# On AP spaces in concern with compact-like sets and submaximality

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*Abstract.* The definitions of AP and WAP were originated in categorical topology by A. Pultr and A. Tozzi, *Equationally closed subframes and representation of quotient spaces*, Cahiers Topologie Géom. Différentielle Catég. **34** (1993), no. 3, 167–183. In general, we have the implications: $T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$, where $KC$ is defined as the property that every compact subset is closed and $US$ is defined as the property that every convergent sequence has at most one limit. And a space is called *submaximal* if every dense subset is open.

In this paper, we prove that: (1) every AP $T_1$-space is US, (2) every nodec WAP $T_1$-space is submaximal, (3) every submaximal and collectionwise Hausdorff space is AP. We obtain that, as corollaries, (1) every countably compact (or compact or sequentially compact) AP $T_1$-space is Fréchet-Urysohn and US, which is a generalization of Hong's result in *On spaces in which compact-like sets are closed, and related spaces*, Commun. Korean Math. Soc. **22** (2007), no. 2, 297–303, (2) if a space is nodec and $T_3$, then submaximality, AP and WAP are equivalent. Finally, we prove, by giving several counterexamples, that (1) in the statement that every submaximal $T_3$-space is AP, the condition $T_3$ is necessary and (2) there is no implication between nodec and WAP.

*Keywords:* AP, WAP, door, submaximal, nodec, unique sequential limit

*Classification:* 54D10, 54D55

## 1. Introduction

The purpose of this paper is to introduce some systemization into the discussion, to point out its importance, and to show some surprising contact with concepts of AP and submaximality which have been studied by several authors (see [2], [3]).

The spaces determined by almost closed subspaces were first introduced by G.T. Whyburn [22] who baptized them accessibility spaces and studied the properties of pseudo-open continuous functions onto accessible spaces. Twenty years later this concept appeared in the paper of A. Pultr and A. Tozzi [19] in the context of categorical topology. Concepts of Whyburn and weakly Whyburn spaces appeared and disappeared repeatedly, under various names. Then they became subjects of an intensive study in context of pseudoradial and related spaces.

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When A. Bella [4] and P. Simon [20] studied topological properties of AP spaces not being aware of the paper of Whyburn, they used the terminology of [19]. The situation changed recently when A.V. Arhangel'skii communicated to the specialists in the field, that the concept of an AP space was first introduced by G.T. Whyburn. As a traditional measure, some authors use the old terms, but we will use the terminology defined by A. Pultr and A. Tozzi, and P. Simon.

Notice that the AP space is a natural generalization of Fréchet-Urysohn and the WAP space contains all sequential spaces.

A. Bella and I.V. Yaschenko [6] discovered that every compact AP space is Fréchet-Urysohn. After a few years, V.V. Tkachuk and I.V. Yaschenko [21] gave a more general result of this, that is, any countably compact AP space is Fréchet-Urysohn.

W. Hong [15] defined the space having the property of *Approximation by Countable Points*, for short, *ACP*. He also defined a WACP space as a generalization of a ACP space. It has shown that WACP implies WAP. He proved that every WACP space has countable tightness.

Section 2 is devoted to spaces in which compact-like sets are closed, and related spaces. It is well known that every compact subset of a $T_2$-space is always closed. However, we may not say that every compact (countably compact, sequentially compact) subset of any space is closed. A topological space $X$ is said to be *KC* (resp. *C-closed, SC-closed*) if every compact (resp. countably compact, sequentially compact) subset of $X$ is closed. A space $X$ has *unique sequential limits*, for short, *US*, if every sequence of points of $X$ may converge to at most one limit. It follows from definitions that every sequentially compact space is countably compact and that every C-closed space is SC-closed. Also we have that every C-closed space is KC.

One can easily prove by definitions that $T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$. A. Wilansky first studied the relationships of the above four properties in [23]. More specially, he proved that no converse implication holds even if the space is compact. W. Hong [16] showed that every C-closed space as well as every SC-closed space is US. Also it was shown that the following properties are equivalent when a space is sequential: (1) US; (2) KC; (3) C-closed; and (4) SC-closed.

