# On exit laws for subordinated semigroups by means of $\mathcal{C}^1$-subordinators

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*Abstract.* We study the integral representation of potentials by exit laws in the framework of sub-Markovian semigroups of bounded operators acting on $L^2(m)$. We mainly investigate subordinated semigroups in the Bochner sense by means of $\mathcal{C}^1$-subordinators. By considering the one-sided stable subordinators, we deduce an integral representation for the original semigroup.

*Keywords:* sub-Markovian semigroup, potential, Bochner subordination, exit law, $\mathcal{C}^1$-subordinator, one-sided stable subordinator

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## Introduction

Let $\mathbb{P} = (P_t)_{t>0}$ be a sub-Markovian semigroup of bounded operators on $L^2(m)$. A $\mathbb{P}$-exit law is a family $\varphi = (\varphi_t)_{t>0}$ of $L^2_+(m)$ satisfying the functional equation

$$(0.1) \qquad P_s\varphi_t = \varphi_{s+t} \qquad (s, t > 0).$$

This notion is first introduced by Dynkin [6] in the framework of potential theory without reference measure. Since, the integral representation of potentials by exit laws was investigated in many papers (cf. [1], [7], [8] and [10]–[15]). Now, let $\beta = (\beta_t)_{t>0}$ be a Bochner subordinator, that is, a vaguely continuous convolution semigroup of sub-probability measures on $[0, +\infty[$. The present paper is devoted to the representation by $\mathbb{P}^\beta$-exit laws, where $\mathbb{P}^\beta$ is the subordinated semigroup of $\mathbb{P}$ by means of $\beta$, i.e.

$$(0.2) \qquad P^\beta_t f := \int_0^\infty P_s f\, \beta_t(ds) \qquad (f \in L^2(m), t > 0).$$

More precisely, we suppose that $\beta$ is a $\mathcal{C}^1$-subordinator (cf. 2.2 below) and we prove the following integral representation: Let $h$ be a $\mathbb{P}^\beta$-pseudo-potential, i.e. $h \geq 0$, $P^\beta_t h \in L^2_+(m)$, $P^\beta_t h \leq h$, and $\lim_{t\to 0} P^\beta_t h = h$. Then there exists a unique $\mathbb{P}^\beta$-exit law $\psi = (\psi_t)_{t>0}$ such that

$$(0.3) \qquad h = \int_0^\infty \psi_s\, ds,$$

where $\psi$ is explicitly given by

$$\psi_t = -\int_0^\infty P_s(P^\beta_{t/2}h)\,\beta'_{t/2}(ds) \qquad (t>0).$$

As an application, we obtain a representation of $\mathbb{P}$-potentials in terms of $\mathbb{P}$-exit laws. Namely, let $u$ be a $\mathbb{P}$-potential, that is $u$ is a $\mathbb{P}$-pseudo-potential and $P_t u \in D(A)$, the domain in $L^2(m)$ of the $L^2(m)$-generator $A$ of $\mathbb{P}$. Then there exist a unique $\mathbb{P}$-exit law $\varphi = (\varphi_t)_{t>0}$ satisfying

$$u = \int_0^\infty \varphi_s\,ds. \tag{0.4}$$

In fact, (0.4) is obtained from (0.3) by considering the one-sided stable subordinator $\eta^\alpha$ of order $\alpha \in ]0,1[$.

A similar problem is investigated in [14] by considering subordinators with complete Bernstein functions instead of $\mathcal{C}^1$-subordinators.

## 1. Preliminaries

Let $(E,\mathcal{E})$ be a standard measurable space and let $m$ be a $\sigma$-finite positive measure on $(E,\mathcal{E})$. We denote by $L^2(m)$ the Banach space of (classes of) square integrable functions defined on $E$, by $\|\cdot\|_2$ the associated norm and by $L^2_+(m)$ the $m$-a.e. non-negative elements of $L^2(m)$. Moreover, in the sequel, equality and inequality holds always $m$-a.e. (i.e. almost everywhere with respect to $m$).

In this section we summarize some known results (cf. [2], [3], [5] and [17]–[19]).

