# A priori estimates for quasilinear parabolic systems with quadratic nonlinearities in the gradient

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*Abstract.* We derive local a priori estimates of the Hölder norm of solutions to quasilinear elliptic systems with quadratic nonlinearities in the gradient. We assume higher integrability of solutions and smallness of its BMO norm but the Hölder norm is estimated in terms of BMO norm of the solution under consideration, only.

*Keywords:* quasilinear parabolic systems, quadratic nonlinearities, regularity, Morrey, VMO spaces

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## 1. Introduction

In this paper we study regularity of weak solutions of systems

$$u_t - \operatorname{div}\big(a(z,u)\nabla u\big) + b(z,u,\nabla u) = 0;\ z \in Q, \tag{1}$$

where $Q = \Omega \times (0,T)$, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $T > 0$ is an arbitrary fixed positive number. By $u_t$ we denote the time derivative of a function $u : Q \to \mathbb{R}^N$, $N > 1$, and by $\nabla u = \{u^i_{x_\alpha}\}^{i\leq N}_{\alpha\leq n}$ its gradient with respect to the space variables.

We assume that $a(z,\eta)$ and $b(z,\eta,\xi)$ satisfy the following conditions:

- (H1) There are positive constants $\lambda, \Lambda$ such that

$$\big(a(z,\eta)\xi,\xi\big) = a^{\alpha\beta}_{ij}(z,\eta)\xi^i_\alpha\xi^j_\beta \geq \lambda|\xi|^2, \quad \xi \in \mathbb{R}^{nN}, \tag{2}$$

$$|(a(z,\eta)p,q)| \leq \Lambda|p||q|, \quad p,q \in \mathbb{R}^{nN}, \tag{3}$$

  for all $(z,\eta) \in Q \times \mathbb{R}^N$.
- (H2) The coefficients $a^{\alpha\beta}_{ij}(.,u) \in \mathrm{VMO}(Q)$ for every $u \in \mathbb{R}^N$, $i,j \leq N$, $\alpha,\beta \leq n$ and, moreover, denoting

$$q^2(R) = \sup_{z^o\in Q, u\in\mathbb{R}^N, r\in(0,R]} \fint_{Q_r(z^o)} |a(z,u) - (a(z,u))_{r,z^o}|^2\,dz,$$

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we have

$$q^2(R) \to 0 \quad \text{for} \quad R \to 0. \tag{4}$$

- (H3) For every $z \in Q$ and $u, v \in \mathbb{R}^N$ it holds

$$|a(z,u) - a(z,v)| \le \omega(|u-v|^2), \tag{5}$$

where $\omega$ is a nondecreasing, bounded and concave function on $[0,\infty)$ with $\lim_{s\to 0+} \omega(s) = 0$.

- (H4) Functions $b(z,\eta,\xi)$ are measurable in $z$ for $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{nN}$, and continuous in $(\eta,\xi)$ for a.a. $z \in Q$. Moreover

$$|b(z,\eta,\xi)| \le b_o|\xi|^2 + b_1, \ (z,\eta) \in Q \times \mathbb{R}^N, \xi \in \mathbb{R}^{nN}, \tag{6}$$

with some constants $b_o > 0, b_1 \ge 0$.[1]

We consider a weak solution $u \in V = L_2((0,T); W_2^1(\Omega))$ of system (1) which satisfies the identity

$$\int_Q (-u\varphi_t + (a(z,u)\nabla u, \nabla\varphi) + b(z,u,\nabla u)\varphi)\, dz = 0; \quad \forall \varphi \in C_0^\infty(Q). \tag{7}$$

The regularity problem for elliptic and parabolic systems with strongly nonlinear terms in the gradient was studied in [1]–[6], [15], [17], [23], [24]. Partial regularity of *bounded* weak solutions of (1) was proved under the smallness assumption

$$2b_o\|u\|_{\infty,Q} < \lambda. \tag{8}$$

More precisely, Hölder continuity of $u$ in a neighborhood of a point $z^o \in Q$ was proved provided that (8) is valid and

$$\liminf_{R\to 0+} \frac{1}{R^n} \int_{Q_R(z^o)} |\nabla u|^2\, dz < \epsilon_o^2 \tag{9}$$

for an $\epsilon_o$ small enough depending on the data only (here and later on $Q_R(z^o)$ is a parabolic cylinder, see the notation at the end of Introduction).

It follows from the above mentioned results that smoothness of *bounded* weak solution $u$ in a neighborhood of $z^o$ holds under the assumption

$$\operatorname{osc}_{Q_R(z^o)} u < \theta \tag{10}$$

for a small enough positive $\theta$ and $R = R(z^o) > 0$ (instead of (8), (9)).

