Ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces

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Abstract. We prove ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces.

1. Introduction

Let X be a Banach space and $u : [0, \infty) \to X$ be a locally integrable function. Let $g : [0, \infty) \to \mathbb{R}_+$ be a locally integrable function such that $\int_0^\infty g(t)\, dt > 0$, where $\mathbb{R}_+ := \{t \geq 0 : t \in \mathbb{R}\}$. We assume the condition

$$\frac{\int_0^t g(r)\, dr}{\int_0^s g(r)\, dr} \to 1 \quad \text{as} \quad t, s \to \infty \quad \text{with} \quad \frac{t}{s} \to 1,$$

and prove that if $\|u(t)\| = O(g(t))$, $t \to \infty$, then the following statements are equivalent:

(i) $x = \lim_{t \to \infty} \left(\int_0^t u(s)\, ds \right) / \left(\int_0^t g(s)\, ds \right)$;

(ii) $x = \lim_{\lambda \downarrow 0} \left(\int_0^\infty e^{-\lambda t} u(t)\, dt \right) / \left(\int_0^\infty e^{-\lambda t} g(t)\, dt \right)$.

This solves the open problem posed in [6]. Then particular choices of the function g will be considered, leading to some generalized Tauberian theorems. Discrete analogues are obtained as well.

2. Results for functions

Let X be a Banach space and $u : [0, \infty) \to X$ be a locally integrable function. The class of all such functions will be denoted by $L^1_{\text{loc}}(\mathbb{R}_+, X)$. For $u \in L^1_{\text{loc}}(\mathbb{R}_+, X)$, $\gamma \geq 1$ and $t > 0$ we define the γ-th order Cesàro integral $\mathfrak{s}^\gamma_t(u)$ over $[0, t]$ as

$$(1) \qquad \mathfrak{s}^\gamma_t(u) := (k_\gamma * u)(t) = \int_0^t k_\gamma(t - s) u(s)\, ds,$$

where $k_\gamma(t) := t^{\gamma-1}/\Gamma(\gamma)$ for $t \in \mathbb{R}_+$. In particular we have $\mathfrak{s}_t^1(u) = \int_0^t u(s)\,ds$. The Laplace integral $\widehat{u}(\lambda)$ for $\lambda \in \mathbb{R}$ is defined as

$$(2) \qquad \widehat{u}(\lambda) := \int_0^\infty e^{-\lambda t} u(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-\lambda t} u(t)\,dt$$

if the limit exists. It is known (see e.g. [1, Proposition 1.4.1]) that if $\widehat{u}(\lambda_0)$ exists then $\widehat{u}(\lambda)$ exists for all $\lambda > \lambda_0$. If μ is a locally finite positive measure on $\mathbb{R}_+$, then we use the notation $\widehat{\mu}(\lambda)$ to denote $\int_0^\infty e^{-\lambda t} d\mu(t)$ when $\int_0^\infty e^{-\lambda t} d\mu(t) < \infty$.

We begin with the following key lemma.

Lemma 2.1. *Let μ be a locally finite positive measure on $\mathbb{R}_+$ such that $\mu[0,\infty) > 0$. If*

$$(C) \qquad \frac{\mu[0,t]}{\mu[0,s]} \to 1 \quad as \quad t, s \to \infty \quad with \quad \frac{t}{s} \to 1,$$

then

$$(C1) \qquad \liminf_{\lambda\downarrow 0} \frac{\mu[0, 1/\lambda]}{\widehat{\mu}(\lambda)} = \liminf_{\lambda\downarrow 0} \frac{\mu[0, 1/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)} > 0.$$

PROOF: By hypothesis there are two constants $G > 1$ and $\delta > 0$ such that if $t > s > G$ and $t/s \leq 1 + \delta$ then

$$0 \leq \frac{\mu(s,t]}{\mu[0,s]} < 1.$$

Thus for $\lambda > 0$ with $1/\lambda > G$ we have $\mu(1/\lambda, (1+\delta)/\lambda] < 2^0 \mu[0, 1/\lambda]$, and

$$\mu((1+\delta)/\lambda,\, (1+\delta)^2/\lambda] < \mu[0, (1+\delta)/\lambda] < 2^1 \mu[0, 1/\lambda].$$

Then for $n \geq 2$ we have inductively

$$\begin{aligned}
\mu((1+\delta)^n/\lambda,\, (1+\delta)^{n+1}/\lambda] \quad &< \quad \mu[0, (1+\delta)^n/\lambda] \\
&= \quad \mu[0, 1/\lambda] + \sum_{k=0}^{n-1} \mu((1+\delta)^k/\lambda,\, (1+\delta)^{k+1}/\lambda] \\
&< \quad \left(1 + \sum_{k=0}^{n-1} 2^k\right) \mu[0, 1/\lambda] = 2^n \mu[0, 1/\lambda].
\end{aligned}$$

