Commutators and associators in Catalan loops

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Abstract. Various commutators and associators may be defined in one-sided loops. In this paper, we approximate and compare these objects in the left and right loop reducts of a Catalan loop. To within a certain order of approximation, they turn out to be quite symmetrical. Using the general analysis of commutators and associators, we investigate the structure of a specific Catalan loop which is non-commutative, but associative, that appears in the original number-theoretic application of Catalan loops.

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1. Introduction

Catalan loops originated from an issue in number theory, studying the relationship between Fermat curves and modular curves. (A more detailed description of the motivation is given in [4].) Catalan loops are two-sided loops that admit a range of possible definitions for commutators and associators. The topic of this paper is the calculation of the commutators and associators of Catalan loops up to a certain order of approximation, and a further investigation of the structure in one of the motivating cases.

A general discussion of commutators and associators in left and right loops is provided in Section 2. In particular, this discussion touches on the diverging commutator conventions in groups. A brief introduction to Catalan loops is given in Section 3. Section 4 covers the formal calculation of the different commutators and associators. In Section 5 we make approximations in order to compare the different commutators and associators in Section 6. Then, Section 7 focuses on one of the smaller motivating cases where the Catalan loop actually turns out to be associative and, hence, a group. Finally, we discuss directions for future work on this topic in Section 8.

2. Commutators and associators

A *right quasigroup* $(Q, \cdot, /)$ is a set Q together with binary operations of *multiplication* (denoted by $x \cdot y$ or juxtaposition xy) and *right division* x/y such that

$$(x \cdot y)/y = x = (x/y) \cdot y$$

for x, y in Q. A *right loop* $(Q, \cdot, /, 1)$ is a right quasigroup $(Q, \cdot, /)$ with an *identity element* 1 such that

$$1 \cdot x = x = x \cdot 1$$

for x in Q. We define a *right loop commutator* to be

$$[x, y]_R = (xy)/(yx)$$

for x, y in Q. Further, we set *right loop associators* to be

$$(x, y, z)_R = (xy \cdot z)/(x \cdot yz)$$

and

$$(x, y, z)_R^* = (x \cdot yz)/(xy \cdot z)$$

for x, y, z in Q.

Dually, a *left quasigroup* $(U, \cdot, \backslash)$ is a set U together with binary operations of *multiplication* (denoted by $x \cdot y$ or juxtaposition xy) and *left division* $y\backslash x$ such that

$$y\backslash(y \cdot x) = x = y \cdot (y\backslash x)$$

for x, y in U. A *left loop* $(U, \cdot, \backslash, 1)$ is a left quasigroup $(U, \cdot, \backslash)$ with an *identity element* 1 such that

$$1 \cdot x = x = x \cdot 1$$

for x in U. We define a *left loop commutator* to be

$$[x, y]_L = (yx)\backslash(xy) \tag{1}$$

for x, y in U. Further, we set *left loop associators* to be

$$(x, y, z)_L = (x \cdot yz)\backslash(xy \cdot z) \tag{2}$$

and

$$(x, y, z)_L^* = (xy \cdot z)\backslash(x \cdot yz)$$

for x, y, z in U. Note that (1) and (2) agree with Bruck's definitions of commutators and associators in a general quasigroup [2, I(2.1)].

A right loop is commutative if and only if

$$[x, y]_R = 1$$

for all x, y. Also, a right loop is associative if and only if

$$(x, y, z)_R = 1$$

for all x, y, z, or equivalently,

$$(x, y, z)_R^* = 1$$

for all x, y, z. Analogous results hold in a left loop.

Note that we have

$$[x,y]_R = xyx^{-1}y^{-1} \tag{3}$$

and

$$[x,y]_L = x^{-1}y^{-1}xy \tag{4}$$

in a group (with $x/y = xy^{-1}$ and $y\backslash x = y^{-1}x$). Traditionally, (3) was used as the commutator definition in the topological literature (e.g. [5, §X.5]), while (4) was used in the algebraic literature (e.g. [3, p. 10]).

3. Catalan loops

Let R be a commutative, unital ring, with a topologically nilpotent element e. In other words, R is complete in the (eR)-adic topology [1, §2.6]. Let E be the annihilator of e in R. Let H be the subgroup of diagonal matrices in $\mathrm{SL}(2,R)$. Consider the set

$$Q' = \left\{ \begin{bmatrix} 1 & ex \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ ex' & 1 \end{bmatrix} \;\middle|\; x, x' \in R \right\}.$$

Define $G = HQ'$. By [4, Proposition 4.2, 4.3], the set Q' forms a loop transversal to the subgroup H in G. The characterization as a loop transversal yields a right loop structure on Q', a so-called *Catalan loop*. Further, by [4, Corollary 5.2; Theorem 5.3] the Catalan loop on Q' is two-sided and may alternatively be represented on $(R/E)^2$, as exhibited in the next paragraph. We use the alternative representation, since it proves to be helpful for our calculations.

