished. Also we wish to make it clear that from [12] it follows that the model does not allow for complete debonding field in finite time. Finally, (2.7) and (2.8) represent respectively the initial displacement field and the initial bonding field. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and S_d are given by

$$\begin{aligned} &u \cdot v = u_i v_i, && |v| = (v \cdot v)^{\frac{1}{2}} \quad \forall\, u, v \in \mathbb{R}^d, \\ &\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, && |\tau| = (\tau \cdot \tau)^{\frac{1}{2}} \quad \forall\, \sigma, \tau \in S_d, \end{aligned}$$

where S_d is the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$\begin{aligned} &H = \left(L^2(\Omega)\right)^d, \ H_1 = \left(H^1(\Omega)\right)^d, \ Q = \left\{\tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\}, \\ &Q_1 = \{\sigma \in Q;\ \operatorname{div}\sigma \in H\}. \end{aligned}$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i\, dx, \quad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij}\, dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every element $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Also, for a regular function $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + \langle \operatorname{div}\sigma, v \rangle_H = \int_\Gamma \sigma\nu \cdot v \, da \;\forall\, v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\; v = 0 \text{ on } \Gamma_1,\; v_\nu = 0 \text{ on } \Gamma_3\}.$$

Since $\operatorname{meas}\Gamma_1 > 0$, the following Korn's inequality holds [7],

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \;\forall\, v \in V, \tag{2.9}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \;\forall\, v \in V. \tag{2.10}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X.$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_1 \in C([0,T];H), \quad \varphi_2 \in C\left([0,T];\left(L^2(\Gamma_2)\right)^d\right) \tag{2.11}$$

and we denote by $f(t)$ the element of V defined by

$$(f(t),v)_V = \int_\Omega \varphi_1(t)\cdot v\,dx + \int_{\Gamma_2} \varphi_2(t)\cdot v\,da \;\forall\, v\in V, \quad \text{for} \quad t\in[0,T]. \tag{2.12}$$

Using (2.11) and (2.12) yields

$$f \in C\left([0,T];V\right).$$

Also we define the functional $j : V \to \mathbb{R}_+$ by

$$j(v) = \int_{\Gamma_3} g\left|v_\tau\right|\,da,$$

where g is assumed to satisfy

$$g \in L^\infty\left(\Gamma_3\right), \; g \geq 0 \; a.e. \text{ on } \Gamma_3. \tag{2.13}$$

In the study of Problem P_1 we assume that the viscosity operator A satisfies

$$\left\{\begin{array}{l}
\text{(a) } A : \Omega\times S_d \to S_d; \\[1em]
\text{(b) there exists } M_A > 0 \text{ such that} \\
\qquad \left|A\left(x,\varepsilon_1\right) - A\left(x,\varepsilon_2\right)\right| \leq M_A\left|\varepsilon_1-\varepsilon_2\right|, \\
\qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega; \\[1em]
\text{(c) there exists } m_A > 0 \text{ such that} \\
\qquad \left(A\left(x,\varepsilon_1\right) - A\left(x,\varepsilon_2\right)\right)\cdot\left(\varepsilon_1-\varepsilon_2\right) \geq m_A\left|\varepsilon_1-\varepsilon_2\right|^2, \\
\qquad \text{for all } \ \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega; \\[1em]
\text{(d) the mapping } x \to A\left(x,\varepsilon\right) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \text{for any } \varepsilon \text{ in } S_d; \\[1em]
\text{(e) } x \to A\left(x,0\right) \in Q.
\end{array}\right. \tag{2.14}$$

The elasticity operator B satisfies

$$
(2.15)\qquad \begin{cases} \text{(a) } B:\Omega\times S_d\to S_d; \\ \\ \text{(b) there exists } M_B>0 \text{ such that} \\ \qquad |B(x,\varepsilon_1)-B(x,\varepsilon_2)|\le M_B\,|\varepsilon_1-\varepsilon_2|\,, \\ \qquad \text{for all } \varepsilon_1,\varepsilon_2 \text{ in } S_d,\ a.e.\ x \text{ in } \Omega; \\ \\ \text{(c) the mapping } x\to B(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for any } \varepsilon \text{ in } S_d; \\ \\ \text{(e) } x\to B(x,0)\in Q. \end{cases}
$$