Hence

$$\begin{aligned}
0 < \int_0^\infty e^{-\lambda t} d\mu(t) \quad &= \quad \int_{[0,1/\lambda]} e^{-\lambda t} d\mu(t) + \sum_{n=0}^\infty \int_{((1+\delta)^n/\lambda,\, (1+\delta)^{n+1}/\lambda]} e^{-\lambda t}\,d\mu(t) \\
&\leq \quad \mu[0, 1/\lambda] + \sum_{n=0}^\infty 2^n \mu[0, 1/\lambda] e^{-(1+\delta)^n} < \infty.
\end{aligned}$$

Therefore

$$\frac{\mu[0,1/\lambda]}{\widehat{\mu}(\lambda)} = \frac{\mu[0,1/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)} \geq \Big(1 + \sum_{n=0}^\infty 2^n e^{-(1+\delta)^n}\Big)^{-1} > 0,$$

completing the proof. $\square$

Theorem 2.2 (cf. [2, Theorem 2.2]). *Suppose* $0 \neq g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ *satisfies condition* (C) *with* $\mu := g(t)\,dt$. *Then for any* $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ *with* $\|u(t)\| = O(g(t))$, $t \to \infty$, *the following statements are equivalent:*

(i) $x = \lim_{t\to\infty} \mathfrak{s}^1_t(u)/\mathfrak{s}^1_t(g) = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds \big)/\big(\int_0^t g(s)\,ds \big)$;

(ii) $x = \lim_{t\to\infty} \mathfrak{s}^\beta_t(u)/\mathfrak{s}^\beta_t(g)$
$\quad = \lim_{t\to\infty} \big(\int_0^t (t-s)^{\beta-1} u(s)\,ds \big)/\big(\int_0^t (t-s)^{\beta-1} g(s)\,ds \big)$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0} \widehat{u}(\lambda)/\widehat{g}(\lambda) = \lim_{\lambda\downarrow 0} \big(\int_0^\infty e^{-\lambda t} u(t)\,dt \big)/\big(\int_0^\infty e^{-\lambda t} g(t)\,dt \big)$.

PROOF: "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" follows from [2, Theorem 2.1].

(iii) $\Rightarrow$ (i): We first note that if $P(t) = \sum_{n=0}^N a_n t^n$ is a polynomial function such that

(3) $$P(t) \geq d > 0 \quad \text{on} \quad [0,1],$$

then

(4) $$x = \lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt}.$$

To see this, put $\widetilde{P}(\lambda) := \int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt$. Then

$$\frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt}$$
$$= \frac{1}{\widetilde{P}(\lambda)} \sum_{n=0}^N a_n \Big(\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt \Big) \cdot \frac{\int_0^\infty e^{-\lambda(n+1)t} u(t)\,dt}{\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt}.$$

Here

(5) $$\lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda(n+1)t} u(t)\,dt}{\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt} = x \qquad \text{(by (iii))},$$

and

(6) $$0 < \frac{\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt}{\widetilde{P}(\lambda)} \leq \frac{1}{d} \qquad \text{(by (3))}.$$

Thus

$$\lim_{\lambda \downarrow 0} \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt}$$

$$= \lim_{\lambda \downarrow 0} \frac{1}{\widetilde{P}(\lambda)} \sum_{n=0}^N a_n \left(\int_0^\infty e^{-\lambda(n+1)t} g(t)\, dt \right) x = x.$$

Next we write

(7)
$$\frac{\int_0^{1/\lambda} u(t)\, dt}{\int_0^{1/\lambda} g(t)\, dt} = \frac{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt},$$

where

$$h(t) := \begin{cases} 0 & \text{if } 0 \le t < e^{-1}, \\ t^{-1} & \text{if } e^{-1} \le t \le 1. \end{cases}$$

For the proof we may assume without loss of generality that $\|u(t)\| \le g(t)$ for all $t \ge 0$. By condition (C), given an $\varepsilon > 0$, there are two constants $G > 1$ and $\delta > 0$ such that

(8)
$$0 \le \frac{\mu(s, t]}{\mu[0, s]} < \varepsilon \quad \text{if} \quad t > s > G \quad \text{and} \quad \frac{t}{s} \le 1 + \delta.$$

It is standard to see that there exists a polynomial function $P(t) = \sum_{n=0}^N a_n t^n$ such that

(a) $h(t) < P(t) \le \varepsilon$ on $[0, e^{-(1+\delta)}]$,
(b) $h(t) < P(t) \le h(e^{-1}) + \varepsilon$ on $[e^{-(1+\delta)}, e^{-1}]$,
(c) $h(t) < P(t) \le h(t) + \varepsilon$ on $[e^{-1}, 1]$.