Let Q be $(R/E)^2$. It will often be convenient to use a vector notation $\mathbf{x} = \langle x, x' \rangle$ for elements of Q. For more complicated pairs $\mathbf{x}$ we will refer to the first component as $[\mathbf{x}]_1$. Then the three binary operations in the *Catalan loop* $(Q, \cdot, /, \backslash, \langle 0, 0\rangle)$ are given as follows. Multiplication:

$$\langle x, x' \rangle \cdot \langle y, y' \rangle = \left\langle x\lambda_m^2 + y\lambda_m, x'\lambda_m^{-1} + y' \right\rangle \tag{5}$$

with $\lambda_m = \lambda_m(\mathbf{x}, \mathbf{y}) = 1 + e^2(yx')$. Right division:

$$\langle x, x' \rangle \,/\, \langle y, y' \rangle = \left\langle x\lambda_r^2 - y\lambda_r, x'\lambda_r^{-1} - y'\lambda_r^{-1} \right\rangle \tag{6}$$

with $\lambda_r = \lambda_r(\mathbf{x}, \mathbf{y}) = 1 - e^2y(x' - y')$. Left division:

$$\langle x, x' \rangle \backslash \langle y, y' \rangle = \left\langle dy - d^{-1}x, \left(d^{-1}y' - dx'\right) - e^2x'y'\left(dy - d^{-1}x\right) \right\rangle, \tag{7}$$

where $d = d(\mathbf{x}, \mathbf{y})$ is the unique recursive solution

$$d = 1 + e^2 \cdot x'(x - y) - e^4 \cdot 2x'^2(x - y) + \dots$$

to the equation

$$d = \left(1 + e^2xx'\right) - e^2d^2x'y.$$

The multipliers λ_m, λ_r and d are known as *fudge factors*.

Now, consider the following two remarks about the right and left division which prove to be useful.

Remark 1. If the respective fudge factors $\lambda_r(\mathbf{x},\mathbf{y}), \lambda_r(\mathbf{y},\mathbf{x})$ are equal to 1, which means

$$e^2 y\left(x'-y'\right) = e^2 x\left(y'-x'\right) = 0,$$

we have

$$\mathbf{x}/\mathbf{y} = \langle x-y, x'-y'\rangle = -\mathbf{y}/\mathbf{x}.$$

Remark 2. If the respective fudge factors $d(\mathbf{y},\mathbf{x})$ and $\lambda_r(\mathbf{x},\mathbf{y})$ are equal to 1 and

$$e^2 y' x'\left(d(\mathbf{y},\mathbf{x})x - d(\mathbf{y},\mathbf{x})^{-1}y\right) = 0,$$

then

$$\mathbf{x}/\mathbf{y} = \mathbf{y}\backslash\mathbf{x}.$$

4. Formal calculations

In this section, we will conduct formal computations of the commutators and associators in a Catalan loop, mainly to introduce the fudge factors we will need later on. Having the respective formulae available will make the subsequent approximations easier. First note

$$\mathbf{xy} = \left\langle x\lambda_1^2 + y\lambda_1, x'\lambda_1^{-1} + y'\right\rangle \quad \text{with} \quad \lambda_1 = 1 + e^2 y x',$$

and

$$\mathbf{yx} = \left\langle y\bar{\lambda}_1^2 + x\bar{\lambda}_1, y'\bar{\lambda}_1^{-1} + x'\right\rangle \quad \text{with} \quad \bar{\lambda}_1 = 1 + e^2 x y'.$$

4.1 The commutators. First we will formally calculate the left and right commutator. The right commutator $[\mathbf{x},\mathbf{y}]_R$ is

$$\begin{aligned}&\left\langle \left(x\lambda_1^2 + y\lambda_1\right)\lambda_R^2 - \left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\lambda_R, \left(x'\lambda_1^{-1} + y'\right)\lambda_R^{-1} - \left(y'\bar{\lambda}_1^{-1} + x'\right)\lambda_R^{-1}\right\rangle \\ &= \left\langle x\left(\lambda_1^2\lambda_R^2 - \bar{\lambda}_1\lambda_R\right) + y\left(\lambda_1\lambda_R^2 - \bar{\lambda}_1^2\lambda_R\right), x'\left(\lambda_1^{-1}\lambda_R^{-1} - \lambda_R^{-1}\right) + y'\left(\lambda_R^{-1} - \bar{\lambda}_1^{-1}\lambda_R^{-1}\right)\right\rangle\end{aligned}$$

with λ_R equal to

$$\begin{aligned}&1 - e^2\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\left[\left(x'\lambda_1^{-1} + y'\right) - \left(y'\bar{\lambda}_1^{-1} + x'\right)\right] \\ &= 1 - e^2\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\left[x'\left(\lambda_1^{-1} - 1\right) - y'\left(\bar{\lambda}_1^{-1} - 1\right)\right].\end{aligned}$$

The left commutator $[\mathbf{x},\mathbf{y}]_L$ is

$$\begin{aligned}&\Big\langle \lambda_L\left(x\lambda_1^2 + y\lambda_1\right) - \lambda_L^{-1}\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right), \\ &\quad \left[\lambda_L^{-1}\left(x'\lambda_1^{-1} + y'\right) - \lambda_L\left(y'\bar{\lambda}_1^{-1} + x'\right)\right] \\ &\quad - e^2\left(y'\bar{\lambda}_1^{-1} + x'\right)\left(x'\lambda_1^{-1} + y'\right)\left([[\mathbf{x},\mathbf{y}]_L]_1\right)\Big\rangle,\end{aligned}$$

with λ_L equal to

$$1+e^2\cdot\left(y'\bar\lambda_1^{-1}+x'\right)\left[\left(y\bar\lambda_1^2+x\bar\lambda_1\right)-\left(x\lambda_1^2+y\lambda_1\right)\right]+O\left(e^4\right).$$