As in [17] we suppose that the adhesion coefficients c_τ and ε_a satisfy the conditions

$$
(2.16)\qquad c_\tau,\varepsilon_a\in L^\infty(\Gamma_3)\,,\ c_\tau,\varepsilon_a\ge 0,\ a.e. \text{ on } \Gamma_3.
$$

We assume that the initial data satisfy

$$
(2.17)\qquad u_0\in V,
$$

$$
(2.18)\qquad \beta_0\in L^2(\Gamma_3)\,;\ 0\le\beta_0\le 1,\ a.e. \text{ on } \Gamma_3.
$$

Next, we define the functional $r:L^2(\Gamma_3)\times V\times V\to\mathbb{R}$ by

$$
r(\beta,u,v)=\int_{\Gamma_3} c_\tau\beta^2 R_\tau(u_\tau)\cdot v_\tau\,da.
$$

Finally, we need the following set for the bonding fields,

$$
\mathcal{O}=\left\{\theta:[0,T]\to L^2(\Gamma_3)\,;\ 0\le\theta(t)\le 1\ \forall\, t\in[0,T],\ a.e. \text{ on } \Gamma_3\right\}.
$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula that Problem P_1 has the following variational formulation.

Problem P_2. Find a displacement field $u:[0,T]\to V$ and a bonding field $\beta:[0,T]\to L^\infty(\Gamma_3)$ such that

$$
(2.19)\qquad \langle A\varepsilon(\dot u(t)),\varepsilon(v)-\varepsilon(\dot u(t))\rangle_Q+\langle B\varepsilon(u(t)),\varepsilon(v)-\varepsilon(\dot u(t))\rangle_Q+j(v)
$$
$$
-j(\dot u(t))+r(\beta(t),u(t),v-\dot u(t))\ge(f(t),v-\dot u(t))_V\quad\forall\ v\in V,\ t\in[0,T],
$$

$$
(2.20)\qquad \dot\beta(t)=-(c_\tau\beta(t)\,|R_\tau(u_\tau(t))|^2-\varepsilon_a)_+\quad a.e.\ t\in(0,T),
$$

$$
(2.21)\qquad u(0)=u_0 \text{ in } \Omega,
$$

$$
(2.22)\qquad \beta(0)=\beta_0 \text{ on } \Gamma_3.
$$

Our main result of this section, which will be established in the next is the following theorem.

Theorem 2.1. *Let $T > 0$ and assume that* (2.11), (2.13), (2.14), (2.15), (2.16), (2.17), *and* (2.18) *hold. Then there exists a unique solution of Problem P_2 which satisfies*

$$u \in C^1([0,T];V),\ \beta \in W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right) \cap \mathcal{O}.$$

3. Existence and uniqueness result

The proof of Theorem 2.1 will be carried out in several steps. In the first step, for a given $\eta \in C([0,T];V)$ we consider the following variational problem.

Problem $P_{1\eta}$. Find $v_\eta : [0,T] \to V$ such that

$$\begin{aligned} &\left\langle A\varepsilon\left(v_\eta(t)\right), \varepsilon\left(w\right) - \varepsilon(v_\eta(t))\right\rangle_Q + \left(\eta(t), w - v_\eta(t)\right)_V + j\left(w\right) - j\left(v_\eta\left(t\right)\right) \\ &\geq \left(f(t), w - v_\eta(t)\right)_V \ \ \forall\, w \in V,\ t \in [0,T]. \end{aligned} \tag{3.1}$$

We show the following result.

Lemma 3.1. *Problem $P_{1\eta}$ has a unique solution and it satisfies $v_\eta \in C([0,T];V)$.*

PROOF: We define the operator $C : V \to V$ by

$$(Cv, w)_V = \langle A\varepsilon(v), \varepsilon(w)\rangle_Q \quad \forall\, v, w \in V.$$

It follows from assumption (2.14) that C is a strongly monotone and Lipschitz continuous operator. The functional j is a continuous semi-norm on V, then by a classical argument of elliptic variational inequalities [2], we deduce that for each $t \in [0,T]$, there exists a unique element $v_\eta(t) \in V$ such that

$$\begin{aligned} &\left\langle A\varepsilon\left(v_\eta(t)\right), \varepsilon\left(w\right) - \varepsilon(v_\eta\left(t\right))\right\rangle_Q + j\left(w\right) - j\left(v_\eta(t)\right) \\ &\geq \left(f(t) - \eta(t), w - v_\eta(t)\right)_V \quad \forall\, w \in V. \end{aligned} \tag{3.2}$$