Unfortunately, description (10) of regular points does not allow to obtain reasonable estimates of the set $\Sigma$ of singular points of the considered solution $u$. As it is known, an appropriate estimate of $\Sigma$ helps to study the solvability of systems (1) (see [9]).

[1] It is sufficient to assume that $b_1 \in L^q(Q)$ for a $q > \frac{n+2}{2}$.

Condition (10) was relaxed in [3], [4] to the assumption

$$[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} < \theta, \tag{11}$$

where $\theta \ll 1$ and where $[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))}$ denotes the seminorm of $u$ in Campanato space $\mathcal{L}^{2,n+2}(Q_R(z^o))$. (Relaxations of the same type were obtained for corresponding elliptic systems in [2], [5], [6].)

A priori local $L_p$ estimate (for $p > 2$) of the gradient of $u$ was obtained in [3] under the assumption (11). In [4] Hölder continuity of $u$ with respect to the parabolic metric $\delta$ and the estimate

$$\|u\|_{C^\alpha(\overline{Q_{\tau R}(z^o)},\delta)} \leq C\left(\|u\|_{2,Q_R}, \|\nabla u\|_{2,Q_R}, R^{-1}, \alpha\right) \tag{12}$$

are proved for any $\alpha \in (0,1)$ and some $\tau \in (0,1)$ provided that condition (11) holds with sufficiently small $\theta$ depending on the data only.

Note that a sufficient condition that ensures (11) can be expressed by the requirement

$$\|\nabla u\|_{L^{2,n}(Q_R(z^o))} < \theta_1 \tag{13}$$

with a small parameter $\theta_1$ (see Proposition 2.1 in [3]).

As it is known, the condition

$$\Phi(R, z^o) = \frac{1}{R^n}\int_{Q_R(z^o)} |\nabla u|^2\, dz < \theta \tag{14}$$

with small enough $\theta$ describes regular points $z^o$ of $u$ for the simplest systems with $b = 0$ in (1). This condition allows to estimate the Hausdorff measure of a singular set of $u$. In this sense, condition (14) is an optimal assumption to estimate the Hölder norm of $u$ in the neighborhood of $z^o$. Let us remark that the *monotonicity* of the function $\Phi(r, \xi^o)$ in the argument $r$ for any $\xi^o \in Q_R(z^o)$ ensures estimate (13) with $\theta_1 = c\theta$ provided that assumption (14) holds. It means that description (13) of regular points of $u$ transforms the problem of the optimal description of regular points to the possibility to state a monotonicity type estimate for the function $\Phi(r, \xi^o)$. We also remark that a priori estimates of $C^\alpha$ norm of $u$ were obtained in [3], [4] under Dirichlet boundary conditions up to the lateral surface of $Q$.

A priori estimate (12) was obtained in [4] for solutions $u$ of system (1) from the space $W^{1,2}(Q)$. The existence of $u_t \in L_2(Q)$ and strong energy estimate were essentially exploited in the proof of (12) by the so called direct method. Actually, parabolic systems with elliptic operators of variational structure were considered in [4].

In this paper, we derive a local a priori estimate (12) under smallness condition (11) for weak solutions $u \in V = L_2((0,T); W_2^1(\Omega))$ under assumptions (H1)–(H4). We also assume that the $L_m$-norm of the gradient of $u$, for some $m > 2$, is finite, but the constant $C$ in (12) does not depend on this norm. For the proof we apply the "$\mathcal{A}$-caloric approximation" method. This method was introduced for parabolic systems in [14] as a development of the "$\mathcal{A}$-harmonic approximation" method for elliptic systems (see [12], [13], [18]). (See [7], [25], [14] for its applications to parabolic problems.)

Moreover, we relax in this paper the assumptions on the smoothness of the matrix $a(z,v)$, namely we do not require the continuity of $a(z,v)$ in $z$. Recall that regularity of weak solutions to linear systems with principle matrices in VMO-space was studied by Chiarenza, Frasca, Longo (see [10]) or Huang (see [19]). It follows from [21] that some smoothness of the principal matrix of parabolic systems is necessary even in the linear case. Regularity of minimizers of certain functionals under VMO smoothness of matrices $a(z,v)$ in $z$ was studied in [11], [26]. Thus, it seems reasonable to study regularity of weak solutions to systems satisfying assumption (H4).