4.2 The right associators. Let us now have a look at the associators in the right loop reduct. Firstly, $\mathbf{xy}\cdot\mathbf{z}$ is

$$\left\langle(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2,(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right\rangle$$

with $\lambda_2=1+e^2z(x'\lambda_1^{-1}+y')$. Secondly,

$$\mathbf{yz}=\left\langle y\lambda_3^2+z\lambda_3,y'\lambda_3^{-1}+z'\right\rangle\quad\text{with}\quad\lambda_3=1+e^2zy',$$

and thus $\mathbf{x}\cdot\mathbf{yz}$ is

$$\left\langle x\lambda_4^2+(y\lambda_3^2+z\lambda_3)\lambda_4,x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right\rangle$$

with $\lambda_4=1+e^2(y\lambda_3^2+z\lambda_3)x'$. Finally, $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ is equal to

$$\begin{aligned}&\Big\langle\left[\left(x\lambda_1^2+y\lambda_1\right)\lambda_2^2+z\lambda_2\right]\lambda_5^2-\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\lambda_5,\\&\quad\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'\right]\lambda_5^{-1}-\left[x'\lambda_4^{-1}+\left(y'\lambda_3^{-1}+z'\right)\right]\lambda_5^{-1}\Big\rangle\\&\quad=\Big\langle x\left(\lambda_1^2\lambda_2^2\lambda_5^2-\lambda_4^2\lambda_5\right)+y\left(\lambda_1\lambda_2^2\lambda_5^2-\lambda_3^2\lambda_4\lambda_5\right)+z\left(\lambda_2\lambda_5^2-\lambda_3\lambda_4\lambda_5\right),\\&\qquad x'\left(\lambda_1^{-1}\lambda_2^{-1}\lambda_5^{-1}-\lambda_4^{-1}\lambda_5^{-1}\right)+y'\left(\lambda_2^{-1}\lambda_5^{-1}-\lambda_3^{-1}\lambda_5^{-1}\right)\Big\rangle,\end{aligned}$$

where λ_5 is

$$\begin{aligned}&1-e^2\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\cdot\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'-x'\lambda_4^{-1}-y'\lambda_3^{-1}-z'\right]\\&=1-e^2\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]\cdot\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}-\lambda_4^{-1}\right)+y'\left(\lambda_2^{-1}-\lambda_3^{-1}\right)\right].\end{aligned}$$

Similarly, we get $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*=$

$$\begin{aligned}&=\Big\langle\left[x\lambda_4^2+\left(y\lambda_3^2+z\lambda_3\right)\lambda_4\right]{\lambda_5^*}^2-\left[\left(x\lambda_1^2+y\lambda_1\right)\lambda_2^2+z\lambda_2\right]\lambda_5^*,\\&\quad-\left[\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'\right]{\lambda_5^*}^{-1}-\left[x'\lambda_4^{-1}+\left(y'\lambda_3^{-1}+z'\right)\right]{\lambda_5^*}^{-1}\right]\Big\rangle\\&=\Big\langle x\left(-\lambda_1^2\lambda_2^2\lambda_5^*+\lambda_4^2{\lambda_5^*}^2\right)+y\left(-\lambda_1\lambda_2^2\lambda_5^*+\lambda_3^2\lambda_4{\lambda_5^*}^2\right)+z\left(-\lambda_2\lambda_5^*+\lambda_3\lambda_4{\lambda_5^*}^2\right),\\&\quad-\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}{\lambda_5^*}^{-1}-\lambda_4^{-1}{\lambda_5^*}^{-1}\right)+y'\left(\lambda_2^{-1}{\lambda_5^*}^{-1}-\lambda_3^{-1}{\lambda_5^*}^{-1}\right)\right]\Big\rangle,\end{aligned}$$

where λ_5^* is

$$\begin{aligned}&1-e^2\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]\left[-\left[\left(x'\lambda_1^{-1}+y'\right)\lambda_2^{-1}+z'-x'\lambda_4^{-1}-y'\lambda_3^{-1}-z'\right]\right]\\&=1+e^2\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]\left[x'\left(\lambda_1^{-1}\lambda_2^{-1}-\lambda_4^{-1}\right)+y'\left(\lambda_2^{-1}-\lambda_3^{-1}\right)\right].\end{aligned}$$

4.3 The left associators. In the left loop reduct, $(\mathbf{x},\mathbf{y},\mathbf{z})_L$ is

$$\begin{aligned}&\Big\langle d\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]-d^{-1}\left[x\lambda_4^2+(y\lambda_3^2+z\lambda_3)\lambda_4\right],\\&\ \Big[d^{-1}\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]-d\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\Big]-\\&\ -e^2\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\cdot\left([(\mathbf{x},\mathbf{y},\mathbf{z})_L]_1\right)\Big\rangle,\end{aligned}$$

where d is

$$\begin{aligned}&1-e^2\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\\&\times\left[x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)\right]+O\left(e^4\right).\end{aligned}$$