Then from (3.2) we deduce that $v_\eta(t)$ is the unique solution of (3.1). Now, let $t_1,\ t_2 \in [0,T]$. In inequality (3.1) written for $t = t_1$, take $w = v_\eta(t_2)$ and also in inequality (3.1) written for $t = t_2$, take $w = v_\eta(t_1)$. Using (2.14)(c), we find after adding the resulting inequalities that

$$\left\|v_\eta\left(t_1\right) - v_\eta\left(t_2\right)\right\|_V \leq \frac{1}{m_A}\left(\left\|f\left(t_1\right) - f\left(t_2\right)\right\|_V + \left\|\eta\left(t_1\right) - \eta\left(t_2\right)\right\|_V\right).$$

As $f \in C([0,T];V)$ and $\eta \in C([0,T];V)$, it follows that $v_\eta \in C([0,T];V)$.

Let now $u_\eta : [0,T] \to V$ be the function defined by

$$u_\eta(t) = \int_0^t v_\eta(s)\,ds + u_0, \quad \text{for} \quad t \in [0,T]. \tag{3.3}$$

u_η satisfies $u_\eta \in C^1([0,T];V)$, $\dot{u}_\eta = v_\eta$. □

Next, we consider the following problem.

Problem $P_{2\eta}$**.** Find a bonding field $\beta_\eta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = -\left(c_\tau \beta_\eta(t) \left|R_\tau\left(u_{\eta\tau}(t)\right)\right|^2 - \varepsilon_a\right)_+ \ \ a.e.\ t \in (0,T), \tag{3.4}$$

$$\beta_\eta\left(0\right) = \beta_0 \text{ on } \Gamma_3. \tag{3.5}$$

We have the following result.

Lemma 3.2. *There exists a unique solution to Problem* $P_{2\eta}$ *and it satisfies*

$$\beta_\eta \in W^{1,\infty}\left(0,T; L^2\left(\Gamma_3\right)\right) \cap \mathcal{O}.$$

Proof: Let $k > 0$ and let

$$X = \left\{\beta \in C\left([0,T]; L^2\left(\Gamma_3\right)\right); \sup_{t\in[0,T]} \left[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}\right] < +\infty\right\}.$$

X is a Banach space for the norm

$$\|\beta\|_X = \sup_{t\in[0,T]} \left[\exp(-kt)\|\beta(t)\|_{L^2(\Gamma_3)}\right],$$

and consider the mapping $\mathcal{T} : X \to X$ given by

$$\mathcal{T}\beta(t) = \beta_0 - \int_0^t \left(c_\tau \beta(s) \left|R_\tau\left(u_{\eta\tau}(s)\right)\right|^2 - \varepsilon_a\right)_+ ds.$$

Using that $|R_\tau(u_{\eta\tau})| \leq L$, it follows that there exists a constant $c_1 > 0$ such that

$$\|\mathcal{T}\beta_1(t) - \mathcal{T}\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_1 \textstyle\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds.$$

Since

$$\begin{aligned}\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds &= \int_0^t e^{ks}(e^{-ks}\left\|\beta_1\left(s\right) - \beta_2(s)\right\|_{L^2(\Gamma_3)})\, ds \\ &\leq \|\beta_1 - \beta_2\|_{X_1} \frac{e^{kt}}{k},\end{aligned}$$

this inequality implies

$$e^{-kt}\left\|\mathcal{T}\beta_1\left(t\right) - \mathcal{T}\beta_2\left(t\right)\right\|_{L^2(\Gamma_3)} \leq \frac{c_1}{k}\left\|\beta_1 - \beta_2\right\|_{X_1}\ \ \forall\, t \in [0,T],$$

and then,

$$\left\|\mathcal{T}\beta_1 - \mathcal{T}\beta_2\right\|_X \leq \frac{c_1}{k}\left\|\beta_1 - \beta_2\right\|_X. \tag{3.6}$$

The inequality (3.6) shows that for $k > c_1$, $\mathcal{T}$ is a contraction. Then we deduce, by Banach fixed point theorem that $\mathcal{T}$ has a unique fixed point β_η which satisfies

