Comment.Math.Univ.Carolin. 52,2 (2011) 303–307

Coronas of ultrametric spaces

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Abstract. We show that, under CH, the corona of a countable ultrametric space is homeomorphic to ω^*. As a corollary, we get the same statements for the Higson's corona of a proper ultrametric space and the space of ends of a countable locally finite group.

Keywords: Stone-Čech compactification, ultrametric space, corona, Higson's corona, space of ends

Classification: 54D35, 54D80, 54G05

Let (X,ρ) be a metric space, $x_0 \in X$, X_d be a set X endowed with the discrete topology, βX_d be the Stone-Čech compactification of X_d. We identify βX_d with the set of all ultrafilters on X and denote by $X^\#$ the set of all ultrafilters on X whose members are unbounded subsets of X. A subset A is bounded if there exists $n \in \omega$ such that $A \subseteq B(x_0, n)$ where $B(x_0, n) = \{x \in X : \rho(x_0, x) \leq n\}$. In what follows, all metric spaces are supposed to be unbounded, so $X^\# \neq \emptyset$. Clearly, $X^\#$ is closed in βX_d.

Given any $r, q \in X^\#$, we say that r, q are *parallel* (and write $r \parallel q$) if there exists $n \in \omega$ such that, for every $R \in r$, we have $B(R,n) \in q$ where $B(R,n) = \bigcup_{x \in R} B(x,n)$. By [5, Lemma 4.1], $\parallel$ is an equivalence on $X^\#$. We denote by $\sim$ the smallest (by inclusion) closed (in $X^\# \times X^\#$) equivalence on $X^\#$ such that $\parallel \subseteq \sim$. By [2, Theorem 3.2.11], the quotient $X^\#/\sim$ is a compact Hausdorff space. It is called the *corona* of X and is denoted by $\check{X}$. To clarify the virtual equivalence $\sim$, we use the slowly oscillating functions.

A function $h : (X,\rho) \to [0,1]$ is called *slowly oscillating* if, for any $n \in \omega$ and $\varepsilon > 0$, there exists a bounded subset V of X such that, for every $x \in X \setminus V$,

$$\operatorname{diam} h(B(x,n)) < \varepsilon,$$

where $\operatorname{diam} A = \sup\{|x-y| : x, y \in A\}$.

By [6, Proposition 1], $p \sim q$ if and only if $h^\beta(p) = h^\beta(q)$ for every slowly oscillating function $h : (X,\rho) \to [0,1]$, where h^β is an extension of h to βX_d. If X is ultrametric we may use only the slowly oscillating functions taking values $0, 1$ [5, Lemma 4.3]. Recall that (X,ρ) is *ultrametric* if $\rho(x,y) \leq \max\{\rho(x,z), \rho(y,z)\}$ for all $x, y, z \in X$.

A metric space X is called *proper* if every ball $B(X,n)$ is compact. In this case $\check{X}$ is homeomorphic to the Higson's corona νX of X (see [1, §6] and [6, p. 154]).

Let (X_1,ρ_1), (X_2,ρ_2) be metric spaces. A bijection $f : X_1 \to X_2$ is called an *asymorphism* if, for any $n \in \omega$, there exists $m \in \omega$ such that, for all $x_1, x_2 \in X_1$ and $y_1, y_2 \in X_2$,

$$\rho_1(x_1,x_2) \le n \Rightarrow \rho_2(f(x_1), f(x_2)) \le m,$$
$$\rho_2(y_1,y_2) \le n \Rightarrow \rho_1(f^{-1}(y_1), f^{-1}(y_2)) \le m.$$

A subset Y of a metric space (X,ρ) is called *large* if there exists $n \in \omega$ such that $B(Y,n) = X$. The metric spaces (X_1,ρ_1), (X_2,ρ_2) are *coarsely equivalent* if there exist large subsets Y_1, Y_2 of X_1, X_2 such that (Y_1,ρ_1), (Y_2,ρ_2) are asymorphic. We show that, in this case, $\check{X}_1$ and $\check{X}_2$ are homeomorphic. Let $f : Y_1 \to Y_2$ be an asymorphism. Since f and f^{-1} are $\prec$-mappings, applying [6, Proposition 1], we conclude that, for any $p,q \in Y_1^{\#}$, $p \sim q$ if and only if $f^{\beta}(p) \sim f^{\beta}(q)$. Then the mapping $\check{f} : \check{Y}_1 \to \check{Y}_2$ defined by $\check{f}([x]) = [f^{\beta}(x)]$, where $[x]$ and $[f^{\beta}(x)]$ are equivalence classes containing x and $f^{\beta}(x)$, is a homeomorphism. To see that $\check{X}_i$ and $\check{Y}_i$, $i \in \{1,2\}$ are homeomorphic, we pick $m_i \in \omega$ such that $X_i = B(Y_i, m_i)$ and, for each $x \in X_i$, choose $h_i(x) \in Y_i$ such that $\rho_i(x, h_i(x)) \le m_i$. Thus the mapping $\check{h}_i : \check{X}_i \to \check{Y}_i$ is a homeomorphism.