Similarly, $(\mathbf{x},\mathbf{y},\mathbf{z})_L^*$ is equal to

$$\begin{aligned}&\Big\langle d^*\left[x\lambda_4^2+(y\lambda_3^2+z\lambda_3)\lambda_4\right]-{d^*}^{-1}\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right],\\&\ \Big[{d^*}^{-1}\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]-d^*\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]\Big]\\&\ -e^2\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]\left[x'\lambda_4^{-1}+(y'\lambda_3^{-1}+z')\right]\cdot\left([(\mathbf{x},\mathbf{y},\mathbf{z})_L^*]_1\right)\Big\rangle,\end{aligned}$$

where d^* is

$$\begin{aligned}&1+e^2\left[(x'\lambda_1^{-1}+y')\lambda_2^{-1}+z'\right]\\&\times\left[\left[(x\lambda_1^2+y\lambda_1)\lambda_2^2+z\lambda_2\right]-\left[x\lambda_4^2+(y\lambda_3^2+z\lambda_3)\lambda_4\right]\right]+O\left(e^4\right)\\&=1+e^2\left(x'\lambda_1^{-1}\lambda_2^{-1}+y'\lambda_2^{-1}+z'\right)\\&\times\left[x\left(\lambda_1^2\lambda_2^2-\lambda_4^2\right)+y\left(\lambda_1\lambda_2^2-\lambda_3^2\lambda_4\right)+z\left(\lambda_2-\lambda_3\lambda_4\right)\right]+O\left(e^4\right).\end{aligned}$$

5. Approximations and comparisons

Having done the formal calculations for the objects of interest, we will now approximate our results and compare the respective commutators and associators. In doing so we will have to make extensive use of the following remark.

Remark 3. Note that if $\lambda=1+O\left(e^4\right)$, then $\lambda^k=1+O\left(e^4\right)$ for any integer k. And thus a multiplication by any power of λ is essentially a multiplication by the identity plus an additional summand of order $O\left(e^4\right)$.

In order to have easy access to versions of the Remarks 1 and 2 which will prove to be useful in the following approximations, we formulate two lemmata.

Lemma 1. *If $\lambda_r(\mathbf{x},\mathbf{y})$ and $\lambda_r(\mathbf{y},\mathbf{x})$ are of order $1+O\left(e^4\right)$, then*

$$\mathbf{x}/\mathbf{y}=-\mathbf{y}/\mathbf{x}$$

up to $O\left(e^4\right)$.

PROOF: Using Remark 3 for $\lambda_r(\mathbf{x},\mathbf{y})$ and $\lambda_r(\mathbf{y},\mathbf{x})$ we get

$$\mathbf{x}/\mathbf{y} = \langle x-y, x'-y'\rangle + O\left(e^4\right)$$

and

$$\mathbf{y}/\mathbf{x} = \langle y-x, y'-x'\rangle + O\left(e^4\right),$$

and are done. □

Lemma 2. *If $\lambda_r(\mathbf{x},\mathbf{y})$ and $d(\mathbf{y},\mathbf{x})$ are of order $1+O\left(e^4\right)$, and $x-y$ is of order $O\left(e^2\right)$, then*

$$\mathbf{x}/\mathbf{y} = \mathbf{y}\backslash\mathbf{x}$$

up to $O\left(e^4\right)$.

PROOF: Using Remark 3 for $\lambda_r(\mathbf{x},\mathbf{y})$ and $d(\mathbf{y},\mathbf{x})$ we get

$$\mathbf{x}/\mathbf{y} = \langle x-y, x'-y'\rangle + O\left(e^4\right)$$

and

$$\mathbf{y}\backslash\mathbf{x} = \left\langle x-y, x'-y'-e^2y'x'(x-y)\right\rangle + O\left(e^4\right).$$

Then we are done, since $e^2y'x'(x-y)$ is already of order $O\left(e^4\right)$ by the second part of the assumption. □

5.1 The commutators. Let us start by having a closer look at the fudge factor

$$\lambda_R = 1 - e^2\left(y\bar\lambda_1^2 + x\bar\lambda_1\right)\left[x'\left(\lambda_1^{-1}-1\right) - y'\left(\bar\lambda_1^{-1}-1\right)\right]$$

of the right commutator $[\mathbf{x},\mathbf{y}]_R$. Since

- $\lambda_1^2 = 1 + e^2 2yx' + O\left(e^4\right)$,
- $\lambda_1^{-1} = 1 - e^2yx' + O\left(e^4\right) \Rightarrow (\lambda_1^{-1}-1) = -e^2yx' + O\left(e^4\right)$,

and similarly

- $\bar\lambda_1^2 = 1 + e^2 2xy' + O\left(e^4\right)$,
- $(\bar\lambda_1^{-1}-1) = -e^2xy' + O\left(e^4\right)$,

we have

$$\lambda_R = 1 - e^4(x+y)\left(xy'^2 - yx'^2\right) + O\left(e^6\right).$$

Thus

$$\lambda_R = 1 + O\left(e^4\right).$$

By Remark 3, the second component of $[\mathbf{x},\mathbf{y}]_R$ now becomes

$$\begin{aligned} &x'\left(\lambda_1^{-1}-1\right) + y'\left(1-\bar\lambda_1^{-1}\right) + O\left(e^4\right) \\ &= e^2\left(xy'^2 - yx'^2\right) + O\left(e^4\right). \end{aligned} \tag{8}$$