It is known that a compact $T_1$-space need not be US ([23]), but we will show that every AP $T_1$-space is US in Section 2. It makes us to improve Corollary 2.15 in [16] by dropping the unnecessary condition "weakly discretely generated" as follows: every countably compact (or compact or sequentially compact) AP $T_1$-space is Fréchet-Urysohn.

Section 3 deals with AP spaces and submaximal spaces. It was proved in [10] that every door space is submaximal. It is well known that every submaximal space is nodec. Giving additional conditions, it was shown that every irreducible submaximal space is a door space ([10]) and that every submaximal $T_3$-space is AP ([6]).

We construct some counterexamples related to the digital line or the product of two real lines equipped with a suitable topology such that submaximal spaces which are not door (Example 3.11, 3.12 and 3.13). As a main theorem in Section 3, we prove that every nodec WAP $T_1$-space is submaximal. This guarantees the following properties are equivalent when a space is nodec and $T_3$: (1) submaximal; (2) AP; and (3) WAP. We also prove that if $X$ is submaximal and collectionwise Hausdorff, then $X$ is AP.

## 2. On compact-like sets

All spaces are assumed to be topological spaces, and our terminologies are standard and follow [3] and [13].

The following definitions of AP and WAP were originated in categorical topology by A. Pultr and A. Tozzi [19]. P. Simon [20] was first to study these properties from a general topological point of view.

**Definition 2.1** ([19]). A space $X$ is said to have the property of *Approximation by Points* (*Weak Approximation by Points*), for short, *AP* (*WAP*), if for every non-closed set $A$ and every (some) point $x \in \overline{A} \backslash A$ there is a subset $B \subset A$ such that $\overline{B} \backslash A = \{x\}$.

Such a set $B$ is also called *almost closed*, and denoted by $B \to x$.

Clearly any AP space is WAP but the converse is not true.

We say that a subset $A$ of a space $X$ is *AP-closed* if for every $F \subset A$ the relation $|\overline{F} \backslash A| \neq 1$ holds.

The following property is well known.

**Proposition 2.2.** *$X$ is a WAP space if and only if every AP-closed subset of $X$ is closed.*

**Definition 2.3** ([15]). A space $X$ is said to have the property of *Approximation by Countable Points*, for short, *ACP*, provided that for every non-closed set $A$ and every point $x \in \overline{A} \backslash A$ there is a countable subset $B \subset A$ such that $\overline{B} \backslash A = \{x\}$.

A topological space $X$ is said to be *KC* if every compact (not necessarily $T_2$) subset of $X$ is closed. Of course, each $T_2$-space is KC. On the other hand, if a space is KC then clearly its singletons are closed, i.e., the space is $T_1$. Under this point of view, the KC property may be envisaged as a kind of separation axiom between $T_1$ and $T_2$ ([5]).

A topological space $X$ is *C-closed* ([17]) (*SC-closed* ([16])) if every countably compact (sequentially compact) subset of $X$ is closed. A space $X$ has *unique sequential limits* ([14]), for short, *US*, if every sequence of points of $X$ may converge to at most one limit.

Since a sequentially compact space is countably compact, every C-closed space is SC-closed. Also since a compact space is countably compact, every C-closed space is KC.

**Theorem 2.4** ([23, Theorem 1]). *$T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$; but no converse implication holds even if the space is compact.*

A compact $T_1$-space need not be US. The simple counterexample is a countably infinite set equipped with the cofinite topology, but we show the following.

**Theorem 2.5.** *If $X$ is an AP $T_1$-space, then $X$ is US.*

Proof: We will show this by the way of contradiction. Suppose that $X$ is an AP $T_1$-space and suppose that there exists a sequence $(x_n : n \in \omega)$ which converges to two distinct points $a$ and $b$ in $X$. Let $I_a = \{n \in \omega : x_n = a\}$ and $I_b = \{n \in \omega : x_n = b\}$. If $I_a$ is infinite, then pick a constant subsequence $(x_{n_k} = a : n_k \in I_a)$ of the sequence $(x_n : n \in \omega)$. Since $X$ is $T_1$, there exists an open neighborhood $U$ of $b$ such that $a \notin U$. Then $U \cap \{x_{n_k} : n_k \in I_a\} \neq \emptyset$ since $b$ is a limit of the subsequence $\{x_{n_k} : n_k \in I_a\}$. This is impossible. Hence $I_a$ is finite. Similarly we have that $I_b$ is finite. Take a set $A = \{x_n : n \in \omega \backslash (I_a \cup I_b)\}$. Then $a \in \overline{A} \backslash A$. Since $X$ is AP, there exists a subset $F$ of $A$ such that $\overline{F} = F \cup \{a\}$. Because $F$ is infinite, $b \in \overline{F} = F \cup \{a\}$, i.e., $b \in F \subset A$. This is a contradiction. Therefore, $X$ is US. □

