The microstructure of Lipschitz solutions for a one-dimensional logarithmic diffusion equation

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Abstract. We consider the initial-boundary-value problem for the one-dimensional fast diffusion equation $u_t = [\mathrm{sign}(u_x) \log |u_x|]_x$ on $Q_T = [0,T] \times [0,l]$. For monotone initial data the existence of classical solutions is known. The case of non-monotone initial data is delicate since the equation is singular at $u_x = 0$. We 'explicitly' construct infinitely many weak Lipschitz solutions to non-monotone initial data following an approach to the Perona-Malik equation. For this construction we rephrase the problem as a differential inclusion which enables us to use methods from the description of material microstructures. The Lipschitz solutions are constructed iteratively by adding ever finer oscillations to an approximate solution.

These fine structures account for the fact that solutions are not continuously differentiable in any open subset of Q_T and that the derivative u_x is not of bounded variation in any such open set. We derive a characterization of the derivative, namely $u_x = d^+ \mathbb{1}_A + d^- \mathbb{1}_B$ with continuous functions $d^+ > 0$ and $d^- < 0$ and dense sets A and B, both of positive measure but with infinite perimeter. This characterization holds for any Lipschitz solution constructed with the same method, in particular for the 'microstructured' Lipschitz solutions to the one-dimensional Perona-Malik equation.

Keywords: logarithmic diffusion, one-dimensional, differential inclusion, microstructured Lipschitz solutions

Classification: 34A05, 35B05, 35B65

1. Introduction

In this work we consider the following one-dimensional evolution equations with logarithmic terms

$$v_t = \left(\frac{v_x}{|v|}\right)_x = [\mathrm{sign}(v) \log |v|]_{xx}, \tag{1.1}$$

and

$$u_t = \frac{u_{xx}}{|u_x|} = [\mathrm{sign}(u_x) \log |u_x|]_x, \tag{1.2}$$

where v and u are real valued functions depending on one space variable x and one time variable t. Due to the logarithmic behavior the equations are singular at $v = 0$ and $u_x = 0$. We are interested in the existence and properties of sign changing solutions of the first equation and non-monotone solutions of the second equation, respectively.

Both equations are limit cases of well-known and related equations, the porous medium equation (1.3) and the p-Laplace equation (1.4).

$$(1.3) \quad v_t = \Delta\left(v^m\right) = \operatorname{div}(mv^{m-1}\nabla v), \quad m > 0$$

$$(1.4) \quad u_t = \Delta_p u =: \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{1}{p-1}\left(|u_x|^{p-2}u_x\right)_x = |u_x|^{p-2}u_{xx}, \quad p > 1.$$

Both equations are of fast diffusion type for $m < 1$ and $p < 2$ respectively and studied intensively in literature, often in the same context [6], [7], [25], [26]. An explicit connection between (1.3) and (1.4) exists for radially symmetric solutions [11], in the one-dimensional case the connection is most direct, since scaled versions can be transformed into each other. Sending $m \to 0$ in a scaled version of (1.3) and $p \to 1$ in (1.4) we obtain the pair of related one-dimensional logarithmic diffusion equations (1.1) and (1.2).

The relation between the equations is preserved as long as $u_x \neq 0$, as one obtains (1.1) by formally differentiating (1.2) and substituting $v = u_x$. We are interested in the evolutionary problems associated with these equations and therefore consider initial values $v_0(x) = v(0,x)$ and $u_0(x) = u(0,x)$. Both logarithmic diffusion equations are only singular in the case $v = u_x = 0$ and regular and parabolic otherwise. Because of this the solutions of (1.1), (1.2) evolve quite differently, depending on whether the initial data v_0, u_0 cross the singular values $v = u_x = 0$ or stay away from them. Most physical settings lead to the default restriction $v \geq 0$ and existence of positive solutions of (1.1) for positive initial data v_0 is known [20], [25]. With $u = \int v\,dx$ one obtains solutions of (1.2) which are monotone as a function of x. For a complete mathematical theory, however, it is of great interest to cover sign-changing initial values v_0 as well, especially since there is no reason for (1.2) to allow only monotone initial data u_0.

Using methods from the study of material microstructures we 'explicitly' construct weak Lipschitz solutions of (1.2) with given non-monotone initial data u_0. All results of this work will be stated and shown for equation (1.2) since the methods apply to this equation only and the results are significantly stronger than for the differentiated problem (1.1). The translation to (1.1) is shortly described after the proofs at the end of Section 3.

For the construction we find an approximate solution, add a sequence of ever finer oscillations and prove convergence to a 'microstructured' Lipschitz solution and thus get the following existence result.

Theorem 1. *Let $u_0 \in C_0^{2+\alpha}([0,l])$, $0 < \alpha < 1$, and $(u_0)_{xx}(0) = (u_0)_{xx}(l) = 0$. Let $Q_T := [0,l] \times [0,T]$ with real numbers $0 < l < \infty$ and $0 < T < \infty$. Then the initial-boundary value problem for (1.2) with homogeneous Dirichlet boundary conditions has infinitely many weak solutions $u \in W^{1,\infty}(Q_T)$ which satisfy the initial condition $u(0,x) = u_0(x)$.*

The microstructure of these solutions can be described in more details. It is known from [20] that non-monotone solutions u of (1.2) cannot be C^1 in the whole domain. Interestingly this is also known for the Perona-Malik equation [12], even if the two equations are not closely related. Furthermore, Lipschitz solutions that are not even C^1 in any open set have been constructed for time independent differential inclusion problems [23], [16], [17], therefore we expect a similar property for our constructed solutions. Indeed this is true in those parts, where the approximate solution is altered by a construction. We prove the following specific non-smoothness result: if $u \in C^1(U)$ for some open set U, then the solution is identical with the approximate solution in U, i.e., no construction process has occurred. Furthermore, there exists a time t_0 after which the construction process is carried out globally, i.e., the solution is nowhere C^1 in $[t_0, T] \times \mathbb{R}$. This implies that u_{xx} does not exist in a classical sense, not even locally. A more refined argument proves the stronger result that no measure-theoretic form of u_{xx} exists, i.e., $u_x \notin \mathrm{BV}$. These proofs lead to a characterization of the derivative as the sum of two continuous, even x-differentiable functions $d^+(t,x)$ and $d^-(t,x)$ on sets A and B

$$u_x = d^+ \cdot \mathbb{1}_A + d^- \cdot \mathbb{1}_B.$$

The sets A and B are both dense and tattered and, most importantly, are not of finite perimeter which is the reason for $u_x \notin \mathrm{BV}$. This characterization depends on the construction and thus transfers to any Lipschitz function constructed by this method, especially to the solutions of the Perona-Malik equation and related problems [27], [28].

The following paragraphs present an outline of the methods and results of this work.

In Section 2 we will give an overview of the development and relations of the methods used.

Section 3 is devoted to the proof of Theorem 1, i.e., the construction of Lipschitz solutions of (1.2) with the methods from material microstructures.

In Section 4 we will prove the intrinsic properties of the 'microstructured' Lipschitz functions, all of which are due to the specific construction.

Section 5 summarizes the results of this work.

2. Background

The particular cases of one-dimensional logarithmic diffusion (1.1), (1.2) are studied thoroughly by Rodriguez and Vázquez in [20]. They show that for non-negative v_0 the Cauchy problem of the differentiated equation (1.1) has infinitely many positive classical solutions which vanish after finite time. Only one of them exists globally in time and is characterized by conservation of mass. Furthermore they show that for sign-changing initial data v_0 there exists no continuous solution, which is one of the few results on sign-changing initial data for logarithmic diffusion equations. However, they also give an example of a weak sign-changing solution of (1.1) with a jump discontinuity. Translating these results to the integrated problem (1.2) with $v = u_x$ they conclude that for non-monotone initial data u_0 there are no C^1-solutions of the Cauchy problem to (1.2), but weak Lipschitz solutions may exist. They also prove that unique solutions for monotone initial data u_0 can be recovered as limits of solutions to the scaled p-Laplace equation (1.4) as $p \to 1$. It remains open if continuous weak solutions for the logarithmic diffusion equation (1.2) exist for given non-monotone initial data u_0.

For the construction of such solutions we use a method established by Zhang [27] for the one-dimensional Perona-Malik model [19]. The method leads to the first and only non-trivial existence result in the one-dimensional case. It has evolved from the study of material microstructures which in turn originated from the study of differential inclusion problems $\nabla u \in K$ with matrix valued derivatives ∇u and non-convex sets K. The basic idea is to consider the rank-one-convex hull of K, where two matrices A and B are rank-one-connected if $\operatorname{rank}(A - B) = 1$. Then each point in the rank-one-convex hull is represented as a convex combination of points in K. As a one-dimensional example one may consider a set K consisting of two points and then approximate K from a point ∇u in between by pushing the value ∇u alternately to a close neighborhood of the left or the right point of the target set K, compare Figure 2.1(a). In fact, it is necessary to consider ∇u inside an 'in-approximation' of K, which is an ε-neighborhood of K intersected with its rank-one-convex hull. For matrix-valued ∇u the pushing is achieved by alternately taking one or the other affine function to approximate u, compare Figure 2.1(b), which is called lamination. The construction is only possible if the affine segments match along an interface which is equivalent to the property of the matrix-valued derivatives to be rank-one-connected.

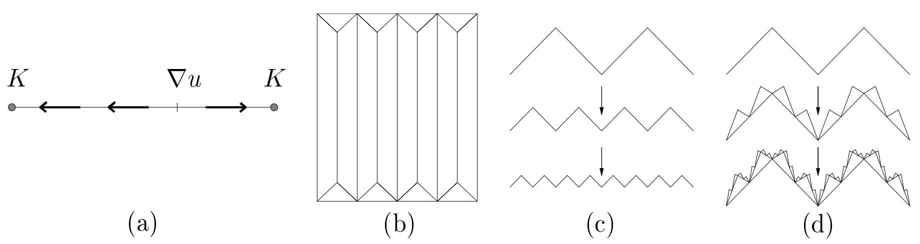

FIGURE 2.1. (a) Convex integration for differential inclusion with a set K consisting of two points (b) Lamination of two affine functions (c) Generating Young measure solutions (d) Generating Lipschitz functions.