Then

$$\frac{\int_0^{1/\lambda} u(t)\, dt}{\int_0^{1/\lambda} g(t)\, dt} - x$$

$$= \frac{\int_0^\infty e^{-\lambda t} \left(h(e^{-\lambda t}) - P(e^{-\lambda t}) \right) u(t)\, dt + \int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt} - x$$

$$=: I_\lambda + II_\lambda - x,$$

and

$$I_\lambda = \frac{\left(\int_0^{1/\lambda} + \int_{1/\lambda}^{(1+\delta)/\lambda} + \int_{(1+\delta)/\lambda}^\infty \right) e^{-\lambda t} \left(h(e^{-\lambda t}) - P(e^{-\lambda t}) \right) u(t)\, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt}$$

$$=: \frac{I_\lambda(1) + I_\lambda(2) + I_\lambda(3)}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt},$$

where

(9)
$$\|I_\lambda(1)\| < \int_0^{1/\lambda} e^{-\lambda t} \varepsilon \, g(t) \, dt \le \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t) \, dt$$

by (c) and the assumption that $\|u(t)\| \le g(t)$ for all $t \ge 0$. On the other hand, (b) implies

$$\|I_\lambda(2)\| < \int_{1/\lambda}^{(1+\delta)/\lambda} e^{-\lambda t} \big(h(e^{-1}) + \varepsilon \big) g(t) \, dt \le (e+\varepsilon) \int_{1/\lambda}^{(1+\delta)/\lambda} g(t) \, dt,$$

where if $\lambda > 0$ is sufficiently small, then by (8)

$$\int_{1/\lambda}^{(1+\delta)/\lambda} g(t) \, dt < \varepsilon \int_0^{1/\lambda} g(t) dt = \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t) \, dt,$$

so that

(10)
$$\|I_\lambda(2)\| < (e+\varepsilon)\varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t) \, dt$$

for all sufficiently small $\lambda > 0$. Finally (a) implies

$$\|I_\lambda(3)\| < \int_{(1+\delta)/\lambda}^\infty e^{-\lambda t} \varepsilon \, g(t) \, dt \le \varepsilon \int_0^\infty e^{-\lambda t} g(t) \, dt.$$

We apply Lemma 2.1 to infer that there exists a constant $\eta > 0$ such that

$$\liminf_{\lambda \downarrow 0} \frac{\int_0^{1/\lambda} g(t) \, dt}{\int_0^\infty e^{-\lambda t} g(t) \, dt} > \eta.$$

Thus if $\lambda > 0$ is sufficiently small, then

$$\int_0^{1/\lambda} g(t) \, dt > \eta \int_0^\infty e^{-\lambda t} g(t) \, dt,$$

so that

(11)
$$\|I_\lambda(3)\| < \frac{\varepsilon}{\eta} \int_0^{1/\lambda} g(t) \, dt = \frac{\varepsilon}{\eta} \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t) \, dt.$$

Consequently

(12)
$$\limsup_{\lambda \downarrow 0} \|I_\lambda\| < \varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}.$$

   Now we write

$$II_\lambda = \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t) \, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t) \, dt} \cdot \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t) \, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t) \, dt}.$$

Similarly as in (12) one may show that if $\lambda > 0$ is sufficiently small, then

$$1 \leq \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t) \, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t) \, dt} < 1 + \varepsilon + (e + \varepsilon)\varepsilon + \frac{\varepsilon}{\eta}.$$

Since $P(t) \geq d > 0$ on $[0,1]$ for some $d > 0$, it follows that

$$\lim_{\lambda \downarrow 0} \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t) \, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t) \, dt} = x.$$

Hence

$$
\begin{aligned}
\|II_\lambda - x\| &\leq \left\| \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t) \, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t) \, dt} - x \right\| \\
&\quad + \left(\varepsilon + (e + \varepsilon)\varepsilon + \frac{\varepsilon}{\eta} \right) \left\| \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t) \, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t) \, dt} \right\| \\
&\longrightarrow \left(\varepsilon + (e + \varepsilon)\varepsilon + \frac{\varepsilon}{\eta} \right) \|x\|
\end{aligned}
$$