A space $X$ is called *weakly discretely generated* ([12]) if for each non-closed subset $A$ of $X$ there exist $x \in \overline{A} \backslash A$ and a subset $D$ of $A$ such that $D$ is discrete and $x \in \overline{D}$. Note that $X$ is weakly discretely generated if it is a sequential $T_1$-space or a compact $T_2$-space.

W.C. Hong proved the following two theorems:

**Theorem 2.6** ([16, Theorem 2.11]). *Every weakly discretely generated AP $T_1$-space is C-closed.*

**Theorem 2.7** ([16, Corollary 2.15]). *Every countably compact (or compact or sequentially compact) weakly discretely generated AP $T_1$-space is Fréchet-Urysohn and US.*

From Theorem 2.6, it is natural to ask whether every weakly discretely generated WAP $T_1$-space is C-closed. But the answer is negative. Note that $\omega_1 + 1$ is a discretely generated WAP $T_1$-space which is not C-closed, and $\omega_1$ is a countably compact subset which is not closed.

By Theorem 2.5, the condition "weakly discretely generated" in Theorem 2.7 can be dropped since every sequential (or countably compact) AP space is Fréchet-Urysohn.

**Corollary 2.8.** *Every countably compact (or compact or sequentially compact) AP $T_1$-space is Fréchet-Urysohn and US.*

## 3. Around AP and submaximality

A space $X$ is called a *door space* ([18]) if every subset of $X$ is open or closed. The term "door" was introduced by Kelley [18, p. 76, Problems C]. Here are some easy facts about door spaces.

**Proposition 3.1.** (1) *The discrete space is a door space.*
(2) *A $T_2$ door space has at most one accumulation point* ([18]).
(3) *In a $T_2$ door space if $x$ is not an accumulation point, then $\{x\}$ is open* ([18]).
(4) *Every subspace of a door space is a door space* ([10, Theorem 2.6]).

A space $X$ is called *submaximal* ([8]) if every dense subset of $X$ is open or, equivalently, every subset with empty interior is closed and discrete. It is clear that every submaximal space is a $T_0$-space.

**Theorem 3.2** ([7, Theorem 3.1]). *Let $X$ be a topological space. Then the following statements are equivalent:*

(1) *$X$ is submaximal;*
(2) *$\overline{A}\backslash A$ is closed, for each $A \subset X$;*
(3) *$\overline{A}\backslash A$ is closed and discrete, for each $A \subset X$.*

A non-empty space $X$ is said to be *irreducible* if it satisfies the following equivalent conditions:

(1) Every two non-empty open subsets of $X$ intersect.
(2) $X$ is not the union of a finite family of closed proper subsets.
(3) Every non-empty open subset of $X$ is dense.
(4) Every open subset of $X$ is connected.

An irreducible space is called sometimes *hyperconnected* (in fact quite often).

**Theorem 3.3** ([10, Theorem 2.7]). *Every door space $X$ is submaximal.*

In general, the converse of Theorem 3.3 is not true ([1, Example 2.8 and 2.9]).

**Theorem 3.4** ([10, Theorem 2.8]). *Every irreducible submaximal space $X$ is a door space.*

A space $X$ is *nodec* ([11]) if every nowhere dense subsets of $X$ is closed.

One can easily show that every submaximal space is nodec. But the converse is not true. The following example is a nodec space which is not submaximal.