We adopt the following notation:

$$\begin{aligned} B_R(x^0) &= \{x \in \mathbb{R}^n : |x - x^0| < R\},\ Q_R(z^0) = B_R(x^0) \times (t^0 - R^2, t^0), \\ (u)_{R,z^0} &= \frac{1}{\omega_n R^{n+2}} \int_{Q_R(z^0)} u(z)\,dz = \fint_{Q_R(z^0)} u(z)\,dz,\ \omega_n = \operatorname{meas} B_1, \\ V_R(z^0) &= L_2\left((t^0 - R^2, t^0); W_2^1(B_R(x^0))\right), z^0 = (x^0, t^0) \in \mathbb{R}^{n+1}. \end{aligned}$$

We will leave out $z^o$and write $Q_R, V_R$ and $(u)_R$ if it does not cause misunderstanding.

## 2. Auxiliary results

We start with a reformulation of the $\mathcal{A}$-caloric lemma due to Duzaar and Mingione (see [14]). (Such a reformulation was proposed in the elliptic case by M. Giaquinta (see Appendix in [13]).

**Definition** ($\mathcal{A}$-caloric function)**.** Let $\mathcal{A}$ be an $nN \times nN$ constant positive definite matrix. A function $h \in V$ is $\mathcal{A}$-caloric if for any $\varphi \in C_0^1(Q)$

$$\int_Q \Big(h\varphi_t - (A\nabla h, \nabla\varphi)\Big) dz = 0.$$

**Lemma 1** ($A$-caloric approximation)**.** *Let $0 < \lambda < \Lambda$ and $n, N \in \mathbb{N}$ (with $n, N \geq 2$) be fixed. Then for any $\varepsilon > 0$ there exists a $C(\varepsilon) = C(\varepsilon, n, N, \lambda, \Lambda) > 0$ such that the following holds: for any bilinear form $A$ on $\mathbb{R}^{nN}$ such that*

$$(A\xi, \xi) \geq \lambda|\xi|^2, \tag{15}$$

$$\|A\| \leq \Lambda, \tag{16}$$

*and for any $u \in V_R(z^o)$ there exist an $A$-caloric $h \in V_R(z^o)$ and $\varphi \in C_0^1(Q_R(z^o))$ so that*

$$\|\nabla\varphi\|_{L_\infty} \leq 1, \tag{17}$$

$$(18)\qquad \fint_{Q_R(z^o)} (|h|^2 + R^2|\nabla h|^2)\,dz \le \fint_{Q_R(z^o)} (|u|^2 + R^2|\nabla u|^2)\,dz,$$

$$(19)\qquad \fint_{Q_R(z^o)} |u-h|^2 \le \varepsilon \fint_{Q_R(z^o)} \left(|u|^2 + R^2|\nabla u|^2\right) dz + C(\varepsilon)R^2 \left| \fint_{Q_R(z^o)} \left(u\varphi_t - (A\nabla u, \nabla\varphi)\right) dz \right|^2.$$

Proof: Let us prove Lemma 1 for $x^o = 0$, $t^o = 0$ and $R = 1$. The full assertion then follows by a standard homotopy argument. For $u \ne 0$ set $\tilde u = u\left(\fint_{Q_1} |u|^2 + |\nabla u|^2\,dz\right)^{-1/2}$, $Q_1 = Q_1(0)$, and fix an $\varepsilon > 0$. Then $\fint_{Q_1}(|\tilde u|^2 + |\nabla \tilde u|^2)\,dz \le 1$ and if, moreover,

$$(20)\qquad \left| \fint_{Q_1} \left(\tilde u\tilde\psi_t - (A\nabla\tilde u, \nabla\tilde\psi)\right) dz \right| \le \delta \|\nabla\tilde\psi\|_{L_\infty}$$

for all $\tilde\psi \in C_0^1(Q_1)$ and some $\delta = \delta(\epsilon)$, then $\tilde u$ satisfies all assumptions of Lemma 4.1 in [14]. Hence there is an $\mathcal{A}$-caloric $\tilde h \in V_1$ such that

$$\fint_{Q_1} \left(|\tilde h|^2 + |\nabla\tilde h|^2\right) dz \le 1,\ \ \fint_{Q_1} |\tilde u - \tilde h|^2\,dz \le \varepsilon.$$

Setting $h = \tilde h\big(\fint_{Q_1}(|u|^2 + |\nabla u|^2)\,dz\big)^{1/2}$ and $\varphi = 0$, we get the inequalities (17)–(19).