(3.4) and (3.5). The regularity $\beta_\eta \in \mathcal{O}$ is a consequence of (3.5) and (2.18), see [17] for details. $\square$

Moreover, we use the Riesz representation theorem to define the function $\Lambda : [0,T] \to V$ by

$$(\Lambda\eta(t), w)_V = \langle B\varepsilon\left(u_\eta(t)\right), \varepsilon(w)\rangle_Q + r\left(\beta_\eta(t), u_\eta(t), w\right), \quad \forall\, w \in V, t \in [0,T]. \tag{3.7}$$

Lemma 3.3. *For each $\eta \in C([0,T];V)$ the function $\Lambda\eta : [0,T] \to V$ belongs to $C([0,T];V)$. Moreover, there exists a unique $\eta^* \in C([0,T];V)$ such that $\Lambda\eta^* = \eta^*$.*

Proof: Let $\eta \in C([0,T];V)$, t_1, $t_2 \in [0,T]$. Using (3.7), there exists a constant $c_2 > 0$ such that

$$\begin{aligned}&\left\|\Lambda\eta\left(t_1\right) - \Lambda\eta\left(t_2\right)\right\|_V\\ &\leq c_2\Big(\left\|B\varepsilon\left(u_\eta\left(t_1\right)\right) - B\varepsilon\left(u_\eta\left(t_2\right)\right)\right\|_Q\\ &\quad + \left\|\beta_\eta^2\left(t_1\right) R_\tau\left(u_{\eta\tau}\left(t_1\right)\right) - \beta_\eta^2\left(t_2\right) R_\tau\left(u_{\eta\tau}\left(t_2\right)\right)\right\|_{L^2(\Gamma_3)}\Big).\end{aligned}$$

Using the properties of the operator R_τ (see [16]) such that

$$\left|R_\tau\left(u_{\eta\tau}\right)\right| \leq L, \;\; \left|R_\tau(a) - R_\tau(b)\right| \leq |a-b| \;\forall\, a,b \in \mathbb{R}^d,$$

(2.10), and that $0 \leq \beta_\eta(t) \leq 1$, $\forall\, t \in [0,T]$, it follows that there exists a constant $c_3 > 0$ such that

$$\begin{aligned}&\left\|\Lambda\eta\left(t_1\right) - \Lambda\eta\left(t_2\right)\right\|_V\\ &\leq c_3\left(\left\|u_\eta\left(t_1\right) - u_\eta\left(t_2\right)\right\|_V + \left\|\beta_\eta\left(t_1\right) - \beta_\eta\left(t_2\right)\right\|_{L^2(\Gamma_3)}\right).\end{aligned} \tag{3.8}$$

Since $u_\eta \in C^1([0,T];V)$ and $\beta_\eta \in W^{1,\infty}(0,T;V)$ we deduce from inequality (3.8) that $\Lambda\eta \in C([0,T];V)$.

Let now $\eta_1, \eta_2 \in C([0,T];V)$. For $t \in [0,T]$ we integrate (3.4) with the initial condition (3.5) to obtain that

$$\beta_{\eta_i}(t) = \beta_0 - \int_0^t \left(c_\tau \beta_{\eta_i}(s)\left|R_\tau\left(u_{\eta_i\tau}(s)\right)\right|^2 - \varepsilon_a\right)_+ ds.$$

Then there exists a constant $c_4 > 0$ such that

$$\begin{aligned}&\left\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\right\|_{L^2(\Gamma_3)}\\ &\leq c_4 \int_0^t \left\|\left(\beta_{\eta_1}(s)\right)\left(\left|R_\tau\left(u_{\eta_1\tau}(s)\right)\right|\right)^2 - \left(\beta_{\eta_2}(s)\right)\left(\left|R_\tau\left(u_{\eta_2\tau}(s)\right)\right|\right)^2\right\|_{L^2(\Gamma_3)} ds.\end{aligned}$$

We use the definition of the truncation operator R_τ and write

$$\beta_{\eta_1}(s) = \beta_{\eta_1}(s) - \beta_{\eta_2}(s) + \beta_{\eta_2}(s).$$

After some elementary calculus we find that there exists a constant $c_5 > 0$ such that

$$\begin{aligned}\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \leq c_5 \int_0^t \|\beta_{\eta_1}(s) - \beta_{\eta_2}(s)\|_{L^2(\Gamma_3)} \\ + c_5 \int_0^t \|u_{\eta_1\tau}(s) - u_{\eta_2\tau}(s)\|_{(L^2(\Gamma_3))^d}\ ds.\end{aligned}$$