Laminations as described above appear as microstructures in elastic materials. Such elastic materials can often be described as minimizers of variational problems with two-well or multi-well potentials, the minima of the potential being the non-convex target set K [3], [14]. A mathematical description of arbitrarily fine microstructures is achieved by considering sequences of laminations (see Figure 2.1(c)). The sequences of derivatives of the lamination functions u do not converge strongly in L^p in general. But a convergence to generalized measure valued functions is achieved, the so called Young measure solutions of the corresponding variational problem [2], [24], [18], [13]. For differential equations it is necessary to convert the problem to an artificial differential inclusion problem in order to allow room for a laminating approximation from within an appropriate lamination convex hull. Then it is possible to refine the lamination construction to achieve a convergence to Lipschitz functions. The basic principle is to add laminations on top of the previous ones (compare Figure 2.1(d)) and to carefully control the derivatives in order to achieve convergence [4], [17], [27], [28], [23]. The nested lamination process results in highly irregular Lipschitz functions, "microstructured' Lipschitz functions', which do not possess higher regularity in general. There exist examples which are not C^1 in any open set [23], [17].

We remark that an alternative, more functional analytic approach to show the existence of Lipschitz solutions has been established by Dacorogna and Marcellini [5]. They use a Baire-category approach, which might provide less clues on the nature of the Lipschitz solutions.

In [27] Zhang has refined this construction method in order to construct Lipschitz solutions of the Perona-Malik equation [19]. In previous publications Lipschitz solutions had only been constructed for steady-state problems with $u_t = 0$. An exception is the pioneering work of Höllig, who described Lipschitz solutions for a specific piecewise linear forward-backward diffusion equation [10]. However, this construction relies heavily on the piecewise affine structure of the equation considered. The flexible method of Zhang is adequate for the quasilinear Perona-Malik equation and applicable to arbitrary initial values which is the main novelty of this construction.

The logarithmic diffusion equation (1.2) is of type $u_t = [\sigma(u_x)]_x$ like the Perona-Malik equation, so that it is possible to transfer the method and to construct Lipschitz solutions of (1.2). As a basis for the construction a start function is found such that its derivative is either in the target set or in its rank-one-convex hull, we call this the approximate solution. Laminations are then added iteratively on those subsets of Q_T on which the approximate solution is not a solution. This creates a sequence converging to a solution in L^∞. It requires some extra care to add laminates to the non-affine approximate solution and still achieve convergence of the derivative [27], [28].

The adaptation of the method to logarithmic diffusion is presented in the next section.

3. Lipschitz solutions for the Dirichlet-Problem of logarithmic diffusion

Following the approach of Zhang [27], we write the logarithmic diffusion equation (1.2) in divergence form so that we can reformulate the problem.

$$u_t = [\sigma(u_x)]_x, \quad \sigma(s) = \operatorname{sign}(s) \cdot \log(|s|) \tag{3.1}$$

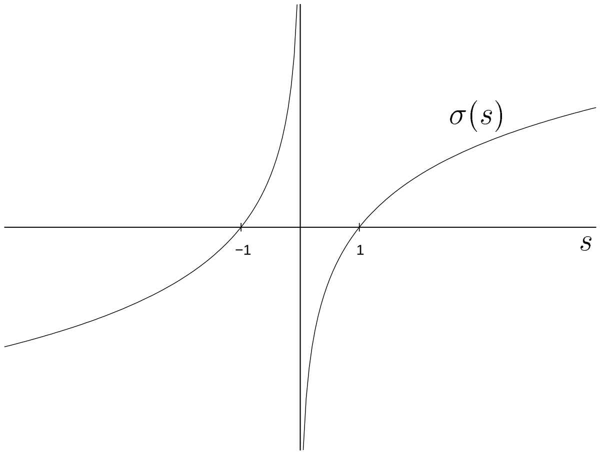

FIGURE 3.2. Singular function $\sigma(s)$ of the logarithmic diffusion equation (3.1).

Now we can state Theorem 1 precisely:

Theorem 1. *Let $u_0 \in C^{2+\alpha}([0,l])$, $0 < \alpha < 1$, with $u(t,0) = 0 = u(t,l)$ and $(u_0)_{xx}(0) = (u_0)_{xx}(l) = 0$. Let $\sigma(s) := \operatorname{sign}(s)\log(|s|)$. Then the Dirichlet problem*

$$\begin{cases} u_t - \sigma(u_x)_x = 0 & (t,x) \in Q_T = [0,T] \times [0,l] \\ u(0,x) = u_0(x) & x \in [0,l] \\ u(t,0) = 0 = u(t,l) & t \in [0,T] \end{cases}$$

has infinitely many weak solutions $u \in W^{1,\infty}(Q_T)$ in the sense that for every $\phi \in C_0^1(Q_T)$:

$$\int_{Q_T} u_t\phi + \sigma(u_x)\phi_x \, dx \, dt = 0.$$

The Dirichlet boundary conditions are chosen for technical reasons and we will assume Dirichlet boundary conditions throughout this section.

In the proof of Theorem 1 we will recursively construct a sequence of functions converging to a solution. The starting function of this sequence is the unique solution to the differential equation $u_t = [\sigma^*(u_x)]_x$ with a monotone function σ^* and the initial data given in Theorem 1. More precisely we choose a strictly monotone interpolation function $\sigma^*(s)$ which bypasses the singularity and agrees with $\sigma(s)$ for $|s| \gg 1$. For large values of $s = u_x$, i.e., $s > y$ with y suitable, we continue linearly to achieve a lower bound on $\sigma^*(u_x)'$. Figure 3.3 illustrates σ^* and relevant regions.

Then we can apply the following theorem.

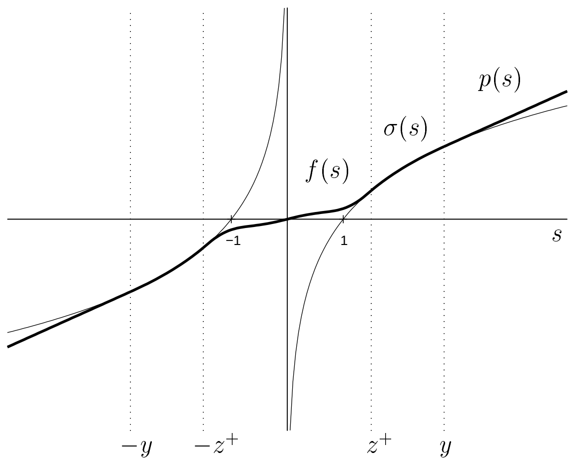

FIGURE 3.3. Monotone interpolation function $\sigma^*(s) \in C^2$ of the approximate equation (3.2). The functions f might be a linear interpolation between $(-z^+, -\log(z^+))$ and $(z^+, \log(z^+))$ with mollified joints. The function p might be a tangent line with a mollified joint.

Theorem 2. *Suppose $\sigma^* \in C^2(\mathbb{R})$ satisfies $0 < \lambda \leq (\sigma^*)'(s) \leq \Lambda$ for some constants $0 < \lambda < \Lambda$. Let $u_0 \in C^{2+\alpha}[0,l]$, $(0 < \alpha < 1)$ fulfill Dirichlet boundary conditions $u_0(0) = u_0(l) = 0$ and the compatibility condition $(u_0)_{xx}(0) = (u_0)_{xx}(l) = 0$. Then the problem*

$$\begin{cases} u_t - \sigma^*(u_x)_x = 0, & (t,x) \in Q_T \\ u(0,x) = u_0(x), & x \in [0,l] \\ u(t,0) = u(t,l) = 0, & t \in [0,T] \end{cases} \tag{3.2}$$

has a unique solution $u^* \in C^{1+\alpha/2, 2+\alpha}(\overline{Q}_T)$ satisfying a maximum principle

$$\max_{(t,x)\in\overline{Q}_T} |u_x^*(t,x)| = \max_{0\leq x\leq l} |(u_0)_x(x)|. \tag{3.3}$$

A proof can be found in the book of Ladyzenskaja, Solonikov and Ural'ceva [15, p. 451, Theorem 6.1]. The corresponding unique function u^* fulfills $|u_x^*| < y$ by (3.3). It is already a solution to (3.1) for $z^+ < |s| = |u_x^*| < y$, so to construct a solution everywhere we only need to modify u^* for $|u_x^*| < z^+$. We call u^* the approximate solution.

We want to modify u^* by adding piecewise affine functions. To allow room for this modification, we rephrase the original problem as a partial differential inclusion problem.

Solutions to the inclusion problem are in a certain large set of functions $\mathcal{F}$. Loosely speaking this set $\mathcal{F}$ 'includes' the approximate solution u^*. It further 'includes' any solutions to (3.1), of which we do not know their existence yet, and — most importantly — it 'includes' a sequence of functions converging from u^* to a solution u.

The following subsections prove Theorem 1 and are organized as follows: In subsection 3.1 we construct such a partial differential inclusion problem whose solutions can be converted into solutions of the original problem (3.1). We define a large set of functions $\mathcal{F}$. This includes solutions to (3.1), a starting function constructed from u^* and a sequence converging to one of the solution.

In subsection 3.2 we define $\mathcal{F}_\varepsilon \subset \mathcal{F}$, where ε is a parameter indicating how close elements of $\mathcal{F}$ are to a solution of (3.1), and we prove that $\mathcal{F}_\varepsilon$ is dense in $\mathcal{F}$ with respect to the L^∞-norm. The proof of the density result is carried out by the repeatedly mentioned construction and thus is the key to all structural results of Section 4.

In subsection 3.3 we use the density result to find a sequence of functions in $\mathcal{F}_{1/2^k}$. The limit of this sequence is a Lipschitz function and yields a solution u of the original problem. We will show that $u \in W^{1,\infty}$ and that there exist infinitely many such solutions.