If, on the contrary, the inequality (20) is not satisfied, there is a nonzero $\tilde\psi \in C_0^1(Q_1)$ such that

$$\left| \fint_{Q_1} \left(\tilde u\tilde\psi_t - (A\nabla\tilde u, \nabla\tilde\psi)\right) dz \right| > \delta \|\nabla\tilde\psi\|_{L_\infty}.$$

After changing (if necessary) the sign of $\tilde\psi$, we have

$$\fint_{Q_1(z^o)} \left(\tilde u\tilde\psi_t - (A\nabla\tilde u, \nabla\tilde\psi)\right) dz > \delta \|\nabla\tilde\psi\|_{L_\infty},$$

and for $\psi = \frac{\tilde\psi}{\|\nabla\tilde\psi\|_{L_\infty}}$ the inequality (17) is satisfied. Thus, also the inequalities

$$\frac{1}{\delta}\fint_{Q_1} (\tilde u\psi_t - (A\nabla\tilde u, \nabla\psi))\,dz > 1,$$
$$\frac{1}{\delta}\fint_{Q_1} (u\psi_t - (A\nabla u, \nabla\psi))\,dz > \left(\fint_{Q_1} (|u|^2 + |\nabla u|^2)\,dz\right)^{1/2}$$

are valid. For this $\psi$ and $h=0$

$$\begin{aligned}\fint_{Q_1} |u-h|^2\,dz = \fint_{Q_1} |u|^2\,dz \le \fint_{Q_1} (|u|^2+|\nabla u|^2)\,dz \\ < \frac{1}{\delta^2}|\left|\fint_{Q_1} u\psi_t - (A\nabla u,\nabla\psi)\,dz\right|^2,\end{aligned}$$

and Lemma 1 holds with $C_\varepsilon = \frac{1}{\delta^2}$, $R=1$, $z^0=0$. □

**Corollary 1.** *Let the assumptions of Lemma* 1 *be satisfied and* $(u)_{Q_R(z^o)}=0$. *Then the assertion of Lemma* 1 *remains true and, moreover, the $\mathcal{A}$-caloric function $h$ satisfying* (18), (19) *can be chosen so that* $(h)_{Q_R(z^o)}=0$ *and* (17) *also holds.*

Proof: For given $A,u$ and $\varepsilon$ find $\tilde h,\varphi$ according to Lemma 1 and set $h=\tilde h-(\tilde h)_{Q_R(z^o)}$. Then $h$ has zero mean value over $Q_R(z^o)$. Relation (17) is obvious, and (18), (19) follow from the inequalities

$$\fint_{Q_R(z^o)} |h|^2\,dz \le \fint_{Q_R(z^o)} |\tilde h|^2\,dz, \quad \fint_{Q_R(z^o)} |u-h|^2\,dz \le \fint_{Q_R(z^o)} |u-\tilde h|^2\,dz.$$

Here we used the fact that for any function $u\in L_2(Q)$ its integral mean value is the minimizer of $F(c)=\int_Q |u-c|^2\,dz$. □

**Corollary 2.** *Let the assumptions of Lemma* 1 *be satisfied. Then for any $\varepsilon>0$ there exists a positive constant $C(\varepsilon)=C(\varepsilon,n,N,\lambda,\Lambda)$ such that the following holds: for any bilinear form $A$ on $\mathbb{R}^{nN}$ satisfying* (15), (16) *and for any $u\in V_R(z^o)$ there exist an $A$-caloric $h\in V_R(z^o)$ and $\varphi\in C^1_0(Q_R(z^o))$ satisfying* (17) *so that*

$$\begin{aligned}\fint_{Q_R(z^o)} \big(|h-(h)_{R,z^o}|^2 + R^2|\nabla h|^2\big)\,dz \\ \le \fint_{Q_R(z^o)} \big(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\big)\,dz,\end{aligned} \tag{21}$$

$$\begin{aligned}\fint_{Q_R(z^o)} |u-h|^2\,dz \le \varepsilon \fint_{Q_R(z^o)} \big(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\big)\,dz \\ + C(\varepsilon)R^2\left|\fint_{Q_R(z^o)} (u\varphi_t - (A\nabla u,\nabla\varphi))\,dz\right|^2.\end{aligned} \tag{22}$$

Proof: Apply Corollary 1 to the function $\tilde u = u-(u)_{R,z^o}$ and find $A$-caloric $\tilde h,\varphi$ so that $\tilde h$ has mean value zero and they satisfy (17), (18), (19). Set $h=\tilde h+(u)_{R,z^o}$. Then $h$ is $A$-caloric. As $(h)_{R,z^o}=(u)_{R,z^o}$, inequality (18) for $\tilde h$ and $\tilde u$ gives

$$\fint_{Q_R(z^o)} \big(|h-(h)_{R,z^o}|^2 + R^2|\nabla h|^2\big)\,dz = \fint_{Q_R(z^o)} \big(|\tilde h|^2 + R^2|\nabla \tilde h|^2\big)\,dz$$

$$\leq \fint_{Q_R(z^o)} \left(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\right) dz.$$

Further, (19) for $\tilde h$ and $\tilde u$ implies

$$\begin{aligned}
&\fint_{Q_R(z^o)} |u-h|^2\,dz = \fint_{Q_R(z^o)} |\tilde u-\tilde h|^2\,dz \\
&\leq \varepsilon \fint_{Q_R(z^o)} \left(|u-(u)_{R,z^o}|^2 + R^2|\nabla u|^2\right) dz \\
&\quad + C(\varepsilon)\left| R\fint_{Q_R(z^o)} \left(u\varphi_t - (A\nabla u,\nabla\varphi)\right) dz\right|^2,
\end{aligned}$$

which accomplishes the proof. □

## 3. Main theorem

We start with the following remark.