**Example 3.5.** Every cofinite topology on an infinite set $X$ is a nodec space which is not submaximal.

Suppose $A$ is infinite in the cofinite topology on $X$. Then $A$ is dense (every non-empty open set misses only finitely many elements of $X$) and so $\operatorname{Int}\overline{A} = \operatorname{Int} X = X$, and $A$ is not nowhere dense. So every nowhere dense subset of $X$ must be finite and thus closed. Hence $X$ is nodec. However, it is not submaximal because every infinite set in $X$ is dense (as before), but only cofinite sets are open. □

**Remark 3.1.** By Example 3.5, every infinite set $X$ with the cofinite topology is a nodec space which is neither submaximal nor WAP.

**Proposition 3.6** ([2, Proposition 2.1])**.** *Every subspace of a submaximal* (*nodec*) *space is a submaximal* (*nodec*) *space.*

A space $X$ is called *ACP*, if for every non-closed subset $A$ of $X$ and each $x \in \overline{A}\backslash A$ there exists a countable subset $B$ of $A$ such that $\overline{B}\backslash B = \{x\}$.

**Proposition 3.7** ([6, Proposition 1.3])**.** *Every submaximal $T_3$-space is AP.*

The following basic diagram exhibits the general relationships among the properties mentioned above:

$$\begin{tikzcd}
\text{door} \arrow[r, shift left] & \text{submaximal} \arrow[l, shift left, "\text{irreducible}"] \arrow[r] \arrow[d, "\substack{T_3 \\ \text{collectionwise}}"', "\substack{\text{or} \\ \text{Hausdorff}}"] & \text{nodec} \\
\text{ACP} \arrow[r] & \text{AP} \arrow[r] & \text{WAP} \arrow[ul, "\text{nodec } T_1"']
\end{tikzcd}$$

**Theorem 3.8.** *Every nodec WAP $T_1$-space is submaximal.*

Proof: Suppose $X$ is not submaximal. Then there exists a non-closed $A \subset X$ such that $\operatorname{Int} A = \emptyset$. Since $X$ is WAP, there exist $x \in \overline{A}\backslash A$ and $F \subset A$ such that $\overline{F} = F \cup \{x\}$. Since $F$ is not closed and since $X$ is nodec, $\operatorname{Int} \overline{F} \neq \emptyset$. Since $\operatorname{Int} F \subset \operatorname{Int} A$ and $\operatorname{Int} A = \emptyset$, $\operatorname{Int} F = \emptyset$. So $x \in \operatorname{Int} \overline{F}$. Since $x$ is an accumulation point of $F$, $\emptyset \neq F \cap (\operatorname{Int} \overline{F}\backslash\{x\}) = \operatorname{Int} \overline{F}\backslash\{x\} \subset F$. Since $X$ is $T_1$, $\operatorname{Int} \overline{F}\backslash\{x\}$ is non-empty open. So $\operatorname{Int} F \neq \emptyset$. This is a contradiction. □

**Corollary 3.9.** *Let $X$ be a nodec $T_3$-space. Then the following are equivalent:*

(1) *$X$ is submaximal;*
(2) *$X$ is AP;*
(3) *$X$ is WAP.*

The following is a submaximal space which is not WAP.

**Example 3.10.** We topologize the set of integers $\mathbb{Z}$ with a base $\{\{2m-1, 2m, 2m+1\}, \{2m+1\} : m \in \mathbb{Z}\}$.

The space $X$ is submaximal but neither door nor $T_1$ ([1, Example 2.8]). Also $X$ is not WAP. For, let $A = \{1\}$. Then $\overline{A} = \{0, 1, 2\}$. If $\emptyset \neq F \subset A$, then $F = \{1\}$ and $\overline{F} = \{0, 1, 2\}$, i.e., $F$ is not almost closed. □

Recall that every door space is submaximal. The following example gives us an information that the converse does not hold though a space $X$ is submaximal and AP (more strongly, ACP).

**Example 3.11.** We topologize the set of integers $\mathbb{Z}$ with a base $\mathcal{B} = \{\{2m, 2m+1\}, \{2m+1\} : m \in \mathbb{Z}\}$. Then the space $X$ is submaximal and AP, but neither door nor $T_1$.

Claim 1: *$X$ is not door.*

Let $A = \{1, 2\}$. Then $A$ is not open and $A$ is not closed because $\operatorname{Int} A = \{1\}$ and $\overline{A} = \{0, 1, 2\}$.