(13)

as $\lambda \downarrow 0$. Combining this with (12) yields

(14)
$$\limsup_{\lambda \downarrow 0} \left\| \frac{\int_0^{1/\lambda} u(t) \, dt}{\int_0^{1/\lambda} g(t) \, dt} - x \right\| < \left(\varepsilon + (e + \varepsilon)\varepsilon + \frac{\varepsilon}{\eta} \right)(1 + \|x\|)$$

which completes the proof, since $\varepsilon > 0$ is arbitrary. □

Theorem 2.3 (cf. [2, Theorem 4.2], [5, Proposition 3.4]). *Let $\alpha \geq 0$. Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{\alpha-1})$, $t \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{t \to \infty} \left(\Gamma(\alpha+1)/t^\alpha \right) \int_0^t u(s) \, ds$;

(ii) $x = \lim_{t \to \infty} \left(\Gamma(\alpha+\beta)/\Gamma(\beta) t^{\alpha+\beta-1} \right) \int_0^t (t-s)^{\beta-1} u(s) \, ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda \downarrow 0} \lambda^\alpha \widehat{u}(\lambda) = \lim_{\lambda \downarrow 0} \lambda^\alpha \int_0^\infty e^{-\lambda t} u(t) \, dt$.

PROOF: "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" follows from [2, Theorem 4.1].

(iii) $\Rightarrow$ (i): Suppose $\alpha > 0$. Then define $g(t) := k_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t \in \mathbb{R}_+$ and $\mu := g(t) \, dt$. It follows that $\|u(t)\| = O(g(t))$, $t \to \infty$, that $\widehat{g}(\lambda) = \lambda^\alpha$ for all $\lambda > 0$, and that $\mu[0,t] = \int_0^t k_\alpha(s) \, ds = (k_1 * k_\alpha)(t) = k_{\alpha+1}(t) = t^\alpha/\Gamma(\alpha+1)$. Hence μ satisfies condition (C), and so (i) follows from Theorem 2.2.

Next suppose $\alpha = 0$. Since $\|u(t)\| = O(t^{-1})$, $t \to \infty$, it follows from standard calculations (see e.g. [8, pp. 204, 206]) that the function $U(t) := \int_0^t u(s) \, ds$ is bounded and feebly oscillating (i.e. $\|U(t) - U(s)\| \to 0$ as t and $s \to \infty$ in such a way that $t/s \to 1$). Thus (i) follows from [5, Proposition 3.4]. The proof is complete. □

Remark. The special case $\alpha = 1$ of Theorem 2.3 states that, under the assumption that u is bounded, the Cesàro limit $\lim_{t\to\infty} (1/t) \int_0^t u(s)\, ds$ exists if and only if the Abel limit $\lim_{\lambda\downarrow 0} \lambda\, \widehat{u}(\lambda)$ exists and they are both equal. This is a classical Tauberian theorem (see e.g. [1, Theorem 4.2.7]). The special case $\alpha = 0$ of Theorem 2.3 states that, under the assumption that $\|u\| = O(t^{-1})$, $t \to \infty$, the limit $\lim_{t\to\infty} \int_0^t u(s)\, ds$ exists if and only if the limit $\lim_{\lambda\downarrow 0} \widehat{u}(\lambda)$ exists and they are both equal. This is another classical Tauberian theorem (see e.g. [1, Theorem 4.2.9]).

Theorem 2.4. *Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$. Then the following statements are equivalent:*
- (i) $x = \lim_{t\to\infty} \left(1/\log t\right) \int_0^t u(s)\, ds$;
- (ii) $x = \lim_{t\to\infty} \left(1/t^{\beta-1} \log t\right) \int_0^t (t-s)^{\beta-1} u(s)\, ds$ *for some/all* $\beta > 1$;
- (iii) $x = \lim_{\lambda\downarrow 0} \left(1/-\log\lambda\right) \widehat{u}(\lambda) = \lim_{\lambda\downarrow 0} \left(1/-\log\lambda\right) \int_0^\infty e^{-\lambda t} u(t)\, dt$.