Remark. The three subsections are largely parallel to the work of Zhang [27]. However, since we need the notation and intermediate estimates for the structural results in Section 4, we cannot shorten this technical part.

3.1 Conversion into partial differential inclusion problem. The original problem (3.1) can be solved by finding a solution to the following partial differential inclusion problem:

$$\text{Find } \Psi \in W^{1,\infty}(Q_T, \mathbb{R}^2) \text{ with } \Psi(t,x) = (\psi(t,x), u(t,x)) \text{ such that}$$

$$D\Psi(t,x) = \begin{pmatrix} \psi_t & \psi_x \\ u_t & u_x \end{pmatrix} \in \left\{ \begin{pmatrix} \sigma(s) & u \\ c & s \end{pmatrix} \;\middle|\; s, c \in \mathbb{R} \right\}. \tag{3.4}$$

The second component u of such a Ψ is a solution of (3.1), as one obtains by comparing matrix entries: The diagonal entries state that $s = u_x$, therefore $\psi_t = \sigma(u_x)$. Also $\psi_x = u$. Thus we have $u_t = \psi_{xt} = \psi_{tx} = \sigma(u_x)_x$ in the distributional sense, so u is a $W^{1,\infty}$ solution of (3.1) in the distributional sense. More specifically, u is a solution to the weak formulation of the problem: $\int_{Q_T} u\phi_t + \sigma(u_x)\phi_x \, dx\, dt = 0$ $\forall \phi \in C_0^2(Q_T)$.

We will now define sets of 2×2 matrices to state Problem (3.4) more precisely and define the set $\mathcal{F}$, in which the approximation process will take place. The diagonal entries of the matrices correspond to points (u_x, ψ_t) in the $\mathbb{R}^2$ plane as indicated in Figure 3.4. In the same figure $z^- = e^{-\log(z^+)}$ is placed as needed for the following definition.

$$
\begin{array}{rcllllll}
\tilde{K} & := & \{(s,\sigma(s)) & \mid & z^- & \le |s| \le & y\} & & & \\
\tilde{E} & := & \{(s,r) & \mid & -z^+ & < s < & -z^-, & -\log(z^+) & < r < & \sigma(s) \text{ or} \\
 & & & & -z^- & \le s \le & z^-, & -\log(z^+) & < r < & \log(z^+) \text{ or} \\
 & & & & z^- & < s < & z^+, & \sigma(s) & < r < & \log(z^+) \quad \}
\end{array}
$$

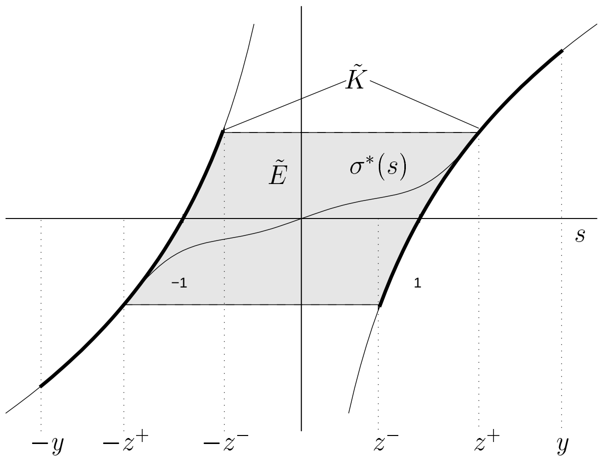

FIGURE 3.4. **Definition of** $\tilde{K}$ (bold curves) **and** $\tilde{E}$ (shaded region).

We define two sets of matrices $K(u)$ and $E(u) \subset M^{2\times 2}(\mathbb{R})$:

$$K(u) := \left\{ \begin{pmatrix} r & u \\ c & s \end{pmatrix} \;\middle|\; (s,r) \in \tilde{K}, |c| \le m \right\} \tag{3.5}$$

$$E(u) := \left\{ \begin{pmatrix} r & u \\ c & s \end{pmatrix} \;\middle|\; (s,r) \in \tilde{E}, |c| < m \right\} \tag{3.6}$$

Notice that $K(0)$ is compact and $E(0)$ is open in the space of lower triangular matrices $V \subset M^{2\times 2}(\mathbb{R})$. We later want to use this compactness and openness respectively, thus we will refer to the projection of matrices onto the space of lower triangular matrices. P_V shall be the orthogonal projection from $M^{2\times 2}(\mathbb{R})$ onto V. Now we can state the partial differential inclusion problem precisely.

$$
\begin{array}{l}
\text{Find } \Psi = (\psi, u) \in W^{1,\infty}(Q_T, \mathbb{R}^2) \text{ such that for all } (t,x) \in Q_T \\
\quad \text{(i)} \;\; D\Psi(t,x) \in K(u), \\
\quad \text{(ii)} \;\; P_V(D\Psi(t,x)) \in K(0) \text{ and } \psi_x = u.
\end{array} \tag{3.4$'$}
$$

Formulations (i) and (ii) are equivalent, the second one will be more useful later since we can use the compactness of $K(0)$. Further, let C^1_{pw} be the set of piecewise C^1 functions, i.e., $\Psi \in C^1_{pw}$ is equivalent to $\Psi \in C$ and there exist at most countably many disjoint open triangles G_i with $|Q_T \setminus \bigcup_{i=1}^{\infty} G_i| = 0$ and $\Psi|_{\bar{G}_i} \in C^1$. Then

$$\mathcal{F} := \left\{ \Psi = (\psi, u) \in C^1_{pw}(Q_T^2) \;\middle|\; \begin{array}{c} D\Psi \in K(u) \cup E(u) \text{ a.e. }, \\ |u| < \|u^*\|_{C^0(\overline{Q}_T)} + 1, \\ m := \|u_t^*\|_{C^0(\overline{Q}_T)} + 1 \end{array} \right\}. \tag{3.7}$$

Notice the strict inequality on the bound for $|u|$ which is necessary to allow modifications of any $\Psi \in \mathcal{F}$. Solutions to the differential inclusion problem (3.4′) which additionally stay within the bounds for u and m, are in $\mathcal{F}$. A priori it is not clear if such solutions do exist. We show first $\mathcal{F} \neq \emptyset$ by constructing an element of $\mathcal{F}$ using the approximate solution u^*: Since $u^* \in C^{1,2}$ according to Theorem 2, we have $u_t^* \in C$ and $u_x^* \in C^1$. By construction $\sigma^* \in C^2$, so $\sigma^*(u_x^*)_x \in C$. We have $u_t^* = \sigma^*(u_x^*)_x \in C(Q_T)$ and Q_T is simply connected, so there exists a vector field $\psi^* \in C^1$ with $(\psi_t^*, \psi_x^*) = (\sigma^*(u_x^*), u^*)$. Now we can define $\Psi^* := (\psi^*, u^*)$ with $D\Psi^* = \begin{pmatrix} \psi_t^* & \psi_x^* \\ u_t^* & u_x^* \end{pmatrix} = \begin{pmatrix} \sigma^*(u_x^*) & u^* \\ u_t^* & u_x^* \end{pmatrix}$. Obviously u^* and u_t^* stay within the bounds which were set in the definition of $\mathcal{F}$. The estimate for $|u_x^*|$ in Theorem 2 gives us $|u_x^*| < s^*$ and therefore $(u_x^*, \sigma^*(u_x^*)) \in \tilde{K} \cup \tilde{E}$. Thus we have $\Psi^* \in \mathcal{F}$ and $\mathcal{F} \neq \emptyset$. Ψ^* will serve as a starting point to construct a sequence in $\mathcal{F}$ converging to a solution of (3.4′) in $W^{1,\infty}$.

3.2 Construction of a dense subset of solutions. For distances we will consider the 1-norm: $\operatorname{dist}(X, Y) = \|X - Y\|_1 = \sum |x_i - y_i|$. If x is a point and K is a set we define: $\operatorname{dist}(x, K) = \min_{k \in K} \|x - k\|_1$.

Theorem 3. *For every $\varepsilon > 0$ the following subset $\mathcal{F}_\varepsilon \subset \mathcal{F}$ is dense in $\overline{\mathcal{F}}^{\infty}$, the closure of $\mathcal{F}$ under the L^∞-norm:*

$$\mathcal{F}_\varepsilon := \left\{ \Psi \in \mathcal{F} \;\middle|\; \int_{Q_T} \operatorname{dist}(D\Psi, K(u))\, dt\, dx < \varepsilon |Q_T| \right\}.$$

We need this density result to find a sequence Ψ_k such that $\Psi_k \in \mathcal{F}_{1/2^k}$. The limit of this sequence will have a weak gradient $D\Psi$ which is in the target set $K(u)$ almost everywhere. The following proof of Theorem 3 is technical, however, subsection 3.2 describes the construction which is essential for all structural results of Section 4 and cannot be omitted.

Proof: Given $\Psi \in \mathcal{F}$, $\varepsilon > 0$ and $0 < \eta < 1$, we need to find $\Psi_\eta \in \mathcal{F}_\varepsilon$ such that $\|\Psi - \Psi_\eta\|_{L^\infty} < \eta$. The proof is divided into two parts: the construction of Ψ_η and the confirmation that $\Psi_\eta \in \mathcal{F}_\varepsilon$.