Further, we have

$$\lambda_1^2 - \bar\lambda_1 = \left[1 + e^2 2yx' + O\left(e^4\right)\right]\left[1 + O\left(e^4\right)\right] - \left[1 + e^2xy'\right]$$

$$= e^2 (2yx' - xy') + O(e^4)$$

and

$$\begin{aligned} \lambda_1 - \bar{\lambda}_1^2 &= \left(1 + e^2 yx'\right) - \left(1 + e^2 2xy'\right) + O\left(e^4\right) \\ &= e^2 (yx' - 2xy') + O\left(e^4\right). \end{aligned}$$

Thus

$$\begin{aligned} &x\left(\lambda_1^2 - \bar{\lambda}_1\right) + y\left(\lambda_1 - \bar{\lambda}_1^2\right) \\ &\quad = e^2 2xy\left[(x' - y') + y^2 x' - x^2 y'\right] + O\left(e^4\right). \end{aligned} \tag{9}$$

Hence applying Remark 3 we see that

$$[\mathbf{x}, \mathbf{y}]_R = e^2 \left\langle 2xy(x' - y') + y^2 x' - x^2 y', xy'^2 - yx'^2 \right\rangle + O\left(e^4\right). \tag{10}$$

Now, we would like to compare $[\mathbf{x}, \mathbf{y}]_R$ and $[\mathbf{y}, \mathbf{x}]_R$ with the help of Lemma 1. All that is left to show is that the fudge factor $\bar{\lambda}_R$ of $[\mathbf{y}, \mathbf{x}]_R$ is of order $1 + O\left(e^4\right)$. But

$$\bar{\lambda}_R = 1 + e^2 \left(x\lambda_1^2 + y\lambda_1\right)\left[x'\left(\lambda_1^{-1} - 1\right) - y'\left(\bar{\lambda}_1^{-1} - 1\right)\right],$$

and by (8) we have

$$\left[x'\left(\lambda_1^{-1} - 1\right) - y'\left(\bar{\lambda}_1^{-1} - 1\right)\right] = O\left(e^2\right).$$

Thus

$$\bar{\lambda}_R = 1 + O\left(e^4\right),$$

whence

$$[\mathbf{x}, \mathbf{y}]_R = -[\mathbf{y}, \mathbf{x}]_R$$

up to $O\left(e^4\right)$ by Lemma 1.

Next, we will make use of Lemma 2 in order to compare the commutators in the left loop reduct with the ones we have just exhibited in the right loop reduct. First, notice that

$$\begin{aligned} [\mathbf{xy}]_1 - [\mathbf{yx}]_1 &= \left(x\lambda_1^2 + y\lambda_1\right) - \left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right) \\ &= O\left(e^2\right) \text{ by (9).} \end{aligned} \tag{11}$$

Secondly, consider the fudge factor of the left commutator $[\mathbf{x}, \mathbf{y}]_L$:

$$\begin{aligned} \lambda_L &= 1 + e^2 \cdot \left(y'\bar{\lambda}_1^{-1} + x'\right)\left[\left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right) - \left(x\lambda_1^2 + y\lambda_1\right)\right] + O\left(e^4\right) \\ &= 1 - e^2 \cdot \left(y'\bar{\lambda}_1^{-1} + x'\right)\left([\mathbf{xy}]_1 - [\mathbf{yx}]_1\right) + O\left(e^4\right) \\ &= 1 + O\left(e^4\right) \text{ by (11).} \end{aligned} \tag{12}$$

Thus the assumptions of Lemma 2 are satisfied, and we conclude that

$$[\mathbf{x},\mathbf{y}]_R = [\mathbf{x},\mathbf{y}]_L$$

up to $O\left(e^4\right)$. Similarly, we consider $[\mathbf{y},\mathbf{x}]_L$ with fudge factor

$$\begin{aligned}\bar{\lambda}_L &= 1 + e^2 \cdot \left(x'\lambda_1^{-1} + y'\right)\left[\left(x\lambda_1^2 + y\lambda_1\right) - \left(y\bar{\lambda}_1^2 + x\bar{\lambda}_1\right)\right] + O\left(e^4\right)\\ &= 1 + e^2 \cdot \left(x'\lambda_1^{-1} + y'\right)\left([\mathbf{xy}]_1 - [\mathbf{yx}]_1\right) + O\left(e^4\right)\\ &= 1 + O\left(e^4\right) \text{ by (11).}\end{aligned} \tag{13}$$

Obviously $[\mathbf{yx}]_1 - [\mathbf{xy}]_1$ is also of order $O\left(e^4\right)$ by (11), and we can apply Lemma 2 again to see that

$$[\mathbf{y},\mathbf{x}]_R = [\mathbf{y},\mathbf{x}]_L$$

up to $O\left(e^4\right)$.