PROOF: Let

$$g(t) := \begin{cases} 0 & \text{if } 0 \le t < 1, \\ t^{-1} & \text{if } t \ge 1. \end{cases}$$

An approximation argument yields that

$$
(15) \qquad
\begin{aligned}
\lim_{t\to\infty} \frac{\int_0^t (t-s)^{\gamma-1} g(s)\, ds}{t^{\gamma-1} \log t} &= \lim_{t\to\infty} \frac{\int_1^t (t-s)^{\gamma-1} s^{-1}\, ds}{t^{\gamma-1} \log t} \\
&= \lim_{t\to\infty} \frac{\int_{1/t}^1 (1-s)^{\gamma-1} s^{-1}\, ds}{\log t} = 1
\end{aligned}
$$

for all $\gamma \ge 1$ and that

$$(16) \qquad \lim_{\lambda\downarrow 0} \frac{\widehat{g}(\lambda)}{-\log\lambda} = \lim_{\lambda\downarrow 0} \frac{\int_1^\infty e^{-\lambda t} t^{-1}\, dt}{-\log\lambda} = \lim_{\lambda\downarrow 0} \frac{\int_\lambda^\infty e^{-t} t^{-1}\, dt}{-\log\lambda} = 1.$$

Since $\|u(t)\| = O(g(t))$, $t \to \infty$, and the measure $\mu := g(t)\, dt$ satisfies condition (C), the desired result follows from Theorem 2.2. □

Remark. If X is a Banach lattice with positive cone X_+ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then statements (i), (ii) and (iii) in Theorem 2.4 are also equivalent. This follows from [2, Theorem 2.2]. (We note that if $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then statement (ii) in Theorem 2.4 implies that $\widehat{u}(\lambda)$ exists for all $\lambda > 0$ (see [3, Lemma 2.5]).)

Fact 2.5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. Consider the following three statements:*
- (i) $x = \lim_{t\to\infty} \int_0^t u(s)\, ds$;
- (ii) $\widehat{u}(\lambda)$ *exists for all* $\lambda > 0$ *and* $x = \lim_{t\to\infty} \left(1/t^{\beta-1}\right) \int_0^t (t-s)^{\beta-1} u(s)\, ds$ *for some/all* $\beta > 1$;
- (iii) $x = \lim_{\lambda\downarrow 0} \widehat{u}(\lambda) = \lim_{\lambda\downarrow 0} \int_0^\infty e^{-\lambda t} u(t)\, dt$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF: Letting $g(t) := \chi_{[0,1]}(t)$ we have

$$\lim_{t \to \infty} \frac{\int_0^t (t-s)^{\gamma-1} g(s)\, ds}{t^{\gamma-1}} = \lim_{t \to \infty} \frac{\int_0^1 (t-s)^{\gamma-1}\, ds}{t^{\gamma-1}} = 1$$

for all $\gamma \geq 1$ and

$$\lim_{\lambda \downarrow 0} \widehat{g}(\lambda) = \lim_{\lambda \downarrow 0} \int_0^1 e^{-\lambda t}\, dt = 1.$$

Thus the desired result follows from [2, Theorem 2.1]. □

Remarks. (a) If $\int_0^\infty \|u(t)\|\, dt < \infty$, then clearly both (i) and (iii) in Fact 2.5 hold. In general (iii) does not imply (i). (For example let $u(t) := \sin t$.) If $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$, or if X is a Banach lattice and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then (iii) implies (i). (See Theorem 2.3 and [2, Theorem 4.2], respectively.)

(b) There exists a continuous function $u : [0, \infty) \to \mathbb{R}$ such that $\inf\{\lambda \in \mathbb{R} : \widehat{u}(\lambda) \text{ exists}\} = 1$ and also such that $\lim_{t \to \infty} (1/t)\int_0^t (t-s)u(s)\, ds \ (\in \mathbb{R})$ exists (see the Remark over Theorem 2.4 in [3], or [7, Example 5]). Thus the hypothesis that $\widehat{u}(\lambda)$ exists for all $\lambda > 0$ cannot be omitted from (ii) in Fact 2.5.

3. Results for sequences

Let $\{x_n\} := \{x_n\}_{n=0}^\infty$ be a sequence in a Banach space X. For $\gamma \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, we define the γ-th order Cesàro sum $\mathfrak{s}_n^\gamma(\{x_i\})$ as

$$(17) \qquad \mathfrak{s}_n^\gamma(\{x_i\}) := \sum_{k=0}^n \binom{n-k+\gamma-1}{n-k} x_k,$$

where $\binom{r}{0} := 1$ and $\binom{r}{n} := r(r-1)\ldots(r-n+1)/n!$ for $r \in \mathbb{R}$ and $n \geq 1$. Thus $\mathfrak{s}_0^\gamma(\{x_i\}) = x_0$ for all $\gamma \in \mathbb{R}$, $\mathfrak{s}_n^0(\{x_i\}) = x_n$ and $\mathfrak{s}_n^1(\{x_i\}) = \sum_{k=0}^n x_k$ for all $n \in \mathbb{N}_0$. The Abel sum $\widehat{\{x_i\}}(r)$ of $\{x_n\}$ is defined as

$$(18) \qquad \widehat{\{x_i\}}(r) := \sum_{n=0}^\infty r^n x_n, \qquad 0 < r < \left(\limsup_{n \to \infty} \|x_n\|^{1/n}\right)^{-1}.$$