Using Gronwall-type inequality, it follows that there exists a constant $c_6 > 0$ such that

$$\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \leq c_6 \int_0^t \|u_{\eta_1\tau}(s) - u_{\eta_2\tau}(s)\|_{(L^2(\Gamma_3))^d}\ ds.$$

Furthermore using (2.10), we obtain

$$\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \leq c_6 d_\Omega \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V\ ds. \tag{3.9}$$

On the other hand, using arguments similar to those in the proof of (3.8), we find that there exists a constant $c_7 > 0$ such that

$$\begin{aligned}&\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \\ &\leq c_7(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)}).\end{aligned}$$

Then, using (3.9) it follows that

$$\begin{aligned}&\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \\ &\leq c_7 \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + c_7 c_6 d_\Omega \textstyle\int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V\ ds.\end{aligned}$$

Moreover keeping in mind that

$$\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \leq \int_0^t \|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V\ ds,$$

we deduce, after combining these two last inequalities, that there exists a constant $c_8 > 0$ such that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c_8 \int_0^t \|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V\ ds.$$

On the other hand using (3.1) we see that

$$\langle A\varepsilon(v_{\eta_1}) - A\varepsilon(v_{\eta_2}), \varepsilon(v_{\eta_2}) - \varepsilon(v_{\eta_1})\rangle_Q + (\eta_1 - \eta_2, v_{\eta_2} - v_{\eta_1})_V \geq 0,$$

which implies by using (2.14)(c) that

$$\|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V \leq \frac{1}{m_A} \|\eta_1(s) - \eta_2(s)\|_V \quad \forall\, s \in [0,T].$$

Hence we deduce that there exists a constant $c_9 > 0$ such that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c_9 \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \, ds \quad \forall\, t \in [0,T]. \tag{3.10}$$

Let now $\alpha > 0$, and denote

$$\|\eta\|_\alpha = \sup_{t\in[0,T]} \left[\exp(-\alpha t)\, \|\eta(t)\|_V\right], \ \forall\, \eta \in C([0,T];V).$$

Clearly $\|\cdot\|_\alpha$ defines a norm on the space $C([0,T];V)$ which is equivalent to the standard norm $\|\cdot\|_{C([0,T];V)}$. Using (3.10) and arguments similar to those in the proof of (3.6), after some calculus we find that there exists a constant $c > 0$ such that

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_\alpha \leq \frac{c}{\alpha} \|\eta_1 - \eta_2\|_\alpha, \forall\, \eta_1, \eta_2 \in C([0,T];V).$$

So for $\alpha > c$, the operator Λ is a contraction on the space $C([0,T];V)$ endowed with the norm $\|\cdot\|_\alpha$. Then by using Banach fixed point theorem it follows that Λ has a unique fixed point $\eta^* \in C([0,T];V)$, which concludes the proof. □

Now, we have all the ingredients to prove Theorem 2.1.

PROOF: *Existence.* Let $\eta^* \in C([0,T];V)$ be the fixed point of Λ and let v_{η^*} be the solution of Problem $P_{1\eta}$ for $\eta = \eta^*$. We show that $(u_{\eta^*}, \beta_{\eta^*})$ is a solution of Problem P_2. Indeed, from (3.3), it follows that $u_{\eta^*} \in C^1([0,T];V)$ and it satisfies the initial condition (2.21). Let β denote the solution of Problem $P_{2\eta}$ for $\eta = \eta^*$, i.e., $\beta = \beta_{\eta^*}$. As $\eta^* = \Lambda\eta^*$, it follows that from (3.1) and (3.7) that (2.19) holds. Clearly, equalities (2.20) and (2.22) hold by Problem $P_{2\eta^*}$. Also the regularity of the bonding field $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{O}$ follows from Lemma 3.2.