Theorem 1. *For a metric space X, the following statements hold:*

(i) *every non-empty open subset of $\check{X}$ contains a copy of ω^*;*
(ii) *every non-empty G_δ-subset of $\check{X}$ has non-empty interior;*
(iii) *if X is ultrametric then $\check{X}$ is zero-dimensional F-space;*
(iv) *if X is countable then $\check{X}$ is of weight $\mathfrak{c}$.*

Proof: We need some notations. Let x_0 be a fixed point of X. Given an unbounded subset P of X and a function $f : \omega \to \omega$, we put

$$\Psi_{P,f} = \bigcup_{i\in\omega} B(P \setminus B(x_0, f(i)), i),$$
$$\overline{\Psi}_{P,f} = \{q \in X^{\#} : \Psi_{P,f} \in q\},$$
$$\check{\Psi}_{P,f} = \{\check{q} \in \check{X} : q \in \overline{\Psi}_{P,f}\},$$

where $\check{q} = \{r \in X^{\#} : r \sim q\}$. By [5, Theorem 2.1], for every $p \in X^{\#}$,

$$\check{p} = \bigcap \{\overline{\Psi}_{P,f} : P \in p, f : \omega \to \omega\},$$

and the family $\Psi_p = \{\check{\Psi}_{P,f} : P \in p, f : \omega \to \omega\}$ is a base of neighbourhoods of $\check{p}$ in $\check{X}$.

(i) Let P be an unbounded subset of X. We choose an injective sequence $(t_n)_{n\in\omega}$ in P such that, for each $n \in \omega$,

$$B(\{t_1, \dots, t_n\}, n) \cap B(t_{n+1}, n+1) = \emptyset,$$

and put $T = \{t_n : n \in \omega\}$. Let q, r be distinct ultrafilters from T^*, $Q \in q$, $R \in r$ and $Q \cap R = \emptyset$. By the choice of T, for each $n \in \omega$, $B(Q,n) \cap B(R,n)$ is bounded, so we can choose $h : \omega \to \omega$ and $h' : \omega \to \omega$ such that $\Psi_{Q,h} \cap \Psi_{R,h'} = \emptyset$. Hence, the mapping from T^* to $\check{X}$, defined by $p \mapsto \check{p}$ is injective. It follows that, for every $f : \omega \to \omega$, $\check{\Psi}_{P,f}$ contains a copy of ω^*.

(ii) Let $p \in X^{\#}$, $\{P_n : n \in \omega\}$ be a decreasing family of members of p, $\{f_n : n \in \omega\}$ be a family of functions $f_n : \omega \to \omega$. It suffices to show that $\bigcap_{n\in\omega} \check{\Psi}_{P_n,f_n}$ has non-empty interior. We choose a sequence $(a_n)_{n\in\omega}$ in X such that $a_n \in P_n \setminus B(x_0, n)$, where x_0 is taken from definition of $\Psi_{P,f}$, put $A = \{a_n : n \in \omega\}$, define a function $f : \omega \to \omega$ by

$$f(i) = \max\{i, f_0(i), \dots, f_i(i)\},$$

and note that

$$A \setminus B(x_0, f(i)) \subseteq P_n \setminus B(x_0, f_n(i))$$

for all $i \geq n$. Since the subset

$$\Psi_{A,f} \setminus \bigcap_{i\geq n} B(A \setminus B(x_0, f(i)), i)$$

is bounded, we get $\check{\Psi}_{A,f} \subseteq \check{\Psi}_{P_n,f_n}$.

(iii) To show that $\check{X}$ is zero-dimensional, we fix $p \in X^{\#}$, $P \in p$ and $f : \omega \to \omega$. By the definition, $\Psi_{P,f}$ is closed. We put $\Phi = \Psi_{P,f}$. Since X is ultrametric, $B(B(x,n), i) = B(x,n)$ for all $x \in X$ and $n \geq i$. It follows that $B(\Phi, i) \setminus \Phi$ is bounded for each $i \in \omega$. Therefore we can define a function $h : \omega \to \omega$ such that $\Psi_{\Phi,h} \subseteq \Phi$, so $\check{\Psi}_{\Phi,h} \subseteq \check{\Psi}_{P,f}$ and $\check{\Psi}_{P,f}$ is open.