Clearly $\widehat{\{x_i\}}(r)$ exists for all $0 < r < 1$ if and only if $\limsup_{n \to \infty} \|x_n\|^{1/n} \leq 1$. Let $\{a_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers such that $\sum_{n=0}^\infty a_n > 0$. We define $u(t) := x_{[t]}$ and $g(t) := a_{[t]}$ for $t \geq 0$, where $[t]$ denotes the largest integer less than or equal to t. Then we have the following

Lemma 3.1. (i) $x = \lim_{n \to \infty} \left(\sum_{k=0}^n x_k\right)/\left(\sum_{k=0}^n a_k\right)$ if and only if $x = \lim_{t \to \infty} \left(\int_0^t u(s)\, ds\right)/\left(\int_0^t g(s)\, ds\right)$.

(ii) *Suppose $\widehat{\{x_i\}}(r)$ and $\widehat{\{a_i\}}(r)$ exist for all $0 < r < 1$. Then*

$$x = \lim_{r \uparrow 1} \frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \lim_{r \uparrow 1} \frac{\sum_{n=0}^{\infty} r^n x_n}{\sum_{n=0}^{\infty} r^n a_n}$$

if and only if

$$x = \lim_{\lambda \downarrow 0} \frac{\widehat{u}(\lambda)}{\widehat{g}(\lambda)} = \lim_{\lambda \downarrow 0} \frac{\int_0^{\infty} e^{-\lambda t} u(t)\, dt}{\int_0^{\infty} e^{-\lambda t} g(t)\, dt}.$$

PROOF: (i) Putting $\delta(t) := t - [t]$ we have $0 \le \delta(t) < 1$, and

$$\frac{\int_0^t u(s)\, ds}{\int_0^t g(s)\, ds} = \frac{\left(1 - \delta(t)\right) \sum_{k=0}^{[t]-1} x_k + \delta(t) \sum_{k=0}^{[t]} x_k}{\left(1 - \delta(t)\right) \sum_{k=0}^{[t]-1} a_k + \delta(t) \sum_{k=0}^{[t]} a_k},$$

so that the first condition of (i) implies the second condition. The converse implication is obvious.

(ii) By an elementary calculation we have

$$\frac{\int_0^{\infty} e^{-\lambda t} u(t)\, dt}{\int_0^{\infty} e^{-\lambda t} g(t)\, dt} = \frac{\sum_{n=0}^{\infty} e^{-\lambda n} x_n}{\sum_{n=0}^{\infty} e^{-\lambda n} a_n}, \qquad \lambda > 0,$$

whence (ii) follows. $\qquad\qquad\square$

Theorem 3.2 (cf. [2, Theorem 3.2]). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} a_n > 0$. Suppose*

(D) $$\frac{\sum_{k=0}^{m} a_k}{\sum_{k=0}^{n} a_k} \to 1 \quad as \quad m, n \to \infty \quad with \quad \frac{m}{n} \to 1.$$

Then for any sequence $\{x_n\}_{n=0}^{\infty}$ in X, with $\|x_n\| = O(a_n)$, $n \to \infty$, the following statements are equivalent:

(i) $x = \lim_{n \to \infty} \mathfrak{s}_n^1(\{x_i\})/\mathfrak{s}_n^1(\{a_i\}) = \lim_{n \to \infty} \left(\sum_{k=0}^{n} x_k \right)/\left(\sum_{k=0}^{n} a_k \right)$;

(ii) $x = \lim_{n \to \infty} \mathfrak{s}_n^\beta(\{x_i\})/\mathfrak{s}_n^\beta(\{a_i\})$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{r \uparrow 1} \widehat{\{x_i\}}(r)/\widehat{\{a_i\}}(r) = \lim_{r \uparrow 1} \left(\sum_{n=0}^{\infty} r^n x_n \right)/\left(\sum_{n=0}^{\infty} r^n a_n \right)$.

PROOF: Condition (D) implies that the function $g(t) = a_{[t]}$ satisfies condition (C) with $\mu := g(t)\, dt$. Hence $\widehat{\{a_i\}}(r)$ and $\widehat{\{x_i\}}(r)$ exist for all $0 < r < 1$. Then "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" follows from [2, Theorem 3.1].