**1.1 Sub-Markovian semigroup.** A bounded operator $N : L^2(m) \to L^2(m)$ is said to be *sub-Markovian* if

$$(0 \le f \le 1) \;\Rightarrow\; (0 \le Nf \le 1), \quad f \in L^2(m).$$

In this case, $N$ can be extended to a pseudo-kernel on $(E,\mathcal{E})$ with respect to the class of $m$-negligible sets. According to a regularization theorem ([5, XIII, 43]), we can assume that $N$ is a sub-Markovian kernel (i.e. $N1 \le 1$) on $(E,\mathcal{E})$.

Therefore, we can apply the potential theory defined by kernels (cf. [5] for example), for such operators.

A *sub-Markovian semigroup* on $E$ is a family $\mathbb{P} := (P_t)_{t\ge 0}$ of sub-Markovian bounded operators on $L^2(m)$ such that $P_0 = I$ (the identity on $E$),

(1) $P_s P_t = P_{s+t}$ for all $s,t>0$,
(2) $\|P_t u\|_2 \le \|u\|_2$ for all $t \ge 0$ and $u \in L^2(m)$,
(3) $\lim_{t\to 0} \|P_t u - u\|_2 = 0$, for every $u \in L^2(m)$.

Let $\mathbb{P}$ be a sub-Markovian semigroup on $E$. The associated $L^2(m)$-*generator* $A$ is defined by

$$Af := \lim_{t\to 0} \frac{1}{t}(P_t f - f)$$

on its domain $D(A)$ which is the set of all functions $u \in L^2(m)$ for which this limit exists in $L^2(m)$. It is known that

(1) $D(A)$ is dense in $L^2(m)$ and $A$ is closed,
(2) if $u \in D(A)$ then $P_t u \in D(A)$ and $A(P_t u) = P_t Au$, for each $t > 0$.

**1.2 Potentials and exit laws.** Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$.

A non-negative measurable function $u$ is said to be $\mathbb{P}$-*excessive* if

(i) $P_t u \le u$ for each $t > 0$,
(ii) $\lim_{t\to 0} P_t u = u$, $m$-a.e.

A $\mathbb{P}$-excessive function $u$ is called a $\mathbb{P}$-*pseudo-potential* if

(iii) $P_t u \in L^2(m)$ for every $t > 0$.

A $\mathbb{P}$-excessive function $u$ is called a $\mathbb{P}$-*potential* if

(iv) $P_t u \in D(A)$ for every $t > 0$.

A $\mathbb{P}$-*exit law* is a family $\varphi := (\varphi_t)_{t>0}$ of elements of $L^2_+(m)$ satisfying the *exit equation*:

$$P_s \varphi_t = \varphi_{s+t} \qquad (s, t > 0). \tag{1.1}$$

In what follows, we consider $\mathbb{P}$-exit laws satisfying

$$\int_t^\infty \varphi_s \, ds \in L^2(m) \qquad (t > 0). \tag{1.2}$$

As it is discussed in our paper [16], condition (1.2) is in fact not restrictive.

The following general result gives a first relation between potentials and exit laws.

**Proposition 1.1.** *Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$ and let $\varphi$ be a $\mathbb{P}$-exit law such that* (1.2) *holds. Then the function*

$$u := \int_0^\infty \varphi_s \, ds \tag{1.3}$$

*is a $\mathbb{P}$-potential. Moreover, we have*

$$\varphi_t = -AP_t u \qquad (t > 0). \tag{1.4}$$

Proof: By Fubini's Theorem and (1.1) we get

$$P_t u = \int_0^\infty P_t \varphi_s \, ds = \int_0^\infty \varphi_{s+t} \, ds = \int_t^\infty \varphi_s \, ds.$$

Therefore, $P_t u \in L^2(m)$ by (1.2) and

$$P_t u = \int_t^\infty \varphi_s \, ds \qquad (t > 0). \tag{1.5}$$

Now from (1.5), we easily deduce that $u$ is $\mathbb{P}$-excessive. Moreover, by (1.5) again we have, for $r, t > 0$

$$\frac{1}{r}(P_{r+t}u - P_t u) = -\frac{1}{r}\int_t^{r+t} \varphi_s \, ds.$$

Hence $P_t u \in D(A)$ and $AP_t u = -\varphi_t$ for each $t > 0$. □

**Remarks 1.2.** (1) In this paper, we will prove the converse of Proposition 1.1. Namely, each $\mathbb{P}$-potential $u$ admits an integral representation by some $\mathbb{P}$-exit law $\varphi$ (i.e. such that (1.3) holds).