5.2 The right associators. It turns out that the associators are related in a very similar fashion. Let us first focus on the right loop, and thus on $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*$. As before, we will first determine the approximation of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$, and then use Lemma 1 to determine $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*$. Consider the fudge factor of $(\mathbf{x},\mathbf{y},\mathbf{z})_R$:

$$\lambda_5 = 1 - e^2\left[x\lambda_4^2 + \left(y\lambda_3^2 + z\lambda_3\right)\lambda_4\right]\left[x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right)\right].$$

Since

$$\begin{aligned}&\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\\ &\quad = \left(1 - e^2yx' + O\left(e^4\right)\right)\left(1 - e^2z(x'+y') + O\left(e^4\right)\right)\\ &\qquad - \left(1 - e^2x'(y+z) + O\left(e^4\right)\right)\\ &\quad = 1 - e^2x'z - e^2y'z - e^2x'y - 1 + e^2x'y + e^2x'z + O\left(e^4\right)\\ &\quad = -e^2y'z + O\left(e^4\right)\end{aligned}$$

and

$$\begin{aligned}&\lambda_2^{-1} - \lambda_3^{-1}\\ &\quad = \left(1 - e^2z(x'+y') + O\left(e^4\right)\right) - \left(1 - e^2zy' + O\left(e^4\right)\right)\\ &\quad = -e^2x'z + O\left(e^4\right),\end{aligned}$$

we have

$$\begin{aligned}x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right) &= -e^2x'y'z - e^2x'y'z + O\left(e^4\right)\\ &= e^2\left(-2x'y'z\right) + O\left(e^4\right)\\ &= O(e^2).\end{aligned}$$

Hence

$$\lambda_5 = 1 + O\left(e^4\right).$$

Using Remark 3, we now directly conclude that the *second component* of $(\mathbf{x}, \mathbf{y}, \mathbf{z})_R$ is

$$\begin{aligned} &x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right) + O\left(e^4\right) \\ &\quad = e^2\left(-2x'y'z\right) + O\left(e^4\right), \end{aligned} \tag{14}$$

and the *first component* of $(\mathbf{x}, \mathbf{y}, \mathbf{z})_R$ is

$$x\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right) + O\left(e^4\right).$$

We will reuse this formula for all the other associators later on. So let us now calculate this in detail. First of all,

$$\begin{aligned} x\big(\lambda_1^2\lambda_2^2 - \lambda_4^2\big) &= x\Big[\left(1 + e^2 2yx' + O\left(e^4\right)\right)\left(1 + e^2\left(2zx' + 2zy'\right) + O\left(e^4\right)\right) \\ &\quad - \left(1 + e^2\left(2yx' + 2zx'\right) + O\left(e^4\right)\right)\Big] \\ &= x\left[1 + e^2\left(2yx' + 2zx' + 2zy'\right) - 1 - e^2\left(2yx' + 2zx'\right) + O\left(e^4\right)\right] \\ &= e^2 2xzy' + O\left(e^4\right). \end{aligned}$$

Secondly,

$$\begin{aligned} y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) &= y\Big[\left(1 + e^2yx'\right)\left(1 + e^2\left(2zx' + 2zy'\right) + O\left(e^4\right)\right) \\ &\quad - \left(1 + e^2 2zy' + O\left(e^4\right)\right)\left(1 + e^2\left(yx' + zx'\right) + O\left(e^4\right)\right)\Big] \\ &= y\big[1 + e^2\left(yx' + 2zx' + 2zy'\right) \\ &\quad -1 - e^2\left(2zy' + yx' + zx'\right) + O\left(e^4\right)\big] \\ &= e^2 yzx' + O\left(e^4\right). \end{aligned}$$

And finally

$$\begin{aligned} z\left(\lambda_2 - \lambda_3\lambda_4\right) &= z\Big[\left(1 + e^2\left(zx' + zy'\right) + O\left(e^4\right)\right) \\ &\quad - \left(1 + e^2zy'\right)\left(1 + e^2\left(yx' + zx'\right) + O\left(e^4\right)\right)\Big] \\ &= z\left[1 + e^2\left(zx' + zy'\right) - 1 - e^2\left(zy' + yx' + zx'\right) + O\left(e^4\right)\right] \\ &= -e^2yzx' + O\left(e^4\right). \end{aligned}$$

Thus the *first component* of $(\mathbf{x}, \mathbf{y}, \mathbf{z})_R$ is

$$\begin{aligned} &\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right) \\ &\quad = e^2\left[2xzy' + yzx' - yzx'\right] + O\left(e^4\right) = e^2\left(2xzy'\right) + O\left(e^4\right). \end{aligned} \tag{15}$$

Finally, we conclude that

$$(\mathbf{x}, \mathbf{y}, \mathbf{z})_R = e^2\big\langle 2xzy', -2x'zy'\big\rangle + O\left(e^4\right) = e^2 2zy'\big\langle x, -x'\big\rangle + O\left(e^4\right).$$

Now, we would like to apply Lemma 1 to find out about $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*$. Therefore we only need to show that

$$\lambda_5^* = 1 + O\left(e^4\right),$$

which is immediate, since we have seen in (14) that

$$x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right) = O\left(e^2\right),$$

and λ_5^* is by definition

$$1 + e^2\left[(x\lambda_1^2 + y\lambda_1)\lambda_2^2 + z\lambda_2\right]\cdot\left[x'\left(\lambda_1^{-1}\lambda_2^{-1} - \lambda_4^{-1}\right) + y'\left(\lambda_2^{-1} - \lambda_3^{-1}\right)\right].$$

So we have

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R^* = (\mathbf{x},\mathbf{y},\mathbf{z})_R$$

up to $O\left(e^4\right)$.