To prove that $\check{X}$ is an F-space, in view of [4, Lemma 1.2.2(b)], it suffices to verify that any two disjoint open F_δ subsets Y, Z of $\check{X}$ have disjoint closures. We may suppose that

$$Y = \bigcup_{n\in\omega} \check{\Psi}_{Y_n,f_n}, \quad Z = \bigcup_{n\in\omega} \check{\Psi}_{Z_n,f_n}.$$

Since $\check{\Psi}_{Y_n,f_n} \cap \check{\Psi}_{Z_m,f_m}$ is bounded for all $m, n \in \omega$, we can choose inductively the sequences of functions $(f'_n)_{n\in\omega}$, $(h'_n)_{n\in\omega}$ such that

$$\check{\Psi}_{Y_n,f_n} = \check{\Psi}_{Y_n,f'_n},\ \check{\Psi}_{Z_m,h_m} = \check{\Psi}_{Z_m,h'_m},\ \Psi_{Y_n,f'_n} \cap \Psi_{Z_m,h'_m} = \emptyset$$

for all $m, n \in \omega$.

For every $n \in \omega$, we put

$$\Psi_n = \bigcap_{i\geq n} B(Y_n \setminus B(x_0, f'_n(i)), i),$$
$$\Psi'_n = \bigcap_{i\geq n} B(Z_n \setminus B(x_0, h'_n(i)), i),$$

and note that

$$\check{\Psi}_n = \check{\Psi}_{Y_n,f_n},\ \check{\Psi}'_n = \check{\Psi}_{Z_n,h_n}.$$

Now suppose that $\check{p} \in \mathrm{cl}_{\check{X}} \bigcup_{n\in\omega} \check{\Psi}_n$ and pick $p' \in \check{p}$ such that $p' \in \mathrm{cl}_{X^\#} \bigcup \overline{\Psi}_n$, so $\bigcup_{n\in\omega} \Psi_n \in p'$. We put $P = \bigcup_{n\in\omega} \Psi_n$ and define a function $f : \omega \to \omega$ by

$$f(i) = \max\{f_0(i), \dots, f_i(i)\}.$$

Since X is ultrametric, $B(\Psi_n, i) = \Psi_n$ for every $i \le n$. Hence, $\Psi_{P,f} \subseteq \bigcup_{n\in\omega} \Psi_n$ and $\check{p} \subseteq \overline{\bigcup_{n\in\omega} \Psi_n}$.

Analogously, for every $\check{q} \in \mathrm{cl}_{\check{X}} \bigcup_{n\in\omega} \check{\Psi}'_n$, we have $\check{q} \subseteq \overline{\bigcup_{n\in\omega} \Psi'_n}$. Since $(\bigcup_{n\in\omega} \Psi_n) \cap (\bigcup_{n\in\omega} \Psi'_n) = \emptyset$, we conclude that $\mathrm{cl}_{\check{X}} Y \cap \mathrm{cl}_{\check{X}} Z = \emptyset$.

(iv) By (i), $\mathrm{w}(\check{X}) \ge \mathfrak{c}$. The family

$$\{\Psi_{P,f} : P \text{ is an unbounded subset of } X, f : \omega \to \omega\}$$

is a base of topology of $\check{X}$, so $\mathrm{w}(\check{X}) \le \mathfrak{c}$. □

Theorem 2. *Let X be an ultrametric space such that $X = B(A, n)$ for some countable subset A of X and $n \in \omega$. Then, under CH, $\check{X}$ is homeomorphic to ω^*.*

Proof: Since X and A are coarsely equivalent, $\check{X}$ and $\check{A}$ are homeomorphic, so we may suppose that X is countable. By Theorem 1, $\check{X}$ is a compact zero-dimensional F-space of weight $\mathfrak{c}$ in which every non-empty G_δ-subset has an infinite interior. Thus, we can apply a characterization [4, Corollary 1.2.4] of ω^* under CH. □

Corollary 1. *Under CH, the Higson's corona νX of a proper ultrametric space X is homeomorphic to ω^*.*

Proof: To apply Theorem 2, we note that νX is homeomorphic to $\check{X}$ and $X = B(A, 1)$ for some countable subset A of X. □

Let G be an infinite discrete group. A subset $A \subseteq G$ is called *almost invariant* if $gA \setminus A$ is finite for every $g \in G$. We denote by $\mathcal{A}$ the family of all infinite almost invariant subsets of G and by εG the set of all maximal filters in $\mathcal{A}$ endowed with the topology defined by the family $\{\{\varphi \in \varepsilon G : A \in \varphi\} : A \in \mathcal{A}\}$ as a base for the open sets. Then εG is the remainder of the Freudental-Hopf compactification of G and every element of εG is called an *end* of G (for this approach to definition of ends see [3]). If G is countable and locally finite, by [7, Theorem 3.1.1] and [5, Proposition 2], there is an ultrametric on G such that $\check{G}$ is homeomorphic to εG. Recall that G is *locally finite* if every finite subset of G generates a finite subgroup.

Corollary 2. *Under CH, the space of ends of a countable locally finite group G is homeomorphic to ω^*.*

Acknowledgment. The author is thankful to the referee who helped to improve the presentation of the paper.

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(*Received* November 17, 2010, *revised* January 12, 2011)