(2) From (1.4), we deduce immediately the unicity of the $\mathbb{P}$-exit in the integral representation (1.3).

(3) The representation by exit laws plays a fundamental role in the framework of potential theory without Green function (cf. [6]–[8] and [10]).

(4) Under some regularity hypothesis on $\mathbb{P}$, the condition $P_t u \in D(A)$ for $t > 0$, is always fulfilled (cf. [7], [8], [10]).

(5) In the next paragraph, we want first to investigate such representation for subordinated semigroups by $\mathcal{C}^1$-subordinators.

(6) The proof of the following useful lemma is given in [16].

**Lemma 1.3.** *Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$ and let $u$ be a $\mathbb{P}$-potential. For $t > 0$, let $\varphi_t$ be defined by* (1.4). *Then $\varphi = (\varphi_t)_{t>0}$ is a $\mathbb{P}$-exit law.*

## 2. Representation for subordinated semigroup

**2.1 Bochner subordination.** For the following classical notions, we refer the reader to [2], [3] and [16]–[18].

We consider $\mathbb{R}$ endowed with its Borel field, we denote by $\lambda$ the Lebesgue measure on $[0, \infty[$ and by $\varepsilon_t$ the Dirac measure at point $t$. Moreover, for each bounded measure $\mu$ on $[0, \infty[$, $\mathcal{L}$ denotes its Laplace transform, i.e. $\mathcal{L}(\mu)(r) := \int_0^\infty \exp(-rs)\, \mu(ds)$ for $r > 0$.

A *Bochner subordinator* is a family $\beta := (\beta_t)_{t>0}$ of sub-probability measures on $\mathbb{R}$ such that

(1) for each $t > 0$, the measure $\beta_t \neq \varepsilon_0$ and $\beta_t$ is supported by $[0, \infty[$,
(2) $\beta_s * \beta_t = \beta_{s+t}$ for all $s, t > 0$,
(3) $\lim_{t \to 0} \beta_t = \varepsilon_0$, vaguely.

In this case the associated potential measure is given by $\kappa := \int_0^\infty \beta_s \, ds$. It is known that $\kappa$ is a Borel measure (cf. [2, Proposition 14.1]).

Let $\mathbb{P}$ be a sub-Markovian semigroup and let $\beta$ be a Bochner subordinator. For every $t > 0$ and for every $f \in L^2(m)$, we may define

$$P_t^\beta f := \int_0^\infty P_s f \; \beta_t(ds) \qquad (t > 0). \tag{2.1}$$

Let $P_0 = I$, then $\mathbb{P}^\beta := (P_t^\beta)_{t>0}$ is a sub-Markovian semigroup on $L^2(m)$. It is said to be *subordinated* to $\mathbb{P}$ in the sense of Bochner by means of $\beta$. We denote by $A^\beta$ the associated generator.

The following two remarks will be used later.

(1) $D(A)$ is a subset of $D(A^\beta)$ (cf. [17, p. 269] for example).
(2) Each $\mathbb{P}$-potential is a $\mathbb{P}^\beta$-potential (for the proof, we can adapt those of [3, p. 185]).

**2.2 $\mathcal{C}^1$-subordinator.** Let $S$ be the Banach algebra of complex Borel measures on $[0,\infty[$, with convolution as multiplication, and normed by the total variation $\|\cdot\|_S$. A Bochner subordinator $\beta = (\beta_t)_{t>0}$ is said to be a $\mathcal{C}^1$-subordinator provided

$t \mapsto \beta_t$ is continuously differentiable from $]0,\infty[$ to $S$ and $\|\beta_t'\|_S < \infty$ for each $t > 0$.

This class of subordinators, is considered in [4]. For the following examples, we will refer also to this paper.