(iii) $\Rightarrow$ (i): By Lemma 3.1 and Theorem 2.2 we have

$$x = \lim_{r \uparrow 1} \frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \lim_{\lambda \downarrow 0} \frac{\widehat{u}(\lambda)}{\widehat{g}(\lambda)} = \lim_{t \to \infty} \frac{\mathfrak{s}_t^1(u)}{\mathfrak{s}_t^1(g)} = \lim_{n \to \infty} \frac{\mathfrak{s}_n^1(\{x_i\})}{\mathfrak{s}_n^1(\{a_i\})},$$

which completes the proof. $\qquad\qquad\square$

Theorem 3.3 (cf. [2, Theorem 4.4], [5, Proposition 3.6]). *Let $\alpha \geq 0$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\|x_n\| = O(n^{\alpha-1})$, $n \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \left(\Gamma(\alpha+1)/(n+1)^{\alpha}\right) \sum_{k=0}^{n} x_k$;

(ii) $x = \lim_{n\to\infty} \left(\Gamma(\alpha+\beta)/(n+1)^{\alpha+\beta-1}\right) \mathfrak{s}_n^{\beta}(\{x_i\})$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{r\uparrow 1} (1-r)^{\alpha} \widehat{\{x_i\}}(r) = \lim_{r\uparrow 1} (1-r)^{\alpha} \sum_{n=0}^{\infty} r^n x_n$.

PROOF: "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" follows from [2, Theorem 4.3].

(iii) $\Rightarrow$ (i): Suppose $\alpha > 0$. Then define $a_n := \binom{n+\alpha-1}{n}$ for $n \geq 0$. It follows (cf. [9, pp. 76–77]) that $(1-r)^{-\alpha} = \sum_{n=0}^{\infty} r^n a_n$ for $0 < r < 1$, and $a_n = n^{\alpha-1}(1 + o(1))/\Gamma(\alpha)$, $n \to \infty$. Thus $\|x_n\| = O(a_n)$, $n \to \infty$. Since

$$\sum_{k=0}^{n} a_k = \binom{n+\alpha}{n} = \frac{n^{\alpha}}{\Gamma(\alpha+1)}\left(1 + o(1)\right), \quad n \to \infty,$$

$\{a_n\}_{n=0}^{\infty}$ satisfies condition (D). Hence (i) follows from Theorem 3.2.

Next suppose $\alpha = 0$. Then the function $u(t) = x_{[t]}$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$, and (iii) implies that $x = \lim_{\lambda\downarrow 0} \int_0^{\infty} e^{-\lambda t} u(t)\, dt$. Hence, by Theorem 2.3, $x = \lim_{t\to\infty} \int_0^t u(s)\, ds = \lim_{n\to\infty} \sum_{k=0}^{n} x_k$. This completes the proof. $\square$

Remark. The special cases $\alpha = 1$ and $\alpha = 0$ of Theorem 3.3 are classical results for sequences corresponding to $\alpha = 1$ and $\alpha = 0$ of Theorem 2.3, respectively. (See e.g. [4, Theorem 3.1], [1, Theorem 4.2.17].)

Theorem 3.4. *Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\|x_n\| = O(n^{-1})$, $n \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \left(1/\log(n+1)\right) \sum_{k=0}^{n} x_k$;

(ii) $x = \lim_{n\to\infty} \left(\Gamma(\beta)/(n+1)^{\beta-1}\log(n+1)\right) \mathfrak{s}_n^{\beta}(\{x_i\})$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0} \left(1/-\log\lambda\right) \widehat{\{x_i\}}(e^{-\lambda}) = \lim_{\lambda\downarrow 0} \left(1/-\log\lambda\right) \sum_{n=0}^{\infty} e^{-\lambda n} x_n$.

PROOF: Define $a_0 := 0$ and $a_n := n^{-1}$ for $n \geq 1$. Hence $\|x_n\| = O(a_n)$, $n \to \infty$, and $\sum_{k=0}^{n} a_k = \log n + O(1)$, $n \to \infty$. It follows that $\{a_n\}_{n=0}^{\infty}$ satisfies condition (D). If $\beta > 1$, then

$$(19) \qquad \mathfrak{s}_n^{\beta}(\{a_i\}) = \sum_{k=1}^{n} \binom{n-k+\beta-1}{n-k} \frac{1}{k}.$$