Claim 2: *$X$ is submaximal.*

Let $X = \mathbb{Z}_o \cup \mathbb{Z}_e$ where $\mathbb{Z}_o = \{2n+1 : n \in \mathbb{Z}\}$ and $\mathbb{Z}_e = \{2n : n \in \mathbb{Z}\}$. For every $n \in \mathbb{Z}_o$, $\{n\}$ is open, and $\mathbb{Z}_e$ is closed and discrete. Let $A$ be any subset of $X$ with $\operatorname{Int} A = \emptyset$. Then $A \subset \mathbb{Z}_e$. Since every subset of a closed and discrete subset is closed, $A$ is closed. Thus $X$ is submaximal.

Claim 3: *$X$ is ACP.*

Let $\overline{A}\backslash A \neq \emptyset$ and $n \in \overline{A}\backslash A$. Then $n \in \mathbb{Z}_e$ $(n = 2m)$. Since $2m \in \{2m, 2m+1\} \in \mathcal{B}$, $\{2m, 2m+1\} \cap A \neq \emptyset$. So $2m+1 \in A$. Take $F = \{2m+1\} \subset A$. Then $\overline{F} = \{2m, 2m+1\} = F \cup \{2m\}$. Since the whole space $X$ is countable, $X$ is ACP. □

Note that every submaximal $T_3$-space is AP (Theorem 3.7). The following example explains that the condition $T_3$ is necessary in the statement.

**Example 3.12.** Let $X = \mathbb{R} \times \{0, 1\}$ be a set. We define a basic open set for $x \in X$ as follows:

(i) every point of $\mathbb{R} \times \{0\}$ is isolated;

(ii) $\langle x, 1\rangle \in \mathbb{R} \times \{1\}$ has a local basis consisting of the form

$$U_K(x) = (U\backslash K) \times \{0\} \cup \{\langle x, 1\rangle\}$$

where $U$ is an open subset of the Euclidean space $\mathbb{R}$ such that $x \in U$ and $K$ is a countable subset of $\mathbb{R}$.

Then $X$ is $T_2$ but not $T_3$.

Claim 1: *$X$ is submaximal.*

If $\operatorname{Int} A = \emptyset$, then $A \subset \mathbb{R} \times \{1\}$. Since $\mathbb{R} \times \{1\}$ is closed and discrete in $X$, $A$ is closed and discrete in $X$. Hence $X$ is submaximal.

Claim 2: *$X$ is not door.*

Let $A = (\mathbb{R}^+ \times \{0\}) \cup (\mathbb{R}^- \times \{1\})$. Then $\operatorname{Int} A = \mathbb{R}^+ \times \{0\} \neq A$ and $\overline{A} = A \cup (\mathbb{R}^+ \cup \{0\}) \times \{1\} \neq A$. Hence $A$ is neither open nor closed. Thus $X$ is not door.

Claim 3: *$X$ is not WAP.*

Suppose $X$ is WAP. Let $A = C \times \{0\}$ where $C$ is the Cantor set on $[0, 1]$. Then $\overline{A} = C \times \{0, 1\}$. $(\overline{A}\backslash A = C \times \{1\}.)$ Since $X$ is WAP, there exist $p \in \overline{A}\backslash A$ and $F \subset A$ such that $\overline{F} = F \cup \{p\}$. Without loss of generality, we may assume that $p = \langle 0, 1\rangle$. Since $p$ is an accumulation point of $F$, for any basic open neighborhood $U_K(0)$ of $p$, $U_K(0) \cap F$ is uncountable. Let $V_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$ and let $B_n = V_n \backslash V_{n+1}$ for each $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $(B_m \times \{0\}) \cap F$ $(= G)$ is uncountable. (If not, $[(\bigcup_{m\in\mathbb{N}} B_m) \times \{0\}] \cap F = [(-1, 1) \times \{0\}] \cap F$ is countable. This is impossible.)

Since $G$ is a subset of $C \times \{0\}$, we can take a sequence $\{C_n \subset [0,1] : n \in \mathbb{N}\}$ such that

- each $C_n$ is a closed interval which can be chosen in each stage of construction of the Cantor set $C$;
- $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$;
- $(C_n \times \{0\}) \cap G$ is uncountable for each $n \in \mathbb{N}$.