We make two preliminary observations. First note the following simplifying identity for distances

$$\operatorname{dist}(D\Psi, K(u)) = \operatorname{dist}(P_V(D\Psi), K(0)) = \operatorname{dist}((\psi_t, u_x), \tilde{K}).$$

For the construction we will consider elements in $P_V(D\Psi)$ and ensure later $(\psi_\eta)_x = u_\eta$. Further $\partial_V E(0)$ will denote the boundary of E in the set of lower triangular matrices V:

$$\partial_V E(0) := \left\{ \begin{pmatrix} r & 0 \\ c & s \end{pmatrix} \;\middle|\; (s,r) \in \partial\tilde{E} \;\text{ or }\; |c| = m \right\}.$$

Second, in the course of the construction we will repeatedly cover a set with at most countably many scaled sets of prescribed shape. This is possible due to the Vitali Covering Principle as established by Saks [21, p. 109]:

Lemma 4 (Vitali Covering Principle)**.** *Let $U, V \subset \mathbb{R}^n$ be bounded, open sets satisfying $|\partial U| = 0 = |\partial V|$. Then there is a sequence $(x_i, r_i) \in \mathbb{R}^n \times (0,\infty)$, $i = 1, 2, \ldots$ such that*

(i) $U_i = x_i + r_i U \subset V$
(ii) $U_i \cap U_j = \emptyset$ *if* $i \neq j$ *and*
(iii) $|V \setminus \bigcup_{i=1}^{\infty} U_i| = 0$.

Let us now proceed with the construction of an approximative function Ψ_η.

(a) *Divide Q_T into triangles G_i.*

Since $\Psi \in \mathcal{F} \subset C^1_{pw}$, there are at most countably many triangular shaped tiles G_i exhausting Q_T such that $\Psi|_{G_i} \in C^1$. We consider each of the G_i individually to construct a function Ψ_η with the property $\int_{Q_T} \operatorname{dist}(D\Psi_\eta, K(u))\, dt\, dx < \varepsilon \cdot |Q_T|$. For this we will repeatedly exhaust tiles with other tiles, an overview is given in Figure 3.5.

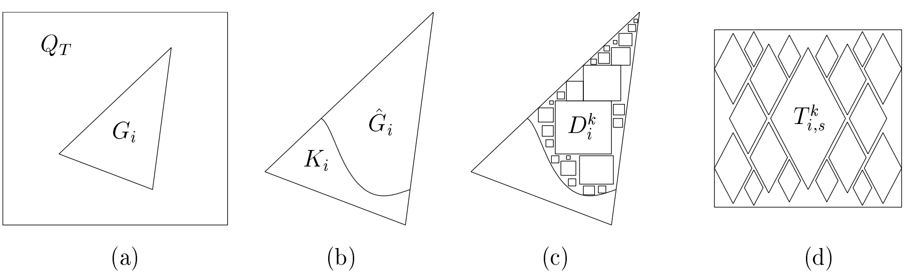

FIGURE 3.5. Repeated exhaustion process for the construction of Ψ_η: (a) Triangles G_i in Q_T (b) Division of G_i into $\hat{G}_i$ and K_i (c) Exhaustion of $\hat{G}_i$ with squares D_i^k (d) Exhaustion of D_i^k with diamonds $T_{i,s}^k$.

(b) *Divide G_i into parts K_i (no construction) and $\hat{G}_i$ (construction).*

We define K_i as the area, where $\operatorname{dist}(D\Psi, K(u))$ is small (including $D\Psi \in K(u)$). As $\Psi \in C^1(\bar{G}_i)$ we may find $\delta_i > 0$ such that the closed set

$$K_i := \left\{ (t,x) \in \overline{G_i} \;\middle|\; \operatorname{dist}(P_V(D\Psi), K(0) \cup \partial_V E(0)) \leq \delta_i \right\} \tag{3.8}$$

satisfies

$$\int_{K_i} \operatorname{dist}(P_V(D\Psi), K(0))\,dt\,dx < \frac{\varepsilon}{2} \cdot |G_i|. \tag{3.9}$$

We may require that the boundary of $\hat{G}_i := G_i \backslash K_i$ has measure zero.

The function Ψ is continuously differentiable in the interior of K_i, that is, $\Psi \in C^1(\mathring{K}_i)$. We do not want to change $D\Psi$ if it is already close to K as on the set K_i, so we define

$$\Psi_\eta := \Psi \qquad \text{on} \quad \bigcup_{i=1}^{\infty} K_i.$$

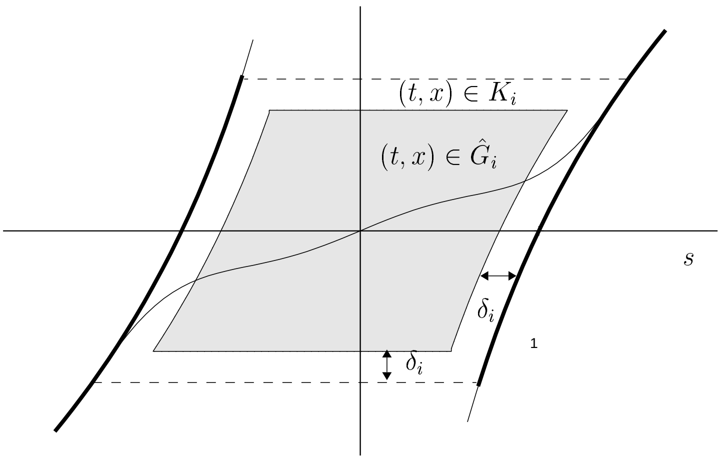

FIGURE 3.6. **Illustration of the definition of K_i and $\hat{G}_i$:** For $(t,x) \in K_i$ the diagonal entries (u_x, ψ_t) of the matrix $D\Psi(t,x)$ as points in $\mathbb{R}^2$ are close to the boundary of $\tilde{E}$. For $(t,x) \in \hat{G}_i$ the corresponding points (u_x, ψ_t) are in the interior region of $\tilde{E}$.

Remark: On $\hat{G}_i$ we have $\operatorname{dist}(P_V(D\Psi), K(0) \cup \partial_V E(0)) \geq \delta_i$, which means

$$\min\{\operatorname{dist}(P_V(D\Psi), K(0)), \operatorname{dist}((\psi_t, u_x), \partial\tilde{E}), \operatorname{dist}(u_t, m)\} \geq \delta_i.$$

From this we can derive estimates for individual entries of the matrix valued derivative $D\Psi$ on G_i.

(i) $\operatorname{dist}((\psi_t, u_x), \tilde{K} \cup \partial\tilde{E}) \geq \delta_i$ (illustration in Figure 3.6) which implies for $(s,r) \in \tilde{K} \cup \partial\tilde{E}$

$$\min_x |u_x - s| \geq \delta_i \qquad \text{and} \qquad \min_x |\psi_t - r| \geq \delta_i, \tag{3.10}$$

as the distance function is based on the 1-norm.

(ii) $\mathrm{dist}(u_t, m) \geq \delta_i$, so

$$|u_t| \leq m - \delta_i. \tag{3.11}$$

For later estimates we make δ_i smaller if necessary, to have:

$$\delta_i < \min\{\eta, \frac{\varepsilon}{2}, 2\zeta\}, \tag{3.12}$$

whereas $\zeta := (\|u^*\|_{C^0(\overline{Q}_T)} + 1) - \|u\|_{C^0(\overline{Q}_T)}$ with u the second part of the function $\Psi = (\psi, u)$. As $\|u\| < \|u^*\| + 1$ (see definition of $\mathcal{F}$) and u is continuous, we have $\zeta > 0$. We need to consider ζ as the initial distance of u to its bound in $\mathcal{F}$ to ensure that modifications stay within that bound and the modified function is still in $\mathcal{F}$.

(c) *Divide $\hat{G}_i$ into squares D_i^k with small variation of derivatives.*

Since $\Psi \in C^1(\overline{G}_i)$ and $\overline{G}_i$ is compact, $D\Psi$ is uniformly continuous on $\overline{G}_i$. Thus there exists a constant η_i such that $\|D\Psi(t,x) - D\Psi(s,y)\|_1 < \rho\delta_i$ if $\|(t,x) - (s,y)\|_1 < \eta_i$. The constant ρ is needed to counterbalance z^+ and will be specified later. Now we cover each $\hat{G}_i$ by at most countably many disjoint squares $\{D_i^k\}_{k=1}^{\infty}$, whose sides are parallel to the coordinate axes and whose side length is smaller than η_i. Let $p_i^k \in D_i^k$ be the center of D_i^k. Then

$$\|D\Psi(t,x) - D\Psi(p_i^k)\| < \rho\delta_i \quad \text{on each} \quad D_i^k. \tag{3.13}$$

(d) *Divide D_i^k into diamond shaped tiles $T_{i,s}^k$.*

On each of the squares D_i^k we want to approximate $\Psi = (\psi, u)$ by a tent-like function $\Psi_\eta = (\psi_\eta, u_\eta)$. In order to do this we divide D_i^k into at most countably many diamond shaped tiles $T_{i,s}^k$. On each of these tiles we first approximate u by adding a piecewise affine tent-like function $g_{i,s}^k$ with average zero, $u_\eta = u + g_{i,s}^k$ on $T_{i,s}^k$. Via the condition $(\psi_\eta)_x = u_\eta$ we will then derive ψ_η.

Construction of a piecewise affine function g:

The construction of $g_{i,s}^k$ will allow us to control the derivative $(u_\eta)_x$. Given $a, b > 0$ and $\delta > 0$ we define a standard tile $T := \{(t,x) \mid |t| \leq 1, |x| \leq \delta(t+1)\}$ and a continuous piecewise affine function $g(t,x)$. For an illustration of T and g see Figure 3.7, for the role of a and b see Figure 3.8.