Uniqueness. Let $(u,\beta) \in C^1([0,T];V) \times W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{O}$ be a solution of Problem P_2 and denote by $\eta \in C([0,T];V)$ the function defined by

$$\begin{aligned}(\eta(t), w)_V = \langle B\varepsilon(u(t)), \varepsilon(w)\rangle_Q + r(\beta(t), u(t), w)\\ \forall\, w \in V,\ t \in [0,T].\end{aligned} \tag{3.11}$$

Inequality (2.19) and equality (3.11) associated with the initial condition $u(0) = u_0$ imply that $\dot{u}$ is a solution of Problem $P_{1\eta}$ and, since this problem has a unique solution denoted v_η, we conclude that

$$\dot{u} = v_\eta.$$

As $v_\eta = \dot{u}_\eta$ and $u_\eta(0) = u(0) = u_0$, then we deduce that

$$u_\eta = u. \tag{3.12}$$

Next, (2.20) and the initial condition $\beta(0) = \beta_0$ imply that β is a solution of Problem $P_{2\eta}$ and, since this problem admits a unique solution β_η, we conclude that

$$\beta = \beta_\eta. \tag{3.13}$$

Using now (3.7) and (3.11)–(3.13) we obtain that $\Lambda\eta = \eta$ and as the operator Λ admits a unique fixed point guaranteed by Lemma 3.3, it follows that

$$\eta = \eta^*. \tag{3.14}$$

The uniqueness of the solution is now a consequence of (3.12)–(3.14). □

References

[1] Awbi B., Chau O., Sofonea A., *Variational analysis of a frictional contact problem for viscoelastic bodies*, Int. Math. J. **1** (2002), no. 4, 333–348.

[2] Brezis H., *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Annales Inst. Fourier **18** (1968), 115–175.

[3] Cangémi L., *Frottement et adhérence: modèle, traitement numérique et application à l'interface fibre/matrice*, Ph.D. Thesis, Univ. Méditerranée, Aix Marseille I, 1997.

[4] Chau O., Fernandez J.R., Shillor M., Sofonea M., *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Computational and Applied Mathematics **159** (2003), 431–465.

[5] Chau O., Shillor M., Sofonea M., *Dynamic frictionless contact with adhesion*, Z. Angew. Math. Phys. **55** (2004), 32–47.

[6] Cocu M., Rocca R., *Existence results for unilateral quasistatic contact problems with friction and adhesion*, Math. Model. Numer. Anal. **34** (2000), 981–1001.

[7] Duvaut G., Lions J.-L., *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[8] Fernandez J.R., Shillor M., Sofonea M., *Analysis and numerical simulations of a dynamic contact problem with adhesion*, Math. Comput. Modelling **37** (2003) 1317–1333.

[9] Frémond M., *Adhérence des solides*, J. Méc. Théor. Appl. **6** (1987), 383–407.

[10] Frémond M., *Equilibre des structures qui adhèrent à leur support*, C.R. Acad. Sci. Paris Sér. II **295**, (1982), 913–916.

[11] Frémond M., *Non-smooth Thermomechanics*, Springer, Berlin, 2002.

[12] Nassar S.A., Andrews T., Kruk S., Shillor M., *Modelling and simulations of a bonded rod*, Math. Comput. Modelling **42** (2005), 553–572.

[13] Raous M., Cangémi L., Cocu M., *A consistent model coupling adhesion, friction, and unilateral contact*, Comput. Methods Appl. Mech. Engrg. **177** (1999), 383–399.

[14] Rojek J., Telega J.J., *Contact problems with friction, adhesion and wear in orthopeadic biomechanics. I: General developements*, J. Theor. Appl. Mech. **39** (2001), 655–677.

[15] Shillor M., Sofonea M., Telega J.J., *Models and Variational Analysis of Quasistatic Contact*, Lecture Notes in Physics, 655, Springer, Berlin, 2004.

[16] Sofonea M., Han W., Shillor M., *Analysis and Approximations of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics, 276, Chapman & Hall / CRC Press, Boca Raton, Florida, 2006.

[17] Sofonea M., Hoarau-Mantel T.V., *Elastic frictionless contact problems with adhesion*, Adv. Math. Sci. Appl. **15** (2005), no. 1, 49–68.
[18] Sofonea M., Arhab R., Tarraf R., *Analysis of electroelastic frictionless contact problems with adhesion*, J. Appl. Math. **2006**, ID 64217, pp.1–25.

Laboratoire de Systèmes dynamiques, Faculté de Mathématiques, USTHB, BP 32 El Alia, Bab-Ezzouar, 16111, Algérie
Email: ttouzaline@yahoo.fr

(*Received* July 6, 2009, *revised* January 11, 2010)