Since

$$\binom{n+\beta-1}{n} = \frac{n^{\beta-1}}{\Gamma(\beta)}\left(1 + o(1)\right), \quad n \to \infty,$$

it follows by an approximation argument that

$$
\begin{aligned}
\mathfrak{s}_n^\beta(\{a_i\}) &= \int_1^n \frac{(n-s)^{\beta-1}}{\Gamma(\beta)} s^{-1}\, ds \cdot \big(1 + o(1)\big) \\
&= \frac{n^{\beta-1}\log n}{\Gamma(\beta)} \cdot \big(1 + o(1)\big), \quad n \to \infty \qquad \text{(by (15))}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\widehat{\{a_i\}}(e^{-\lambda}) &= \sum_{n=1}^\infty e^{-\lambda n} n^{-1} = \int_1^\infty e^{-\lambda t} t^{-1}\, dt \cdot \big(1 + o(1)\big) \\
&= -\log\lambda \cdot \big(1 + o(1)\big), \quad \lambda \downarrow 0 \qquad \text{(by (16))}.
\end{aligned}
$$

Hence the desired result follows from Theorem 3.2. $\qquad\qquad\square$

Remark. If X is a Banach lattice and $\{x_n\}_{n=0}^\infty \subset X_+$, then statements (i), (ii) and (iii) in Theorem 3.4 are also equivalent. This follows from [2, Theorem 3.2]. (We note that statement (ii) in Theorem 3.4 implies that $\widehat{\{x_i\}}(r)$ exists for all $0 < r < 1$.)

Fact 3.5. *Let $\{x_n\}_{n=0}^\infty$ be a sequence in X. Consider the following three statements:*

(i) $x = \lim_{n\to\infty} \sum_{k=0}^n x_k$;

(ii) $x = \lim_{n\to\infty} \big(\Gamma(\beta)/(n+1)^{\beta-1}\big)\, \mathfrak{s}_n^\beta(\{x_i\})$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{r\uparrow 1} \widehat{\{x_i\}}(r) = \lim_{r\uparrow 1} \sum_{n=0}^\infty r^n x_n$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF: By letting $a_0 := 1$ and $a_n := 0$ for $n \geq 1$, the desired result follows as in Fact 2.5. We may omit the details. $\qquad\qquad\square$

Remark. In general (iii) does not imply (i) in Fact 3.5. (For example let $x_n := (-1)^n$.) If $\{x_n\}$ satisfies $\|x_n\| = O(n^{-1})$, $n \to \infty$, or if X is a Banach lattice and $\{x_n\} \subset X_+$, then (iii) implies (i). (See Theorem 3.3 and [2, Theorem 4.4], respectively.)

4. A counterexample

The following example shows that condition (D) is essential in Theorem 3.2. (See also Example 3 in [6].)

Example. Define $\{a_n\}_{n=0}^\infty$ by

$$
a_n := \begin{cases} n & \text{if } n \in \{2^k,\, 2^k+1\} \text{ for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}
$$

Thus $\{a_n\}$ does not satisfy condition (D). Next define $\{x_n\}_{n=0}^{\infty}$ by

$$x_n := \begin{cases} n & \text{if } n = 2^k \text{ for some } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $x_n = O(a_n)$, $n \to \infty$. An elementary calculation yields

$$\frac{1}{2} = \liminf_{n \to \infty} \frac{\sum_{k=0}^n x_k}{\sum_{k=0}^n a_k} < \limsup_{n \to \infty} \frac{\sum_{k=0}^n x_k}{\sum_{k=0}^n a_k} = \frac{2}{3},$$

so that $\lim_{n \to \infty} \mathfrak{s}_n^1(\{x_i\})/\mathfrak{s}_n^1(\{a_i\})$ does not exist. Nevertheless we have

$$\frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \frac{\sum_{n=1}^{\infty} 2^n r^{2^n}}{(1+r)\sum_{n=1}^{\infty} 2^n r^{2^n} + r \sum_{n=1}^{\infty} r^{2^n}} \to \frac{1}{2} \quad \text{as } r \uparrow 1.$$

Remark. Let $0 \neq g \in L_{\mathrm{loc}}^1(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\widehat{g}(\lambda)$ exists for all $\lambda > 0$ and that $x = \lim_{\lambda \downarrow 0} \widehat{u}(\lambda)/\widehat{g}(\lambda)$ implies $x = \lim_{t \to \infty} \left(\int_0^t u(s)\,ds\right)/\left(\int_0^t g(s)\,ds\right)$ for all $u \in L_{\mathrm{loc}}^1(\mathbb{R}_+, X)$ with $\|u(t)\| = O(g(t))$, $t \to \infty$. Then in view of Theorem 2.2 it would be natural to ask the following question: Does the measure $\mu := g(t)\,dt$ satisfy condition (C) of Lemma 2.1? The author could not solve this problem.

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