(1) One-sided stable subordinator: For each $\alpha \in ]0,1[$ and $t > 0$, let $\eta_t^\alpha$ be the unique probability measure on $[0,\infty[$ such that $\mathcal{L}\,(\eta_t^\alpha)(r) = \exp(-tr^\alpha)$ for $r > 0$. Then $\eta^\alpha := (\eta_t^\alpha)_{t>0}$ is a convolution semigroup on $[0,\infty[$ called *the one-sided stable subordinator of index $\alpha$*. $\eta^\alpha$ is a $\mathcal{C}^1$-subordinator for each $\alpha \in ]0,1[$.

(2) Gamma subordinator: For $t > 0$, let $g_t(s) := 1_{]0,\infty[}(s)(1/\Gamma(t))\, s^{t-1}\exp(-s)$ and $\beta_t := g_t \cdot \lambda$. Then $\gamma := (\gamma_t)_{t>0}$ is a subordinator, called the $\Gamma$-subordinator. Moreover $\gamma$ is a $\mathcal{C}^1$-subordinator.

(3) Compound Poisson subordinator: Let $q$ be an arbitrary probability measure on $[0,\infty[$ and let $c > 0$. Put

$$\beta_t := e^{-ct}\sum_{j=0}^{\infty}\frac{(ct)^j}{j!}q_j \qquad (t > 0),$$

where $q_0 := \varepsilon_0$ and $q_j := \{q\}^{*j}$. Then $\beta$ is a $\mathcal{C}^1$-subordinator, called the *compound Poisson subordinator*. Moreover, the Bernstein function of $\beta$ is given by

$$k(r) = c\mathcal{L}(\varepsilon_0 - q)(r) \qquad (r > 0).$$

This construction includes many explicitly known Bochner subordinators. Thus, for $q = \varepsilon_1$, we obtain the Poisson subordinator with jump c. Similarly, for $q = \sum_{j=1}^{\infty}\frac{(1-b)^j}{cj}\varepsilon_j$ where $0 < b < 1$ and $c = -\log(b)$, we obtain the *negative Binomial subordinator*.

(4) Let $(b_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ be any two sequences satisfying

$$0 < b_n < 1; \quad a_n > 0; \quad \lim_{n\to\infty} b_n = 1; \quad \sum_{n=0}^{\infty} a_n < \infty,$$

and define $k(r) = \sum_{n=0}^{\infty} a_n r^{b_n}$, $r > 0$. Then $k$ is the Bernstein function of some Bochner subordinator which is not a $\mathcal{C}^1$-subordinator.

(5) $(\varepsilon_t * \beta_t)_{t>0}$ is not a $\mathcal{C}^1$-subordinator, even when $\beta$ is a $\mathcal{C}^1$-subordinator.

(6) If $\beta^1$, $\beta^2$ are $\mathcal{C}^1$-subordinators then so is $\beta^1 * \beta^2$.

(7) Let $\beta$ be a $\mathcal{C}^1$-subordinator with Bernstein function $f$. Suppose that $\|\beta_t'\|_S < c/t$ for some constant $c > 0$ when $t \downarrow 0$. $f$ is bounded if and only if $\beta$ is a compound Poisson family.

**Lemma 2.1.** *Let $\beta$ be a $\mathcal{C}^1$-subordinator. Then*

$$\beta'_{s+t} = \beta'_s * \beta_t \qquad (s, t > 0) \tag{2.2}$$

*and*

$$\beta_t = -\beta'_t * \kappa \qquad (t > 0), \tag{2.3}$$

*where $\beta'_t := \frac{\partial}{\partial t}\beta_t$ and $\kappa = \int_0^\infty \beta_t \, dt$.*

Proof: Let $\beta$ be a $\mathcal{C}^1$-subordinator. Since $\mathcal{L}(\beta_t)(r) = \exp(-tf(r))$, by differentiation with respect to $t$ under the integral sign, we obtain

$$\mathcal{L}(\beta'_t) = \frac{\partial}{\partial t}\mathcal{L}(\beta_t)(r) = -f(r)\exp(-tf(r)) \qquad (t, r > 0). \tag{2.4}$$