We specify $g(t,x)$ for the triangular upper left quarter of T first, i.e., $0 \leq x \leq \delta(t+1)$ and $-1 \leq t \leq 0$. Then we extend in t-direction as an even function and

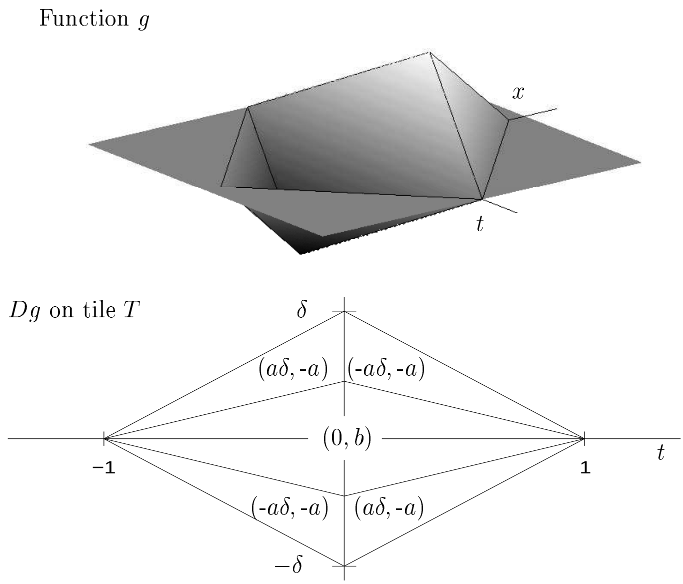

FIGURE 3.7. **Function g:** Piecewise affine function $g(t,x)$, diamond shaped tile T and values of $Dg(t,x)$ in parts of T.

in x-direction as an odd function. Outside T the function is zero.

$$
\begin{aligned}
g(t,x) &= \begin{cases} bx, & 0 \le x \le \frac{a\delta(t+1)}{a+b}, \qquad -1 \le t \le 0 \\ a\delta(t+1) - ax, & \frac{a\delta(t+1)}{a+b} \le x \le \delta(t+1), \qquad -1 \le t \le 0 \end{cases} \\
g(-t,x) &= g(t,x) \qquad \text{(even in } t\text{-direction)} \\
g(t,-x) &= -g(t,x) \qquad \text{(odd in } x\text{-direction)} \\
g(t,x) &= 0 \qquad \forall (t,x) \notin T
\end{aligned}
\tag{3.14}
$$

Straightforward calculations show that g has the following properties (compare Figure 3.7):

(3.15)

(i) $g \in C^1_{pw}$ and $g|_{\partial T} = 0$

(ii) $|g(t,x)| \le \frac{ab}{a+b}\delta \le \frac{a+b}{4}\delta$

(iii) $g_x \in \{-a, b\}$ and $g_t \in \{0, \pm a\delta\}$ a.e. in T

(iv) line integral of g in x-direction across T is zero: $\int_{-\delta(t+1)}^{\delta(t+1)} g(t,x)\,dx = 0$

(v) line integral of g_t in x-direction across T is zero: $\int_{-\delta(t+1)}^{\delta(t+1)} g_t(t,x)\,dx = 0$

We need scaled versions of g later in every tile $T^k_{i,s}$. So we define $T(p,\mu,\mu\delta)$ as the diamond shaped tile T centered at point $p = (t(p), x(p))$ with width μ and height $\mu\delta$, the standard tile is $T(0,1,\delta)$. Further $g(a,b,\mu,\mu\delta,p,t,x)$ is defined as a scaled version of g on the tile $T(p,\mu,\mu\delta)$ with parameters a and b and variables t and x. The function $g(a,b,\mu,\mu\delta,p,t,x)$ has analogous properties to g.

Remark: We mention that the roles of b and $-a$ can be exchanged. A function $\tilde{g}$ with the same properties can be constructed with $\tilde{g}_x = -a$ in the interior diamond and $\tilde{g}_x = b$ in the exterior triangles of the tile T.

For the local construction of $g^k_{i,s}$ we first decompose D^k_i into at most countably many diamond shaped tiles $T^k_{i,s}$ centered at $p^k_{i,s}$ with width $\mu^k_{i,s} < 1$ and height $\mu^k_{i,s} \cdot \rho\delta_i$. For the exact definition of a^k_i and b^k_i we write $\tilde{K}$ as the union of a positive and a negative part, whereas $\tilde{K}_+ := \{(s,\sigma(s)) \mid \ (z^- - \varepsilon) \le s \le (y+\varepsilon)\}$ and $\tilde{K}_- := \{(s,\sigma(s)) \mid -(s,\sigma(s)) \in \tilde{K}_+\}$, compare Figure 3.8. Then for each D^k_i with center p^k_i we can find a^k_i and b^k_i such that

$$\begin{aligned} &\operatorname{dist}((\psi_t(p^k_i), u_x(p^k_i) - a^k_i), \tilde{K}_-) = \tfrac{\delta_i}{2}\,, \\ &\operatorname{dist}((\psi_t(p^k_i), u_x(p^k_i) + b^k_i), \tilde{K}_+) = \tfrac{\delta_i}{2}\,. \end{aligned} \tag{3.16}$$

We recall that the distance to the upper or lower boundary of $\tilde{E}$ is larger than δ_i, see (3.10), and thus sufficiently large to have the above points still inside $\tilde{E}$.

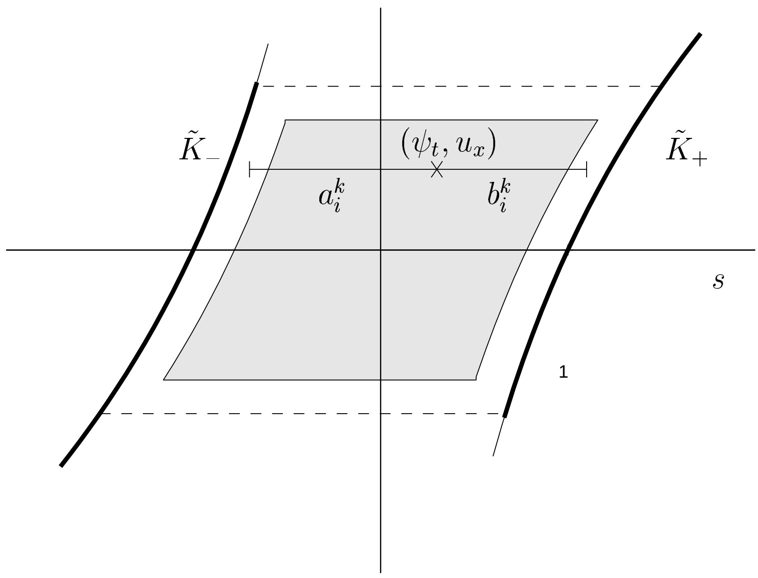

FIGURE 3.8. **Push $u_x(p^k_{i,s})$ close to $\tilde{K}$:** $\delta_i/2$ is the distance of $u_x(p^k_{i,s})+b^k_i$ to the positive part $\tilde{K}_+$ and of $u_x(p^k_{i,s})-a^k_i$ to the negative part $\tilde{K}_-$.

Now we define the piecewise affine functions $g_{i,s}^k$ on each tile $T_{i,s}^k$ by

$$g_{i,s}^k(t,x) = g(a_i^k, b_i^k, \mu_{i,s}^k, \mu_{i,s}^k \cdot \rho\delta_i, p_{i,s}^k, t, x).$$

We remark that for all i, k we have $a_i^k + b_i^k \leq 2z^+$ (compare Figure 3.8), so with the bounds on the functions $g_{i,s}^k$ (3.15)(ii) we get

$$|g_{i,s}^k| \leq z^+ \cdot \rho \cdot \frac{\delta_i}{2}\,. \tag{3.17}$$

We unite all functions $g_{i,s}^k$ of disjoint support to a joint piecewise affine function u_g, integrate to get the potential function ψ_g and then define Ψ_η as

$$\begin{aligned} &\Psi_\eta = (\psi_\eta, u_\eta) := (\psi + \psi_g, u + u_g) \\ &\text{with } \ \psi_g(t,x) := \int_0^x u_g(t,\tau)\, d\tau \ \text{ and } \ u_g(t,x) := \sum_{i,k,s} g_{i,s}^k. \end{aligned} \tag{3.18}$$

We mention that this definition does not contradict the previous definition of Ψ_η on tiles K_i. One of the properties of $g_{i,s}^k$ is that the integration in x-direction across complete tiles T is zero (3.15)(iv). Thus for points outside the collection of tiles, $\psi_g(t,x) = 0$. Since K_i is the set where no tiles are added, we still have $\Psi_\eta = \Psi$ on K_i. More precisely

(3.19)
- (i) $u_g| - K_i = 0$ since we only used tent functions g on the complement of $\bigcup K_i$,
- (ii) $\psi_g|_{K_i} = \int_0^x u_g(t,\tau)\, d\tau = 0$ since if $(t,x) \in K_i$ then it is outside of $\bigcup T_{i,s}^k$, and we have the integration property (3.15)(iv).

A tedious calculation with $\rho = 1/(3z^+)$ confirms $\Psi_\eta \in \mathcal{F}_\varepsilon$ and $\|\Psi - \Psi_\eta\|_{L^\infty} < \eta$, for details we refer to [22], [27]. This completes the proof of the density result Theorem 3. □

3.3 A sequence converging to a Lipschitz solution. The density result is sufficient to construct a sequence Ψ_k converging in $\overline{\mathcal{F}}^\infty$ such that $\operatorname{dist}(P_V(D\Psi_k), K(0))$ becomes arbitrarily small. To prove the convergence of the derivatives $D\Psi_k$ and thus achieve the limit derivative to be in K we will use the following lemma established by B. Kirchheim ([14, Lemma 3.27]).