**Remark 1.** Let $b \equiv 0$ in system (1). It is well known that there exists $p_0 = p_0(\lambda,\Lambda,n) > 2$ such that $\nabla u \in L_p(Q')$, $, Q' \subset\subset Q$, $p \in [2,p_0)$, for any solution $u \in V(Q)$ (see [8], [20]). It appears that an $L_p$ estimate ($p > 2$) for the gradient is also valid for system (1) with $b \neq 0$ provided that some smallness assumption holds.

To describe this assumption we introduce the following definition:

**Definition.** Let $\theta$ and $R_o$ be fixed positive numbers. We say that "$u$ satisfies condition $S_{\theta,R_o}$" in a cylinder $Q_R(z^o) \subset Q$ provided that

- $\nabla u \in L_m(Q_R(z^o))$ with an exponent $m > 2$,
- $[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} \leq \theta$ with some $R = R(z^o) \leq R_o$.

**Theorem 1.** *Let assumptions* (H1)–(H4) *hold and let $u \in V$ satisfy* (7). *Then there exist parameters $\theta, R_o$ such that if $u$ fulfills condition $S_{\theta,R_o}$ in a cylinder $Q_R(z^o)$ with $R \leq R_o$ then $u \in C^\alpha(Q_{\tau R}(z^o))$ for some $\tau \in (0,1)$, any $\alpha \in (0,1)$ and*

$$\|u\|_{C^\alpha(\overline{Q_{\tau R}(z^o)})} \leq C_1\left(\|u\|_V, \alpha, R^{-1}\right). \tag{23}$$

*The constant $C_1$ does not depend on the norm $\|\nabla u\|_{m,Q_R}$. It neither depends on the exponent $m$ from condition $S_{\theta,R_0}$ provided that $m \geq p_0$ (see Remark 1).*

**Remark 2.** Further, we consider $u$ on $Q_R(z^o)$ satisfying $S_{\theta_o,R_o}$ with some $\theta_0$ we specify below. Then

$$\int_Q \Big(-u\varphi_t + (a(z,u)\nabla u,\nabla\varphi) + b(z,u,\nabla u)\varphi\Big)\,dz = 0 \tag{24}$$

holds for any test function $\varphi \in W_2^1(Q_R) \cap \mathcal{L}^{2,n+2}(Q_R)$, spt $\varphi \subset Q_R$. Indeed, $\mathcal{L}^{2,n+2}(Q_R) \hookrightarrow L_s(Q_R)$ for any $s < \infty$, and we have

$$\int_{Q_R} |\nabla u|^2 |\varphi| \, dz \leq \left( \int_{Q_R} |\nabla u|^m \, dz \right)^{2/m} \cdot \left( \int_{Q_R} |\varphi|^{\frac{m}{m-2}} \right)^{\frac{m-2}{m}} < \infty.$$

Moreover, $u \in V_R(z^o) \cap L_s(Q_R(z^o))$ for any $s < \infty$. Applying the Steklov average approximations for the test functions $\varphi$ in (24), one can prove that $u$ has finite norm

$$\sup_{(t^o - R^2, t^o)} \|u(\cdot, t)\|_{2, B_R(x^o)} + \|\nabla u\|_{2, Q_R(z^o)},$$

and $u$ is a continuous in $t$ function in $L_2(B_R(x^o))$-norm. (See the details in paragraph 4, Chapter 3, [22].)

**Proposition 1.** *There exist $\theta_o, R_o$ such that if $u$ satisfies condition $S_{\theta_o, R_o}$ in $Q_R(z^o)$ and (24) then*

$$\left( \fint_{Q_r} |\nabla u|^p \, dz \right)^{2/p} \leq C_2 \fint_{Q_{ar}} (1 + |\nabla u|)^2 \, dz, \quad \forall Q_{ar} \subset Q_R(z^o), \tag{25}$$

*with any $p \in (2, \min(p_o, m))$, where $p_o = p_o(\lambda, \Lambda, n) > 2$, $a > 2$ is an absolute constant. The constant $C_2$ depends on $\lambda, \Lambda, n, b_0, b_1$ and $\theta_o$ but does not depend on $\|\nabla u\|_{m, Q_R}$.*

Proposition 1 is a consequence of Theorem 2.1 in [3]. We should remark that a priori estimate (25) was derived for smooth solutions in [3] but we take into account Remark 2 and assert that the proof of Theorem 2.1 is valid under the condition $S_{\theta_o, R_o}$.