5.3 The left associators. Now, we will apply Lemma 2 to the results in the right loop reduct we just obtained to derive approximations for $(\mathbf{x},\mathbf{y},\mathbf{z})_L$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_L^*$. Therefore, we need to show that the assumptions of Lemma 2 are satisfied. First, notice that

$$\begin{aligned}
&\left[(\mathbf{xy}\cdot\mathbf{z})\right]_1 - \left[(\mathbf{x}\cdot\mathbf{yz})\right]_1 \\
&\quad = x\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right) \\
&\quad = O\left(e^2\right)
\end{aligned} \tag{16}$$

by (15). Secondly, since this factor appears in both of the fudge factors

$$\begin{aligned}
d = 1 &- e^2\left[x'\lambda_4^{-1} + (y'\lambda_3^{-1} + z')\right] \\
&\times\left[x\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right)\right] + O\left(e^4\right)
\end{aligned}$$

and

$$\begin{aligned}
d^* = 1 &+ e^2\left(x'\lambda_1^{-1}\lambda_2^{-1} + y'\lambda_2^{-1} + z'\right) \\
&\times\left[x\left(\lambda_1^2\lambda_2^2 - \lambda_4^2\right) + y\left(\lambda_1\lambda_2^2 - \lambda_3^2\lambda_4\right) + z\left(\lambda_2 - \lambda_3\lambda_4\right)\right] + O\left(e^4\right),
\end{aligned}$$

we conclude that d and d^* are of order $1 + O\left(e^4\right)$. So applying Lemma 2 to $(\mathbf{x},\mathbf{y},\mathbf{z})_R$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_L$, and $(\mathbf{x},\mathbf{y},\mathbf{z})_R^*$ and $(\mathbf{x},\mathbf{y},\mathbf{z})_L^*$ respectively, we get

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R = (\mathbf{x},\mathbf{y},\mathbf{z})_L \quad\text{and}\quad (\mathbf{x},\mathbf{y},\mathbf{z})_R^* = (\mathbf{x},\mathbf{y},\mathbf{z})_L^*$$

up to $O\left(e^4\right)$.

6. Symmetry theorems

Having made the approximations, we now observe the following symmetries.

Theorem 1. *To within e^4 in a Catalan loop, we have*

$$[\mathbf{x},\mathbf{y}]_R = [\mathbf{x},\mathbf{y}]_L = -\,[\mathbf{y},\mathbf{x}]_L = -\,[\mathbf{y},\mathbf{x}]_R$$

and

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R = (\mathbf{x},\mathbf{y},\mathbf{z})_L = -(\mathbf{x},\mathbf{y},\mathbf{z})_L^* = -(\mathbf{x},\mathbf{y},\mathbf{z})_R^* .$$

Now set $\overleftarrow{\mathbf{x}} = \big\langle x', x \big\rangle$.

Theorem 2. *To within e^4 in a Catalan loop, we have*

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R = \big(\mathbf{x}, \overleftarrow{\mathbf{z}}\,, \overleftarrow{\mathbf{y}}\,\big)_R$$

and

$$\big(\overleftarrow{\mathbf{x}}\,, \mathbf{y}, \mathbf{z}\big)_R = -\overleftarrow{(\mathbf{x},\mathbf{y},\mathbf{z})}_R.$$

7. The special case of $e = 2$

Let $R = \mathbb{Z}/2^{n+1}\mathbb{Z}$ (as in the motivating case, see [4]). Then $e = 2$ is certainly nilpotent in R, with annihilator $E = \{0, 2^n\}$. Furthermore,

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R = e^3 zy' \big\langle x, -x' \big\rangle + O\left(e^4\right)$$

and

$$[\mathbf{x},\mathbf{y}]_R = e^2 \big\langle x'y^2 - y'x^2, xy'^2 - yx'^2 \big\rangle + e^3 \big\langle xy\left(x' - y'\right), 0 \big\rangle + O\left(e^4\right).$$

Now we will have a closer look at the Catalan loop on

$$G = \left(\left(\mathbb{Z}/2^{3+1}\mathbb{Z}\right) / \left\{0, 2^3\right\}\right)^2 = \left(\mathbb{Z}/2^3\mathbb{Z}\right)^2,$$

which turns out to be a group. It is convenient to set $\mathbf{n} = \langle n, n \rangle$ for integers n.

Theorem 3. *The Catalan loop $(G, \cdot, \mathbf{0})$ is a group.*

Proof: Note that

$$(\mathbf{x},\mathbf{y},\mathbf{z})_R = e^3 zy' \big\langle x, -x' \big\rangle + O\left(e^4\right) = \mathbf{0},$$

since $e^3 = 8 = 0$ in $\mathbb{Z}/2^3\mathbb{Z}$. Thus we have associativity, and are done, since the Catalan loop $(G, \cdot, /, \backslash, \mathbf{0})$ is a two-sided loop. $\square$

Considering the commutator

$$[\mathbf{x},\mathbf{y}]_R = e^2 \big\langle x'y^2 - y'x^2, xy'^2 - yx'^2 \big\rangle,$$

we see that $(G,\cdot,\mathbf{0})$ is certainly *non-abelian*, since

$$[(0,1),(1,0)]_R = (4,4) \neq \mathbf{0}.$$

It turns out that the commutator subgroup $[G,G]$ of G only consists of $\mathbf{0}$ and $\mathbf{4}$ as we will see in the following:

Proposition 1. $[G,G] = \{\mathbf{0},\mathbf{4}\}$.