Lemma 5. *Let $\Omega \subset \mathbb{R}^m$ be bounded and open. For a Lipschitz mapping $f : \Omega \to \mathbb{R}^n$ and $k \in \mathbb{N}$, let $r(f,k)$ be the supremum of all $r > 0$ such that there is a compact set $K \subset \Omega$ with $|\Omega \backslash K| < 2^{-k}$ and*

$$|f(x+y) - f(x) - \langle Df(x), y\rangle\,| \leq \frac{1}{k}|y| \quad \textit{if} \quad x \in K \ \ \textit{and} \ \ |y| \leq kr.$$

By Rademacher's Theorem, $r(f,k) > 0$ (e.g. [8, p. 281, Theorem 6]). *Consider a sequence $f_k : \Omega \to \mathbb{R}^n$ of uniformly Lipschitz mappings and suppose* $0 < r_k <$

$\min\{1/k^2, r(f_k,k)\}$ *for all* k. *Let* $B_\infty(f,r)$ *denote the ball around* f *with radius* r *with respect to the* L^∞ *norm. If* $f \in \bigcap_k B_\infty(f_k, r_k)$*, then* $\lim_{k\to\infty} Df_k(x) \to Df(x)$ *for a.e.* $x \in \Omega$ *(pointwise limit).*

We take $\Psi_1 = \Psi^* \in \overline{\mathcal{F}}^\infty$ as a starting point for the sequence and r_1 arbitrary. Given any $\Psi_{k-1} \in \overline{\mathcal{F}}^\infty$ and r_{k-1}, $k \geq 2$, we use the density result of Theorem 3 with $\varepsilon = 1/2^k$ and $\eta = r_{k-1}/2$ to find Ψ_k with the following properties:

- $\Psi_k \in \mathcal{F}_{1/2^k}$, i.e., $\displaystyle\int_{Q_T} \operatorname{dist}(D\Psi_k, K(u))\,dt\,dx < \frac{1}{2^k}|Q_T|$;
- $\operatorname{dist}(\Psi_k, \Psi_{k-1}) < \dfrac{r_{k-1}}{2}$, i.e., $\Psi_k \in B_\infty(\Psi_{k-1}, \dfrac{r_{k-1}}{2}) \cap \mathcal{F}_{1/2^k}$.

Now we find r_k as needed in the lemma above

$$r_k := \min\left\{ r(\Psi_k, k), \frac{r_{k-1}}{3}, \frac{1}{k^2} \right\}.$$

This guarantees $B_\infty(\Psi_k, r_k) \subset B_\infty(\Psi_{k-1}, r_{k-1})$, and therefore $\lim_{k\to\infty} \Psi_k \in \bigcap B_\infty(\Psi_k, r_k)$. The sequence $\Psi_k = (\psi_k, u_k)$ is uniformly Lipschitz, as $D\Psi_k \in \mathcal{F}$ and $\mathcal{F}$ is bounded (see (3.7)). Further Ψ_k is a Cauchy sequence in $\overline{\mathcal{F}}^\infty$ by definition, so $\Psi_k \to \Psi \in \overline{\mathcal{F}}^\infty$. We now apply Lemma 5 to get $\lim_{k\to\infty} D\Psi_k(t,x) = D\Psi(t,x)$ for a.e. $(t,x) \in Q_T$.

As $D\Psi_k(x) \to D\Psi(x)$ a.e., we know that the identity $(\psi_k)_x = (u_k)$ is still true for the limit function $\psi_x = u$ almost everywhere. Our candidate for a solution to Theorem 1 is this u, but we need to confirm the following statements:

(i) $D\Psi \in K(u)$ a.e. in Q_T;
(ii) $u \in W^{1,\infty}$;
(iii) existence of infinitely many such u;
(iv) boundary conditions, i.e., $u(t,0) = u(t,l) = 0$ $\forall t$;
(v) $\forall \phi \in C_0^1(Q_T)\ :\ \int_{Q_T} [u_t\phi + \sigma(u_x)\phi_x]\,dx\,dt = 0$.

(i) We know $\lim_{k\to\infty} \int_{Q_T} \operatorname{dist}(D\Psi_k, K(u))\,dt\,dx = 0$. Recall the simplified notation for the distance function. As $(\psi_k)_x = u$ and $|(u_k)_t| < m$ only the two diagonal entries of $D\Psi_k(t,x)$ are relevant and we have $\operatorname{dist}(D\Psi_k, K(u)) = \min_{(s,t)\in\tilde{K}}\{|(\psi_k)_t - s| + |(u_k)_x - t|\}$. Since $\tilde{K}$ is compact, the distance function is Lipschitz. It is also positive and uniformly bounded from above since the pair $((\psi_k)_t, (u_k)_x)$ is inside $\tilde{K} \cup \tilde{E}$. We therefore have by Lebesgue's theorem

$$0 = \lim_{k\to\infty} \int_{Q_T} \operatorname{dist}(D\Psi_k, K(u))\,dt\,dx = \int_{Q_T} \lim_{k\to\infty} \operatorname{dist}(D\Psi_k, K(u))\,dt\,dx.$$

With positivity and Lipschitz continuity we get

$$\operatorname{dist}(D\Psi, K(u)) = \operatorname{dist}(\lim\nolimits_{k\to\infty} D\Psi_k, K(u)) = \lim\nolimits_{k\to\infty} \operatorname{dist}(D\Psi_k, K(u)) = 0 \text{ a.e.,}$$
$$\Rightarrow D\Psi \in K(u) \text{ a.e. in } Q_T.$$

(ii) Since the sequence Ψ_k is uniformly Lipschitz, its limit Ψ is also Lipschitz. As ∂Q_T is piecewise C^1 we have $\Psi \in W^{1,\infty}(Q_T)$, compare [8, p. 279, Theorem 4].

(iii) When constructing the sequence Ψ_k we can split at any Ψ_{k-1} to continue on infinitely many different sequences in the following way. Given Ψ_{k-1}, choose any $\Psi_{k_1} \in B_\infty(\Psi_{k-1}, r_{k-1})$. Now take $s_1 = \frac{1}{3}\operatorname{dist}(\Psi_{k-1}, \Psi_{k_1})$. Choose $\Psi_{k_2} \in B_\infty(\Psi_{k-1}, s_1)$. Since $\Psi_{k_1} \notin B_\infty(\Psi_{k-1}, s_1)$, we have $\Psi_{k_1} \neq \Psi_{k_2}$. Analogously we can choose $s_2 = \frac{1}{3}\operatorname{dist}(\Psi_{k-1}, \Psi_{k_2})$ and $\Psi_{k_3} \in B_\infty(\Psi_{k-1}, s_2)$ and so on, to construct an infinite collection of Ψ_{k_i} with $\Psi_{k_i} \neq \Psi_{k_j}$, each leading to a solution Ψ^i. As $B_\infty(\Psi_{k_i}, s_i) \cap B_\infty(\Psi_{k_j}, s_j) = \emptyset \ \ \forall i \neq j$ and each solution $\Psi^i \in B_\infty(\Psi_{k_i}, s_i)$, we know $\Psi^i \neq \Psi^j \quad \forall i \neq j$ and have found infinitely many different solutions $\Psi^i = (\psi^i, u^i)$.

Remark: Formally we just know $\Psi^i \neq \Psi^j$, but this could be due to $\psi^i \neq \psi^j$ and we could have $u^i = u^j$. This scenario is avoided if we take instead of the ball $B_\infty(\Psi_{k-1}, s_i)$ the slightly smaller ball $B_{\|.\|_u}(\Psi_{k-1}, s_i)$ with the half-norm $\|\Psi\|_u = \|(\psi, u)\|_u = \|u\|_\infty$.

(iv) Since the construction on tiles only changes the values on the interior of tiles and all tiles are inside Q_T, we have $u|_{\partial Q_T} = u^*|_{\partial Q_T}$, thus $u(t,0) = u^*(t,0) = 0 = u^*(t,l) = u(t,l)$.

(v) We know that for almost every $(t,x) \in Q_T$

$$D\Psi = \begin{pmatrix} \psi_t & \psi_x \\ u_t & u_x \end{pmatrix} \in \begin{pmatrix} \sigma(u_x) & u \\ u_t & u_x \end{pmatrix}.$$

Thus for all $\phi \in C_0^2(Q_T)$ we get by partial integration:

$$\begin{aligned} &\int_{Q_T} [u_t\phi + \sigma(u_x)\phi_x]\, dx\, dt = -\int_{Q_T} u\phi_t dx dt + \int_{Q_T} \sigma(u_x)\phi_x\, dx\, dt \\ &= -\int_{Q_T} \psi_x\phi_t dx dt + \int_{Q_T} \psi_t\phi_x\, dx\, dt = \int_{Q_T} \psi\phi_{tx} dx dt - \int_{Q_T} \psi\phi_{xt}\, dx\, dt = 0. \end{aligned}$$

The extension to test functions $\phi \in C_0^1(Q_T)$ is achieved with the definition of a linear map $L : C_0^1(Q_T) \to \mathbb{R}$,

$$L(\phi) = \int_{Q_T} [u_t\phi + \sigma(u_x)\phi_x]\, dx\, dt.$$

As $u \in W^{1,\infty}$, L is well defined, bounded and continuous. Also $L \equiv 0$ on $C_0^2(Q_T)$ which is dense in $C_0^1(Q_T)$. Thus continuity implies $L \equiv 0$ on $C_0^1(Q_T)$. $\square$

Let us add three general comments on the just proven result:

Remark I: Although it took a lot of effort to explicitly construct elements of $\mathcal{F}_\varepsilon$, these are not the only functions which may appear in such a sequence. In principle, the sequence Ψ_k may contain any other piecewise affine function in $\mathcal{F}$. In particular any other possible Lipschitz solution is contained in $\mathcal{F}$. However, we are especially interested in solutions which are achieved solely with the construction given, which means, they are limits of sequences of which every single element has been constructed as explained in this section. All structural results of the next section apply to such solutions.

Remark II: The restriction to Dirichlet boundary conditions is only technical. Since the constructive process only takes place in the interior of the domain, any boundary condition for which an approximate solution u^* can be found, is possible. This includes generalized Neumann boundary conditions of the type $\sigma(u_x) = 0$, for details see [22].

Remark III: The existence result translates to the differentiated logarithmic diffusion equation (1.1). However, the translation is not as smooth as one might hope. Given a sign-changing initial datum v_0, we integrate it to obtain a non-monotone u_0. Theorem 1 gives a solution u of (1.2) for that initial value. Differentiation yields $v = u_x$ which is a weak solution of (1.1) in the interior of the domain. Since, however, $v \in L^\infty(Q_T)$, it is not necessarily defined for $\{0\} \times [0, l]$ which is a set of zero measure in $\mathbb{R}^2$. In this case v does not attain the initial value in a classical sense. A weak interpretation could be that v at point $(0, x)$ has an approximate limit of $v_0(x)$. However, this is only achieved for $|v_0(x)| > c > 1$. The construction does not recover the initial data v_0 close to the singular value $v_0 = 0$. The reason is that the constructed solutions avoid the singular value $v = u_x = 0$. v jumps from positive to negative values and u zigzags.