In what follows, $\theta_o$ is fixed by Proposition 1, and we assume further that

$$[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} \leq \theta \tag{26}$$

holds with some $\theta \leq \theta_o$ that will be chosen later.

The smallness assumption (26) does not allow to derive Caccioppoli inequality for $u$, but with the help of (25) we can prove

**Proposition 2.** *Let $u$ satisfy condition $S_{\theta, R_o}$ in $Q_R(z^o)$ and (24). Then the inequality*

$$\begin{aligned} \int_{Q_r(\zeta^o)} |\nabla u|^2 \, dz \leq \frac{C_3}{r^2} \int_{Q_{a_1 r}(\zeta^o)} |u - (u)_{a_1 r}|^2 \, dz \\ + C_4 \theta \int_{Q_{a_1 r}(\zeta^o)} |\nabla u|^2 \, dz + c_5 r^{n+4}, \end{aligned} \tag{27}$$

*holds for all $Q_{a_1 r}(\zeta^o) \subset Q_{R_1}(z^o)$. Here $R_1 = \frac{R}{\sqrt{n}}$, $a_1 = 2a$ and $a > 2$ is fixed in (25).*

Proof: We put $\varphi = (u-(u)_{2r,\zeta^o})h^2$ in (24), where $h$ is a smooth cut-off function supported in $Q_{2r}(\zeta^o)$, $h=1$ in $Q_r(\zeta^o)$, and derive for $u$ the inequality

$$\int_{Q_r(\zeta^o)} |\nabla u|^2\,dz \le \frac{c}{r^2}\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r}|^2\,dz$$
$$+ c\left(\int_{Q_{2r}(\zeta^o)} b_0|\nabla u|^2|u-(u)_{2r,\zeta^o}|\,dz + b_1\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|\,dz\right).$$

Using Hölder and Young inequality we estimate the last term on the right hand side:

$$\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|\,dz \le c\left(\frac{1}{r^2}\int_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|^2\,dz + r^{n+4}\right). \tag{28}$$

We explain now how to estimate the strongly nonlinear term $I$

$$I = \int_{Q_{2r}(\zeta^o)} |\nabla u|^2|u-(u)_{2r,\zeta^o}|\xi^2.$$

Estimates (25), (26) imply that the following chain of inequalities is valid:

$$\begin{aligned} I &\le \left(\fint_{Q_{2r}(\zeta^o)} |\nabla u|^p\,dz\right)^{2/p}\left(\fint_{Q_{2r}(\zeta^o)} |u-(u)_{2r,\zeta^o}|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} |Q_{2r}| \\ &\le C\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right).[u]_{\mathcal{L}^{p',n+2}(D_{R/\sqrt{n}}(z^o))} \\ &\le C(p)\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right).[u]_{\mathcal{L}^{2,n+2}(D_{R/\sqrt{n}}(z^o))} \\ &\le C(p)\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right).[u]_{\mathcal{L}^{2,n+2}(Q_R(z^o))} \\ &\le C(p)\theta\left(\int_{Q_{2ar}(\zeta^o)} |\nabla u|^2\,dz\right). \end{aligned} \tag{29}$$

Here $p' = \frac{p}{p-2}$ and the parabolic cube $D_{R_1}(z^0) \subset Q_R(z^o)$. The inequality is valid due to isomorphism of $\mathcal{L}^{s,n+2}(D_{R_1})$ and $\mathcal{L}^{2,n+2}(D_{R_1})$ for different $s>1$. Due to (28), (29) we arrive at (27). □

Now we can exclude in (27) the term with $\theta$.

**Proposition 3.** *There exists $\theta_1 \le \theta_o$ such that the assumptions of Proposition 2 with $\theta \le \theta_1$ ensure that the inequality*

$$\int_{Q_r(\zeta^o)} |\nabla u|^2\, dz \le \frac{C_6}{r^2} \int_{Q_{lr}(\zeta^o)} |u - (u)_{lr,\zeta^o}|^2\, dz + c_7 r^{n+4} \tag{30}$$

*holds with an absolute constant $l > 2$.*

Proof: We rewrite (27) changing cylinders $Q_r$ by parabolic cubes $D_r$ and apply parabolic version of Lemma 0.5 of [16]. □