Let $s, t, r > 0$, using (2.4), we get

$$\begin{aligned}
\mathcal{L}(\beta'_s * \beta_t)(r) &= \mathcal{L}(\beta'_s)(r)\mathcal{L}(\beta_t)(r) \\
&= -f(r)\exp(-sf(r))\exp(-tf(r)) \\
&= -f(r)e^{-(s+t)f(r)} \\
&= \mathcal{L}(\beta'_{s+t})(r).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathcal{L}(-\beta'_s * \kappa)(r) &= -\mathcal{L}(\beta'_s)(r)\mathcal{L}(\kappa)(r) \\
&= f(r)\exp(-sf(r))\frac{1}{f(r)} \\
&= \mathcal{L}(\beta_t)(r).
\end{aligned}$$

We deduce (2.2) and (2.3) by the injectivity of Laplace transform. $\square$

**Proposition 2.2.** *Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$, let $\beta$ be a $\mathcal{C}^1$-subordinator and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of $\beta$. Then $P_t^\beta(L^2(m)) \subset D(A^\beta)$ and*

$$A^\beta P_t^\beta u = \int_0^\infty P_s u \, \beta_t^{'}(ds) \qquad (t > 0, u \in L^2(m)). \tag{2.5}$$

Proof: Let $\beta$ be a $\mathcal{C}^1$-subordinator. For each $u \in L^2(m)$, we have

$$\Big\| \int_0^\infty P_s u \, \beta_t^{'}(ds) \Big\|_2 \leq \|u\|_2 \, \|\beta_t^{'}\|_S \qquad (t > 0).$$

Therefore the function $x \mapsto \int_0^\infty P_s u \, \beta_t^{'}(ds)$, is well defined and lies in $L^2(m)$. Moreover, following [4, Theorem 4], the differentiation with respect to $t$ under the integral sign is justified in $P_t^\beta u$ and by (2.1) we have

$$\int_0^\infty P_s u \, \beta_t'(ds) = \frac{\partial}{\partial t} P_t^\beta u = A^\beta P_t^\beta u \qquad (t > 0, u \in L^2(m)).$$

□

**Theorem 2.3.** *Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$, let $\beta$ be a $\mathcal{C}^1$-subordinator and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of $\beta$. For each $\mathbb{P}^\beta$-pseudo-potential $h$, there exists a unique $\mathbb{P}^\beta$-exit law $\psi = (\psi_t)_{t>0}$ such that*

$$h = \int_0^\infty \psi_s \, ds, \tag{2.6}$$

*where $\psi$ is explicitly given by*

$$\psi_t = - \int_0^\infty P_s(P_{t/2}^\beta h) \, \beta_{t/2}'(ds) \qquad (t > 0).$$

*Moreover, if $h \in L_+^2(m)$, then $\psi$ is on the form*

$$\psi_t = - \int_0^\infty P_s h \, \beta_t'(ds) \qquad (t > 0). \tag{2.7}$$

Proof: Let $\beta$ be a $\mathcal{C}^1$-subordinator and let $h$ be a $\mathbb{P}^\beta$-pseudo-potential.

Step 1: We prove that $h$ is a $\mathbb{P}^\beta$-potential. Indeed, for all $s, t > 0$ we have

$$P_{s+t}^\beta h = P_s^\beta (P_t^\beta h) \in P_s^\beta (L^2(m))$$

by hypothesis. Hence $P_{s+t}^\beta h \in D(A^\beta)$ by Proposition 2.2. We conclude that for all $t > 0$ we have $P_t^\beta h = P_{t/2+t/2}^\beta h \in D(A^\beta)$ and therefore $h$ is a $\mathbb{P}^\beta$-potential.

Step 2: From the first step we may define

$$\psi_t := -A^\beta(P_t^\beta h) \qquad (t>0). \tag{2.8}$$

If we apply Lemma 1.3 for $\mathbb{P}^\beta$ instead of $\mathbb{P}$, we deduce that $\psi = (\psi_t)_{t>0}$ is a $\mathbb{P}^\beta$-exit law.