4. Properties of Lipschitz solutions with microstructure

The procedure of Section 3 results in Lipschitz solutions which display arbitrarily fine structure, we refer to them as 'Lipschitz solutions with microstructure'. The aim of this section is to give a more detailed description of this microstructure.

The functions of the converging sequence are C^1 in tile parts of decreasing size but not differentiable across the boundaries of the tiles, so we expect the limit function to be nowhere C^1, which implies that the ridge lines, which are jump parts of the derivative, are dense in the domain. Further, the derivatives in the sequence $(u_l)_x$ jump across the tile boundaries. For the limit functions these jumps become dense and $u_x \notin \mathrm{BV}$. However, from the construction process evolve two disjoint dense sets A and B, on each of which u_x is continuous and even differentiable in the x-direction. The main result of this section is the following characterization of u_x:

$$u_x = d^+ \cdot \mathbb{1}_A + d^- \cdot \mathbb{1}_B,$$

where $d^\pm$ are continuous and differentiable in x-direction, and thus u_x has some regularity, if restricted to one of those dense sets A or B.

The results do not depend on the fast diffusion equation, but are inherent to the construction of Section 3. They transfer to similar constructions, in particular to solutions for the Perona-Malik equation from [27].

4.1 Solutions are nowhere C^1 for large t.

Theorem 6. *For all functions u obtained as the limit of a sequence u_l constructed with the method described in Section 3 there exists $t_0 > 0$ such that for all open sets $U \subset [t_0, T] \times [0, l]$ we have $u \notin C^1(U)$.*

The key to the existence of the bounding time t_0 is the following lemma concerning the approximate equation and approximate solution.

Lemma 7. *Any solution $u \in C^{1,2}$ of the Dirichlet problem*
(4.1)

$$\begin{cases} u_t - \sigma^*(u_x)_x = 0 & (t,x) \in Q_T = [0,T] \times [0,l], \\ & \sigma^* \in C^2, \sigma^*(0) = 0, 0 < \lambda < {\sigma^*}'(s) < \Lambda \\ u(0,x) = u_0(x) & x \in [0,l],\ u_0 \in C^{2+\alpha} \\ u(0) = u(l) = 0 = u_{xx}(0) = u_{xx}(l) & \forall t \in [0,T] \end{cases}$$

has the property

$$u_x \overset{t\to\infty}{\longrightarrow} 0 \quad \textit{uniformly.}$$

This property is obtained when studying the limits of the energy functional $E(t) := \int_0^l S^*(u_x)\,dx = \int_0^l \int_0^{u_x(t,x)} \sigma^*(s)\,ds\,dx$, for details of the proof see [22]. Notice, that the conditions for σ^* are more restrictive than in Theorem 2 since we require $\sigma^*(0) = 0$. But as the construction of a solution in Section 3 uses a symmetric σ^*, this lemma is applicable here.

The key to the non-differentiability property is the appearance of arbitrarily small tiles T everywhere in $[t_0, T] \times [0, x]$, which implies that the boundaries of the tiles are dense in $[t_0, T] \times [0, l]$. More specifically, the following lemma holds.

Lemma 8. *Let u be a function obtained as the limit of a sequence u_l with $u_0 = u^*$ and every element u_l constructed with the method of Section 3. Let t_0 be such that $u^*_x \le 1 \ \ \forall t > t_0$ (t_0 exists by Lemma 7). Then for a.e. $(t,x) \in [t_0,T] \times [0,l]$ there exists a sequence $(T^k_{i,s})_{l'}$ such that $(t,x) \in (T^k_{i,s})_{l'}$ $\forall l'$ and $\operatorname{diam}(T^k_{i,s})_{l'} \overset{l'\to\infty}{\longrightarrow} 0$.*

Proof: For any point (t,x), $t \ge t_0$ there exists $\tilde{\delta}$ with $\operatorname{dist}((u^*_x(t,x), \sigma^*(u^*_x(t,x))), \partial\tilde{E}) > \tilde{\delta}$, compare Figure 3.4. By the construction in Section 3 u is the limit of a sequence of which each u_{j+1} is constructed by adding piecewise affine functions u_g to u_j. We may write

$$u = u^* + \sum_{l=1}^{\infty} (u_g)_l, \quad u_j = u^* + \sum_{l=1}^{j} (u_g)_l. \tag{4.2}$$

The piecewise affine functions are added on tiles $T^k_{i,s}$. Let $\Delta T^k_{i,s}$ be a connected subset of the set $\{(t,x) \in T^k_{i,s} \,|\, (u_g)_l \in C^1\}$, thus $(u_g)_l$ is affine on $\Delta T^k_{i,s}$. Then $\Delta T^k_{i,x}$ is either a triangle or a center diamond, compare Figure 3.7. If at step l

a new tile $(T^k_{i,s})_l$ is constructed inside an old tile $(T^k_{i,s})_{l-j}$ from step $l-j$ the following inclusions hold

$$(T^k_{i,s})_l \subset (D^k_i)_l \subset (G_i)_l \subset (\Delta T^k_{i,s})_{l-j}. \tag{4.3}$$

This nested construction of diamonds T and squares D implies $\operatorname{diam}(T^k_{i,s})_l < \frac{1}{2}\operatorname{diam}(T^k_{i,s})_{l-j}$, so if a sequence of nested tiles exists, then $\operatorname{diam}(T^k_{i,s})_l \overset{l\to\infty}{\longrightarrow} 0$. Observe first that there is at least one tile around a.e. point (t,x).

$$\begin{aligned} t \geq t_0 &\Rightarrow \operatorname{dist}((u_x(t,x), \sigma^*(u_x(t,x))), \partial\tilde{E}) > \tilde{\delta} \Rightarrow \exists\, l : (\delta_i)_l < \tilde{\delta} \\ &\Rightarrow (t,x) \notin (K_i)_l \Rightarrow \text{ a first tile } T^k_{i,s} \text{ around } (t,x) \text{ will be constructed.} \end{aligned}$$

Let ∂T be the inner and outer boundaries of tile T, more specifically, the subset of T where the piecewise affine function g is not differentiable. We assume that (t,x) is not a boundary point of any tile and not in the residual from the repeated exhaustion processes, i.e., we assume

$$(t,x) \notin \left(\bigcup_{l,i,k,s} \partial(T^k_{i,s})_l \right) \cup \left(\bigcup_{l,i,k,s} (Q_T \backslash (T^k_{i,s})_l) \right).$$

Since the union of boundaries and residual sets is of zero measure, the assumption is true for almost every $(t,x) \in Q_T$.

We proceed with a contradiction argument. Let us assume the sequence of decreasing tiles around (t,x) is finite, i.e., there exists l_1 such that $(t,x) \in \Delta(T^k_{i,s})_{l_1}$ but for all $l > l_1$ we have $(t,x) \notin \bigcup_{i,k,s} \Delta(T^k_{i,s})_l$. The tile-parts are compact and $u_g \in C^1(\Delta T^k_{i,s})$. Further the distance of $(u_l)_x$ to $\tilde{E}$ is positive, so there exists $\tilde{\delta}'$ such that the distance of the point $((u_g)_x, \sigma^*(u^*_x))$ to the boundary of $\tilde{E}$ is larger $\tilde{\delta}'$. Thus the argument from above applies and a further tile will be constructed. This is a contradiction to the assumption that at l_1 the last tile was constructed. □

Proof of Theorem 6: The theorem is proved by contradiction, so let us assume $u \in C^1(U)$ for some open set $U \subset [t_0, T] \times [0, l]$. By Lemma 8 we can find $(T^k_{i,s})_{l_0} \subset U$. On this tile we can write

$$u = u^* + \sum_{l=1}^{l_0-1} (u_g)_l + (u_g)_{l_0} + \sum_{l=l_0+1}^{\infty} (u_g)_l =: u^* + \phi^1 + g + \phi^2. \tag{4.4}$$

Since the tiles T of the sequence u_j are nested, we have $\phi^1_x \equiv c$ on $(T^k_{i,s})_{l_0}$ and by the integration property of the piecewise affine functions g in (3.15) we know

$$\int_{(T^k_{i,s})_{l_0}} \phi^2_x \, dx = 0. \tag{4.5}$$

Further, with the bound on the variation of the derivative $D\Psi$ on squares D, compare (3.13), and the fact that Dg is constant, we get the following bound for the variation of u_x^* inside $(T_{i,s}^k)_{l_0} \subset (D_i^k)_{l_0}$:

$$\|u_x^*(t,x) - u_x^*(s,y)\| \leq \|D\Psi(t,x) - D\Psi(s,y)\| \leq \rho\delta_i \leq \delta_i \leq \frac{1}{2^{l_0}}. \tag{4.6}$$

The limit function u is a solution of $u_t = \sigma(u_x)_x$ which implies $(u_x, \sigma(u_x)) \in \tilde{K}_+ \cup \tilde{K}_-$ almost everywhere (compare Figure 3.8). Since we assumed $u \in C^1((T_{i,s}^k)_{l_0})$ and $(T_{i,s}^k)_{l_0} \subset U$, the derivative u_x is continuous and thus the points $(u_x, \sigma(u_x))$ are only on one branch of $\tilde{K}$. Without loss of generality let $(u_x, \sigma(u_x)) \in \tilde{K}_+$ which implies $u_x \in [z^-, z^+]$. Now we choose M to be one component $\Delta(T_{i,s}^k)_{l_0}$ of the tile $(T_{i,s}^k)_{l_0}$ such that $g_x|_M = -a$. Since $u_x > z^-$ we have

$$\int_M u_x \geq \int_M z^- \geq |M|z^- > 0. \tag{4.7}$$

By construction we know that on M we have $(u^* + \phi^1 + g)_x \in (-z^+, -z^- + \delta_i) \subset (-z^+, -z^- + \frac{1}{2^{l_0}})$. From the integration property of ϕ^2 in (4.5) we get:

$$\begin{aligned}\int_M u_x &= \int_M u_x^* + \phi_x^1 + g_x + \phi_x^2 = \int_M u_x^* + \phi_x^1 + g_x \\ &\leq \int_M (-z^- + \frac{1}{2^{l_0}}) = |M|(-z^- + \frac{1}{2^{l_0}}) < 0\end{aligned} \tag{4.8}$$

$$\Rightarrow \quad 0 < |M|z^- \overset{(4.7)}{\leq} \int_M u_x \overset{(4.8)}{<} 0. \tag{4.9}$$