**Remark 3.** From (30) and (26) it follows that

$$\frac{1}{r^n} \int_{Q_r(\zeta^o)} |\nabla u|^2\, dz \le C_8 ⨍_{Q_{lr}(\zeta^o)} |u - (u)_{lr,\zeta^o}|^2\, dz + C_8 r^4, \tag{31}$$

and thus also

$$\|\nabla u\|^2_{L^{2,n}(Q_{R_2}(z^0))} \le C_8(\theta^2 + R^4),\ R_2 = \tau R, \tag{32}$$

with a $\tau \in (0,1)$. We put

$$C_8(\theta^2 + R_o^4) = \kappa^2(\theta, R_o) \to 0 \quad \text{for} \quad \theta, R_o \to 0. \tag{33}$$

Hence

$$\|\nabla u\|_{L^{2,n}(Q_{R_2}(z^0))} \le \kappa(\theta, R_o),\ R_2 = \tau R < R_o. \tag{34}$$

Proof of Theorem 1: Let $R_3 = \frac{R_2}{2}$, $\zeta^o \in Q_{R_3}(z^o)$, $r \le \frac{R_3}{l}$, where $l > 2$ is fixed in (30). Then $Q_r(\zeta^o) \subset Q_{R_3}(z^o)$. Put $A = (a(z,u^o))_{r,\zeta^o}$, $u^o = (u)_{r,\zeta^o}$ and estimate

$$L_r(\zeta^o) = \left| \int_{Q_r(\zeta^o)} \big(u\varphi_t - (A\nabla u, \nabla\varphi)\big)\, dz \right|$$

for all $\varphi \in C_0^1(Q_r(\zeta^o))$ with $C_\varphi = \|\nabla\varphi\|_{\infty, Q_r(\zeta^o)} \le 1$. (Note that it implies that $\max_{Q_r(\zeta^o)} |\varphi| \le C_\varphi r$.) Hence, (6) and (34) supply the inequalities

$$\begin{aligned}
L_r(\zeta^o) &\le C_\varphi \Big\{ \int_{Q_r(\zeta^o)} |a(z,u) - a(z,u^o)||\nabla u|\, dz \\
&\quad + \int_{Q_r(\zeta^o)} |a(z,u^o) - A||\nabla u|\, dz \\
&\quad + b_o r^{1+n/2} \Big( \int_{Q_r(\zeta^o)} |\nabla u|^2\, dz \Big)^{1/2} \Big( \frac{1}{r^n} \int_{Q_r(\zeta^o)} |\nabla u|^2\, dz \Big)^{1/2} + b_1 r^{n+3} \Big\} \\
&\le C_\varphi \Big\{ \Big[ \omega^{1/2}(\theta) + q(R_o) + b_o \kappa(\theta, R_o) \Big] r^{1+n/2} \Big( \int_{Q_r(\zeta^o)} |\nabla u|^2\, dz \Big)^{1/2} + C r^{n+3} \Big\}.
\end{aligned}$$

We put

$$T(\theta, R_o) = cC_\varphi\big[\omega^{1/2}(\theta) + q(R_o) + b_o\kappa(\theta, R_o)\big], \tag{35}$$

and we get

$$L_r(\zeta^o) \le T(\theta, R_o)r^{1+n/2}\bigg(\int_{Q_r(\zeta^o)} |\nabla u|^2\, dz\bigg)^{1/2} + Cr^{n+3}. \tag{36}$$

From Corollary 2 and (36) it follows that for $u$ there is an $\mathcal{A}$-caloric $h$ in $V_r(\zeta^o)$ such that

$$\begin{aligned}\fint_{Q_r(\zeta^o)} |u-h|^2\, dz &\le \varepsilon \fint_{Q_r(\zeta^o)} \big(|u-(u)_r|^2 + r^2|\nabla u|^2\big)\, dz \\ &\quad + C(\varepsilon)\bigg(T^2(\theta, R_o)r^2 \fint_{Q_r(\zeta^o)} |\nabla u|^2\, dz + r^4\bigg).\end{aligned} \tag{37}$$

Now we use Campanato's inequality for $h$ and $\rho \le r < 1$, (37), (21) and Proposition 3 to get

$$\begin{aligned}\Phi(\rho, \zeta^o) = \Phi(\rho) &= \int_{Q_\rho(\zeta^o)} |u-(u)_\rho|^2\, dz \\ &\le 2\int_{Q_\rho(\zeta^o)} |h-(h)_\rho|^2\, dz + 2\int_{Q_\rho(\zeta^o)} |u-h-((u)_\rho-(h)_\rho)|^2\, dz \\ &\le 2\int_{Q_\rho(\zeta^o)} |h-(h)_\rho|^2\, dz + 2\int_{Q_\rho(\zeta^o)} |u-h|^2\, dz \\ &\le C\left(\frac{\rho}{r}\right)^{n+4}\int_{Q_r(\zeta^o)} |h-(h)_r|^2\, dz + 2\int_{Q_r(\zeta^o)} |u-h|^2\, dz \\ &\le C\left(\frac{\rho}{r}\right)^{n+4}\int_{Q_r(\zeta^o)} |h-(h)_r|^2\, dz + 2\varepsilon\int_{Q_r(\zeta^o)} \big(|u-(u)_r|^2 + r^2|\nabla u|^2\big)\, dz \\ &\quad + 2C(\varepsilon)T^2(\theta, R_o)r^2\int_{Q_r(\zeta^o)} |\nabla u|^2\, dz + Cr^{n+4}.\end{aligned}$$