Step 3: We prove the representation (2.6): For $s,t>0$,

$$\begin{aligned}
P_{s+t}^\beta h &= \int_0^\infty P_r(P_s^\beta h)\,\beta_t(dr)\\
&\overset{(2.3)}{=} -\int_0^\infty P_r(P_s^\beta h)\,(\beta_t' * \kappa)(dr)\\
&= -\int_0^\infty\int_0^\infty P_{r+\ell}(P_s^\beta h)\,\beta_t'(dr)\,\kappa(d\ell)\\
&= -\int_0^\infty\int_0^\infty\int_0^\infty P_{r+\ell}(P_s^\beta h)\,\beta_t'(dr)\,\beta_q(d\ell)\,dq\\
&= -\int_0^\infty\left(\int_0^\infty P_r(P_s^\beta h)(\beta_t' * \beta_q)(dr)\right)dq\\
&\overset{(2.2)}{=} -\int_0^\infty\int_0^\infty P_r\left(P_s^\beta h\right)\,\beta_{t+q}'(dr)\,dq\\
&\overset{(2.5)}{=} -\int_0^\infty A^\beta\left(P_{t+q}^\beta P_s^\beta h\right)\,dq\\
&= -\int_0^\infty A^\beta\left(P_{t+q+s}^\beta h\right)\,dq\\
&\overset{(2.8)}{=} \int_0^\infty \psi_{t+s+q}\,dq\\
&= \int_{t+s}^\infty \psi_q\,dq.
\end{aligned}$$

Therefore, we obtain the representation

$$P_t^\beta h = \int_t^\infty \psi_s\,ds \qquad (t>0) \tag{2.9}$$

in $L^2(m)$. Now, by letting $t\downarrow 0$ in (2.9), we obtain (2.6).

Moreover if $h\in L^2_+(m)$, then (2.7) is immediate from (2.5) and (2.8). □

**Remarks 2.4.** Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$.

(1) Let $\beta$ be a $\mathcal{C}^1$-subordinator. From (2.9) and Proposition 1.1, we deduce that each $\mathbb{P}^\beta$-pseudo-potential is a $\mathbb{P}^\beta$-potential.

(2) Let $h \in L^2_+(m)$. By application of (2.7), we obtain the following formulas:

(i) If $h$ is a $\mathbb{P}^{\eta^{\frac{1}{2}}}$-potential then

$$h = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty P_r h r^{\frac{-3}{2}} \left(1 - \frac{2s^2}{4r}\right) \exp\left(\frac{-s^2}{4r}\right) dr\, ds.$$

(ii) If $h$ is a $\mathbb{P}^\gamma$-potential then

$$h = \int_0^\infty \frac{1}{\Gamma(s)} \int_0^\infty P_r h \left(\frac{\Gamma'(s)}{\Gamma(r)} - \log r\right) r^{s-1} \exp(-r)\, dr\, ds.$$

## 3. Application to the original semigroup

For each $\alpha \in ]0,1[$ let $\eta_t^\alpha$ be the one-sided stable subordinator. Following [19, p. 263], the measure $\eta_t^\alpha$ has a density, denoted by $\rho_t^\alpha$, with respect to $\lambda$ where

$$\rho_t^\alpha(s) = \frac{1}{\pi} \int_0^\infty r^\alpha \exp(rs\cos\theta - tr^\alpha \cos\alpha\theta) \sin(sr\sin\theta - tr^\alpha \sin\alpha\theta + \theta)\, dr$$

for all $s, t > 0$ and for some $\theta \in [\frac{\pi}{2}, \pi]$.

Let $q_t^\alpha(s) = \frac{\partial}{\partial t}\rho_t^\alpha(s)$ we have

$$q_t^\alpha(s) := \frac{-1}{\pi} \int_0^\infty \exp(sr\cos\theta - tr^\alpha\cos\alpha\theta) \sin(sr\sin\theta - tr^\alpha \sin\alpha\theta + \alpha\theta + \theta) r^\alpha\, dr$$

For all $s, t > 0$, we denote

$$\begin{aligned}
\Upsilon_t^\alpha(s) &:= \int_0^s \rho_t^\alpha(r)\, dr, \\
q_t^\alpha(s) &:= \frac{\partial}{\partial t}\rho_t^\alpha(s), \\
\Lambda_t^\alpha(s) &:= \int_0^s q_t^\alpha(r)\, dr.
\end{aligned}$$