Since the $C_n$'s are closed subsets of the compact space $C$ with finite intersection property, we can choose $y \in \bigcap_{n\in\mathbb{N}} C_n$. Since $(U \times \{0\}) \cap G$ is uncountable for any open neighborhood $U$ of $y$ in the subspace $C$ of the Euclidean space $\mathbb{R}$, $\langle y, 1\rangle \in \overline{G} \subset \overline{F}$, $\langle y, 1\rangle \neq \langle 0, 1\rangle$ and $\langle y, 1\rangle \notin F$. This is a contradiction to $\overline{F} = F \cup \{\langle 0, 1\rangle\}$. Thus $X$ is not WAP. □

We now give an example of a submaximal AP $T_2$-space which is neither $T_3$ nor door. It is obtained by replacing the word "countable subset" $K$ with "finite subset" $K$ of $\mathbb{R}$ in Example 3.12.

**Example 3.13.** Let $X = \mathbb{R} \times \{0, 1\}$ be a set. We define a basic open set $B(x)$ for $x \in X$ as follows:

(i) every point of $\mathbb{R} \times \{0\}$ is isolated;
(ii) $\langle x, 1\rangle \in \mathbb{R} \times \{1\}$ has a local basis consisting of the form

$$U_K(x) = (U \backslash K) \times \{0\} \cup \{\langle x, 1\rangle\},$$

where $U$ is an open subset of the Euclidean space $\mathbb{R}$ such that $x \in U$ and $K$ is a finite subset of $\mathbb{R}$.

Then $X$ is $T_2$ but not $T_3$. One can show that $X$ is submaximal but not door by the same argument of Example 3.12.

Claim: *$X$ is ACP.*

Let $p \in \overline{A} \backslash A$. Then $p = \langle x, 1\rangle \in \mathbb{R} \times \{1\}$. For each $U_n = (x - \frac{1}{n}, x + \frac{1}{n})$, there exists $\langle x_n, 0\rangle \in (U_n \times \{0\}) \cap A$. Then $x_n \to x$ (in the usual topology). Let $p_n = \langle x_n, 0\rangle$ and let $F = \{p_n : n \in \mathbb{N}\}$. Then $F$ is a countable subset of $A$ such that $\overline{F} = F \cup \{p\}$. Therefore $X$ is ACP. □

A space $X$ is said to be *collectionwise Hausdorff* provided that for each closed and discrete subset $A$ of $X$ the points in $A$ can be separated by pairwise disjoint open subsets of $X$. It follows from the definition that every collectionwise Hausdorff space is Hausdorff.

**Theorem 3.14.** *If $X$ is submaximal and collectionwise Hausdorff, then $X$ is AP.*

Proof: Let $p \in \overline{A} \backslash A$. Since $\mathrm{Int}(\overline{A} \backslash A) = \emptyset$, $\overline{A} \backslash A$ is closed and discrete in $X$. Take a family $\{V_x : x \in \overline{A} \backslash A\}$ of pairwise disjoint open subsets of $X$ such that $x \in V_x$ for each $x \in \overline{A} \backslash A$. Let $F = \overline{V_p} \cap A$. Then $p \in \overline{F}$ because $U \cap F = U \cap \overline{V_p} \cap A \supset (U \cap V_p) \cap A \neq \emptyset$ for every open neighborhood $U$ of $p$.

Since $V_x \cap V_p = \emptyset$ for all $x \in \overline{A} \backslash A$ with $x \neq p$, $V_x \cap \overline{V_p} = \emptyset$. Hence $V_x \cap F = \emptyset$, i.e., $x \notin \overline{F}$ for all $x \in \overline{A} \backslash A$ with $x \neq p$.

To prove that $F$ is almost closed, it is sufficient to show that $\overline{F} \cap A = F$. Since $\overline{V_p}$ is closed in $X$, $F$ is closed in $A$. Hence $\overline{F} \cap A = \overline{F}^A = F$.

Therefore $X$ is AP. $\square$

**Remark 3.2.** The space $X$ in Example 3.12 is a submaximal $T_2$-space which is neither AP nor collectionwise Hausdorff.

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