After rearranging the formula we get

$$\begin{aligned}\Phi(\rho) &\le C\left(\left(\frac{\rho}{r}\right)^{n+4} + 2\varepsilon\right)\int_{Q_r(\zeta^o)} \big(|u-(u)_r|^2 + r^2|\nabla u|^2\big)\, dz \\ &\quad + 2C(\varepsilon)\left[T^2(\theta, R_o)r^2\int_{Q_r(\zeta^o)} |\nabla u|^2\, dz + Cr^{n+4}\right] \\ &\le C\left(\left(\frac{\rho}{r}\right)^{n+4} + 2\varepsilon\right)\Phi(r) \\ &\quad + \left[C\left(\left(\frac{\rho}{r}\right)^{n+4} + 2\varepsilon\right) + 2C(\varepsilon)T^2(\theta, R_o)\right] r^2\int_{Q_r(\zeta^o)} |\nabla u|^2\, dz + Cr^{n+4}\end{aligned} \tag{38}$$

$$\leq C\left(\left(\frac{\rho}{r}\right)^{n+4}+2\varepsilon\right)\Phi(r)$$
$$+\left[C\left(\left(\frac{\rho}{r}\right)^{n+4}+2\varepsilon\right)+2C(\varepsilon)T^2(\theta,R_o)\right]\left(\Phi(lr)+r^{n+4}\right)+Cr^{n+4}.$$

As (38) is evidently true for $\rho\in(r,lr]$ and

$$\Phi(r)=\int_{Q_r(\zeta^o)}|u-(u)_r|^2\,dz\leq\int_{Q_r(\zeta^o)}|u-(u)_{lr}|^2\,dz$$
$$\leq\int_{Q_{lr}(\zeta^o)}|u-(u)_{lr}|^2\,dz=\Phi(lr),$$

we "rename" $lr$ by $r$ and simplify the constants to obtain

$$\Phi(\rho)\leq C_9\left[\left(\frac{\rho}{r}\right)^{n+4}+\left\{\varepsilon+C(\varepsilon)T^2(\theta,R_o)\right\}\right]\Phi(r)+C(\varepsilon)r^{n+4}, \tag{39}$$

for all $\zeta^o\in Q_{R_3}(z^o)$ and $0<\rho\leq r\leq R_3$. By Campanato's algebraic lemma, for any $\alpha\in(0,1)$ there exists an $\epsilon_0=\epsilon_0(C_9,n,\alpha)>0$ so that

$$\Phi(\rho,\zeta^o)\leq C_{10}\rho^{n+2+2\alpha}\left[\frac{\Phi(r,\zeta^o)}{r^{n+2+2\alpha}}+1\right] \tag{40}$$

provided that

$$C_9\{\varepsilon+C(\varepsilon)T^2(\theta,R_o)\}<\epsilon_0. \tag{41}$$

Now, we can fix $\varepsilon\leq\frac{\epsilon_0}{2C_9}$ and then fix $\theta\leq\theta_1$ and $R_o$ (we reduce $R_o$ if needed) to supply the inequality

$$C_9C(\varepsilon)T^2(\theta,R_o)<\frac{\epsilon_0}{2}\,. \tag{42}$$

From (40) it follows that

$$\begin{aligned}[u]^2_{\mathcal{L}^{2,n+2+2\alpha}(Q_{R_3}(z^o))}&\leq C_{11}(\|\nabla u\|_{2,Q_R(z^o)},R^{-1},\alpha),\\ \langle u\rangle_{C^\alpha(Q_{R_3}(z^o))}&\leq C_{12}(\|\nabla u\|_{2,Q_R(z^o)},R^{-1},\alpha),\end{aligned} \tag{43}$$

provided that $u$ satisfies condition $S_{\theta,R_o}$ in $Q_R(z^o)$, $R\leq R_o$ for the chosen $\theta$ and $R_o$. Note that the constants $C_{11},C_{12}$ depend on $\alpha$ and do not depend on $\|\nabla u\|_{m,Q_R}$. □

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