Let $u$ be a $\mathbb{P}$-potential. Then $u$ is a $\mathbb{P}^\beta$-potential and therefore Theorem 2.3 may be applied for such function. In particular, if we take $\beta_t = \eta_t^\alpha$, the one-sided stable subordinator of index $\alpha \in ]0,1[$, we obtain the following result:

**Corollary 3.1.** *Let $u$ be a $\mathbb{P}$-potential. Then*

$$P_t u = \int_0^\infty \psi_r^t\, dr \qquad (t > 0), \tag{3.1}$$

*where*

$$\psi_r^t = -\int_0^\infty P_{s+t} u\, q_r^\alpha(s)\, ds \qquad (r > 0). \tag{3.2}$$

Proof: Let $u$ be a $\mathbb{P}$-potential and let $t>0$ be fixed. Then $P_t u$ is a $\mathbb{P}$-potential and therefore a $\mathbb{P}^{\eta^\alpha}$-potential. Using Theorem 2.3, there exists a unique $\mathbb{P}^{\eta^\alpha}$-exit law $\psi^t = (\psi^t_s)_{s>0}$ such that

$$P_s^{\eta^\alpha} P_t u = \int_s^\infty \psi_r^t \, dr \qquad (s>0), \tag{3.3}$$

where $\psi_r^t$ is given by (3.2). Letting $s \downarrow 0$ in (3.3), we obtain (3.1). □

**Lemma 3.2.** *Let $\alpha \in ]0,1[$. For each $t>0$, $s \mapsto \Upsilon_t^\alpha(s)$ is an increasing bounded continuous function from $]0,\infty[$ to $[0,1]$. Moreover for all $s>0$, we have*

$$\lim_{t\to\infty} \Upsilon_t^\alpha(s) = 0 \tag{3.4}$$

*and*

$$\lim_{t\to 0} \Upsilon_t^\alpha(s) = 1. \tag{3.5}$$

Proof: The proof is adapted from [19, p. 263].

Since for all $t>0$, $\eta_t^\alpha$ is a probability measure on $]0,\infty[$, it follows that

$$s \mapsto \Upsilon_t^\alpha(s) = \int_0^s \eta_t^\alpha(dr)$$

is an increasing bounded continuous function from $]0,\infty[$ into $[0,1]$.

On the other hand by the change of variables $r = t^{-1/\alpha} v$, $z = t^{1/\alpha} u$, we get

$$\begin{aligned}
\Upsilon_t^\alpha(s) &= \int_0^s \rho_t^\alpha(z)\, dz \\
&= \frac{1}{\pi} \int_0^s \int_0^\infty r^\alpha e^{rz\cos\theta + tr^\alpha \cos\alpha\theta} \sin(zr\sin\theta - tr^\alpha \sin\alpha\theta + \theta)\, dr\, dz \\
&= \frac{1}{\pi} \int_0^s \int_0^\infty v^\alpha e^{t^{\frac{-1}{\alpha}} vz\cos\theta + tr^\alpha \cos\alpha\theta} \sin(zt^{\frac{-1}{\alpha}} v\sin\theta - v^\alpha \cos\alpha\theta + \theta)\, dv\, dz \\
&= \frac{1}{\pi} \int_0^{st^{\frac{-1}{\alpha}}} \int_0^\infty v^\alpha e^{uv\cos\theta + v^\alpha \cos\alpha\theta} \sin(uv\sin\theta - v^\alpha \cos\alpha\theta + \theta)\, dv\, du \\
&= \int_0^{st^{\frac{-1}{\alpha}}} \rho_1^\alpha(v)\, dv = \Upsilon_1^\alpha(st^{\frac{-1}{\alpha}}).
\end{aligned}$$

Therefore (3.4) and (3.5) hold. □

**Lemma 3.3.** *Let $\alpha \in ]0,1[$. For each $s>0$, $t \mapsto \Upsilon_t^\alpha(s)$ is a differentiable function on $]0,\infty[$. Moreover for all $s>0$, we have*

$$\Lambda_t^\alpha(s) = \frac{\partial}{\partial t}\Upsilon_t^\alpha(s), \tag{3.6}$$

$$\int_0^\infty \Lambda_t^\alpha(s)\,dt = -1, \tag{3.7}$$

$$\lim_{s\to 0}\Lambda_t^\alpha(s) = \lim_{s\to\infty}\Lambda_t^\alpha(s) = 0 \qquad (t>0). \tag{3.8}$$