PROOF: First, we will show that

$$[\mathbf{x},\mathbf{y}]_R = 4\cdot\left\langle x'y^2 - y'x^2, xy'^2 - yx'^2\right\rangle$$

is either $\mathbf{0}$ or $\mathbf{4}$. In other words, we need to prove that

$$x'y^2 - y'x^2 \quad \text{and} \quad xy'^2 - yx'^2 \tag{17}$$

are either both even or both odd. In the following, the equalities are all to be taken modulo 2.

(1) Assume $x'y = y'x$. This is the case if and only if $x'y^2 = y'x^2$ and $xy'^2 = yx'^2$, since $x = x^2$. Thus both sums in (17) yield an even result, since a sum is even if and only if its two summands are either both even or both odd.

(2) Otherwise, $x'y \neq y'x$ iff $x'y^2 \neq y'x^2$ and $xy'^2 \neq yx'^2$. Thus both sums in (17) yield an odd result.

So $[G,G]$ is generated by $\mathbf{0}$ and $\mathbf{4}$. But any multiplication involving only these two elements is componentwise addition, since the fudge factors involved are equal to 1. Hence $[G,G]$ is equal to $\{\mathbf{0},\mathbf{4}\}$. □

Theorem 4. *The abelianization $G/[G,G]$ is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_4$.*

PROOF: First, note that $(0,1)[G,G]$ is an element of order 8 in $G/[G,G]$ and generates the set

$$\langle (0,1)[G,G]\rangle = \left\{(0,n)[G,G] \;\middle|\; n = 0,1,\dots,7\right\},$$

since $(0,1)^n = (0,n)$ in G and $[G,G] = \{\mathbf{0},\mathbf{4}\}$ by Proposition 1. As an abelian group of order 2^5, the commutator quotient $G/[G,G]$ is then either isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or to $\mathbb{Z}_8 \oplus \mathbb{Z}_4$. Secondly, consider the quotient of $G/[G,G]$ by $\langle (0,1)[G,G]\rangle$. Note that in this quotient, the second component of a representative of any coset can be chosen to be 0, while the first component can be chosen to be between 0 and 3. Hence the quotient of $G/[G,G]$ by $\langle (0,1)[G,G]\rangle$ consists of the elements

$$((n,0)[G,G])\,\langle (0,1)[G,G]\rangle$$

with $n = 0,1,2,3$. Now, we can see that $((1,0)[G,G])\langle (0,1)[G,G]\rangle$ is an element of order 4 in $(G/[G,G])/\langle (0,1)[G,G]\rangle$, since $(1,0)^n = (n,0)$ in G. Hence $G/[G,G]$ is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_4$. □

Finally, let us consider the center of G:

$$Z(G) = \left\{ \mathbf{x} \in G \;\middle|\; [\mathbf{x},\mathbf{y}]_R = \mathbf{0} \text{ for all } \mathbf{y} \in \left(\mathbb{Z}/2^3\mathbb{Z}\right)^2 \right\}.$$

Proposition 2. $Z(G) = \left\{ \mathbf{x} = \langle x, x' \rangle \in G \;\middle|\; x \text{ and } x' \text{ are even} \right\}$.

Proof: From the formula for $[\mathbf{x},\mathbf{y}]_R$ we see that

$$\mathbf{x} \in Z(G) \quad \text{iff} \quad 4 \cdot \left\langle x'y^2 - y'x^2, xy'^2 - yx'^2 \right\rangle = \mathbf{0}.$$

Or equivalently:

$$\mathbf{x} \in Z(G) \quad \text{iff} \quad x'y^2 - y'x^2 \text{ and } xy'^2 - yx'^2 \text{ are even for all } \mathbf{y}.$$

Using the equivalences given in Case (1) of the proof of Proposition 1, we have

$$\mathbf{x} \in Z(G) \quad \text{iff} \quad x'y \equiv y'x \mod 2 \text{ for all } \mathbf{y}, \tag{18}$$

since $x \equiv x^2 \mod 2$ for all x. Choosing $\mathbf{y} = \mathbf{1}$ in (18) now shows that both components of $\mathbf{x}$ have to be equal modulo 2. Then setting $\mathbf{y} = (0,1)$ in (18) yields that $\mathbf{x}$ is necessarily even. This condition is certainly sufficient, and we are done. □

Theorem 5. *The central quotient $G/Z(G)$ is isomorphic to the Vierergruppe.*

Proof: By Proposition 2 the central quotient is of order

$$\left(2^3 \cdot 2^3\right) / \left(2^2 \cdot 2^2\right) = 4.$$

Thus $G/Z(G)$ is either the cyclic group of order four or the Vierergruppe. But $G/Z(G)$ is not cyclic, since G is not abelian. □

8. Future work

Given the approximations for the commutators and associators, and the symmetries observed in Section 6, it becomes of interest to find formulae using higher orders of approximation. In order to do that, one may refer to the general formulae given in Section 4. Eventually, we hope for a pattern to be recognized in order to characterize the structure of Catalan loops following the analysis of Section 7 for the group case.

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