Proof: Since $t \mapsto \rho_t^\alpha(s)$ is differentiable on $[0,\infty[$, using a derivation theorem under the integral sign with respect to $t$, the function $t \mapsto \Upsilon_t^\alpha(s)$ is differentiable and

$$\frac{\partial}{\partial t}\Upsilon_t^\alpha(s) = \frac{\partial}{\partial t}\left(\int_0^s \rho_t^\alpha(z)\,dz\right) = \int_0^s q_t^\alpha(z)\,dz = \Lambda_t^\alpha(s).$$

Hence (3.6) holds. Moreover by Lemma 3.2, we have

$$\int_0^\infty \Lambda_t^\alpha(s)\,dt = \int_0^\infty \frac{\partial}{\partial t}\Upsilon_t^\alpha(s)\,dt = \lim_{t\to\infty}\Upsilon_t^\alpha(s) - \lim_{t\to 0}\Upsilon_t^\alpha(s).$$

Therefore (3.7) holds.

If we take $\theta_\alpha = \frac{\pi}{1+\alpha}$, then by the derivation theorem under the integral sign with respect to $t$, we obtain

$$q_t^\alpha(s) = \frac{1}{\pi}\int_0^\infty r^\alpha \exp\left((rs+tr^\alpha)\cos\theta_\alpha\right)\sin\left((sr-tr^\alpha)\sin\theta_\alpha\right)\,dr.$$

It follows that $s \to q_t^\alpha(s)$ is integrable on $]0,\infty[$. Hence by differentiation of $\int_0^\infty \eta_t^\alpha(ds) = \int_0^\infty \rho_t^\alpha(s)\,ds = 1$ with respect to $t$, we obtain (3.8). $\square$

**Theorem 3.4.** *Let $\mathbb{P}$ be a sub-Markovian semigroup on $L^2(m)$. Then, for each $\mathbb{P}$-potential $u$ there exists a unique $\mathbb{P}$-exit law $\varphi$ such that*

$$u = \int_0^\infty \varphi_s\,ds. \tag{1.3}$$

Proof: Let $u$ be a $\mathbb{P}$-potential. By Lemma 1.3, the family $\varphi := (\varphi_t)_{t>0}$ defined by (1.4), i.e.

$$\varphi_t := -AP_tu \qquad (t>0) \tag{1.4}$$

is a $\mathbb{P}$-exit law.

On the other hand, there exists by Corollary 3.1, a unique $\mathbb{P}^{\eta^\alpha}$-exit law $\psi^t$ (given by (3.2)) such that (3.1) holds. Using an integration by parts we obtain

$$\psi_s^t = \left[-P_{r+t}h\Lambda_s^\alpha(r)\right]_0^\infty + \int_0^\infty \frac{\partial}{\partial r}P_{r+t}u\Lambda_s^\alpha(r)\,dr \qquad (s>0)$$

and by Lemma 3.2 we get

$$\psi_s^t = -\int_0^\infty \varphi_{r+t}\,\Lambda_s^\alpha(r)\,dr \qquad (s>0). \tag{3.9}$$

Now by (3.2), (3.10), (3.1) and Fubini's Theorem we get

$$\begin{aligned} P_t u &= \int_0^\infty \int_0^\infty -\varphi_{r+t}\,\Lambda_s^\alpha(r)\,dr\,ds \\ &= \int_0^\infty -\varphi_{r+t}\left(\int_0^\infty \Lambda_s^\alpha(r)\,dr\right)\,ds \\ &= \int_0^\infty \varphi_{r+t}\,dr \\ &= \int_t^\infty \varphi_r\,dr. \end{aligned}$$

We conclude as in the proof of Proposition 1.1. □

**Remark 3.5.** In this paper, we have used a representation for the subordinated structure (Theorem 2.3), in order to obtain a representation for the original one (Theorem 3.4). A similar idea is already investigated in [9, Theorem 2].

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