This is a contradiction, so u cannot be in $C^1(U)$. □

Remark: For the proof of $u \notin C^1(U)$ it was only necessary to find a single tile $T_{i,s}^k \subset U$. Therefore the result holds for any open set $U \subset Q_T$ in which at least one tile is used for the construction. More generally, it is sufficient to find a single tile T with $T \cap U \neq \emptyset$ since by Lemma 8 the intersection will contain a full tile. This implies

$$u \in C^1(U) \quad \Rightarrow \quad u|_U = u^*|_U.$$

4.2 Singular measure theoretic second derivative - $u_x \notin$ BV. By the proof in the previous section we see that u_x is neither positive everywhere nor negative everywhere. Further it does not attain values close to zero because $|u_x| > z^-$. This implies that the derivative jumps from positive to negative values and the jump is always at least $2z^-$. Intuition tells us the variation of u_x should not be bounded, not even locally. More precisely, a function $u \in L^1(U)$, $U \subset \mathbb{R}^n$ open, has bounded variation in U iff

$$\sup\left\{ \int_U u \operatorname{div}\varphi\, dx \;\middle|\; \varphi \in C_0^1(U), |\varphi| \le 1 \right\} < \infty.$$

We write $\mathrm{BV}(U)$ to denote the space of functions of bounded variation.

Theorem 9. *Let u be obtained as the limit of a sequence u_l constructed with the method described in Section 3. Let $U \subset [t_0, T] \times [0, l]$ be open. Then $u_x \notin \mathrm{BV}(U)$.*

To prove unbounded variations of a function on a subset of $\mathbb{R}^2$ is not straight forward. The fact that the derivative jumps infinitely often in any open set is sufficient in the one-dimensional case, because of the concept of essential variation [9, 5.10]. But in the two-dimensional case a function of bounded variation may have infinitely many jumps in any open set, as shown by the counterexample 3.53 in [1, 3.5]. To prove the above theorem we want to use the *Characterization of* BV *by sections* from [1] which implies (compare [1, p. 196, l. 7–11]):

(4.10) *Let $u \in L^1(\Omega \subset \mathbb{R}^N)$. Then $u \in \mathrm{BV}(\Omega)$ only if for any $\nu \in S^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega_\nu$ the restriction $u_y^\nu \in \mathrm{BV}(\Omega_y^\nu)$,*

where: $\Omega_\nu :=$ projection of Ω onto the hyperplane orthogonal to ν,
$\Omega_y^\nu :=$ section of Ω corresponding to $y \in \Omega_\nu$: $\{y + t\nu \in \Omega, t \in \mathbb{R}\}$,
$u_y^\nu :=$ restriction of u to Ω_y^ν, i.e., $u_y^\nu(t) = u(y + t\nu)$,
$\mathcal{H}^{N-1} := (N-1)$-dimensional Hausdorff measure.

We want to show that for the fixed direction $\nu = (0, 1) \in S^2$ and for $\mathcal{L}^{N-1}$-a.e. $t \in [0, T]$ the restriction of the derivative $(u_x)_t^\nu$ is not in $\mathrm{BV}([0, l])$ and so $u_x \notin \mathrm{BV}$ by (4.10). For this, one needs to access values of the derivative u_x on the line $t \times [0, l]$. This is a set of measure zero which implies that u_x is not necessarily defined there. The final argument of the proof will be the same as in the previous section, though the setup is more complicated. Recall

$u_l \to u$ which is a weak solution of $u_t = [\sigma(u_x)]_x$,
$(u_l)_x \to u_x$ p.w. $\mathcal{H}^2$-a.e.,
$u_l \in C^1_{pw} \;\Rightarrow\; u_l \in C^1$ $\mathcal{H}^2$-a.e.,
$u_x \in I^+ \cup I^-$ with $I^+ = [z^-, z^+]$ and $I^- = -I^+$.

We use the following notation.

$$\begin{aligned}
(u_x)|_t &:= \text{restriction of } u_x \text{ to the specific time t, i.e.,} (u_x)|_t(x) = u_x(t,x),\\
V_l &:= \Big\{ \bigcup U \subset [t_0,T] \times [0,x] \mid u_l \in C^1(U) \Big\} = \text{open interior tile parts},\\
V &:= \bigcap_{l=1}^{\infty} V_l,\\
V^+ &:= \{(t,x) \in V \mid \exists\, l_0 : \forall\, l > l_0 \ (u_l)_x(t,x) \in I^+\},\\
V^- &:= \{(t,x) \in V \mid \exists\, l_0 : \forall\, l > l_0 \ (u_l)_x(t,x) \in I^-\},\\
\tilde{V} &:= V \backslash (V^+ \cup V^-),\\
M &:= \{(t,x) \in [t_0,T] \times [0,l] \mid (u_l)_x(t,x) \to u_x(t,x)\},\\
E &:= V \cap M,\\
R &:= \{t \in [t_0,T] \mid \mathcal{H}^1(t \times [0,l] \backslash E) = 0\}.
\end{aligned}$$

The sets V_l, V, M, and E cover almost all of $[t_0,T] \times [0,l]$, i.e., $\mathcal{H}^2([t_0,T] \times [0,l] \backslash .) = 0$ for V_l, V, M and E. Also $\mathcal{H}^1([t_0,T] \backslash R) = 0$, consider $\mathcal{H}^2([t_0,T] \times [0,l] \backslash E) = \int_{t_0}^T \mathcal{H}^1(t \times [0,l] \backslash E)\,dt$, and $\tilde{V} \cap M = \emptyset$ which implies $\mathcal{H}^2(\tilde{V}) = 0$.

The values of $(u_l)_x$ converge in V^+ and V^-. To see this, we first note from the definition of $V^\pm$ that $(u_l)_x$ stays positive or negative respectively after some step l_0. Then, however, the further variation of $(u_l)_x$ is bounded by δ_i in each step, compare Figure 3.8 and equation (3.16). By the estimate on δ_i in (3.12) and the choice of the sequence Φ_l at the beginning of Section 3.3, we see that the series of $(\delta_i)_l$ converges, so $(u_l)_x$ is a Cauchy sequence and we get

$$\begin{aligned}
&\forall (t,x) \in V^+ : \ (u_l)_x(t,x) \to d^+ \text{ with some value } d^+ \in I^+,\\
&\forall (t,x) \in V^- : \ (u_l)_x(t,x) \to d^- \text{ with some value } d^- \in I^-.
\end{aligned} \tag{4.11}$$

Proof of Theorem 9: Let $U \subset [t_0,T] \times [0,l]$ be open, $t \in R$ such that $\mathcal{H}^1(U|_t) > 0$. Consider $(u_x)|_t$ and an open Interval $J \subset U|_t$. We will show that u_x attains values in I^+ and I^- for certain points in J, so the variation of $(u_x)|_t$ is at least $2z^-$ in J. Then we can cover $U|_t$ with countably many intervals J to show that the essential variation on the line $t \times [0,l]$ is not bounded. This implies $(u_x)|_t \notin \mathrm{BV}(U|_t)$ (compare [9, 5.10]). With the characterization of BV functions in (4.10) we then get $(u_x) \notin \mathrm{BV}(U)$.

Let l_0 be so large that the diameter of the tiles T at step l_0 is less than $|J|/2$. Since $t \in R$, we have $\mathcal{H}^1(U|_t \backslash E) = 0$ and the interval J is covered up to an $\mathcal{H}^1$-zero set by restrictions of connected components of V_{l_0}. There is a tile $(T^k_{i,s})_l$ such that its restriction to time t is contained in J. This restriction has three connected components which are intersections with three tile parts, two boundary triangles and the middle diamond. We call the first component W_l^- and the second one W_l^+, such that $(u_l)_x|_{W_l^\pm} \in I^\pm$ respectively (compare Figure 4.9). The third component mirrors the first and is thus irrelevant for the argument. Since $t \in R$, the horizontal line parts of the tile T are not exactly on the line $t \times [0, l]$ but above or below. Further, this also accounts for the fact that the sets $V^+ \cap M$ and $V^- \cap M$ cover J up to a set of $\mathcal{H}^1$-measure zero. If

$$(V^+ \cap M \cap J) \neq \emptyset \quad \text{and} \quad (V^- \cap M \cap J) \neq \emptyset \tag{4.12}$$

the convergence in $V^\pm$, see (4.11), gives that u_x attains values in I^+ and I^- and the jump is confirmed. Assume (4.12) was false. Without loss of generality let

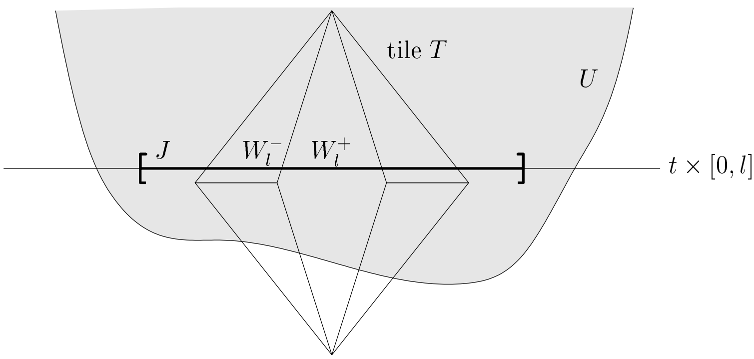

FIGURE 4.9. **Jump along a line:** Intersection of the line $t \times [0, l]$ with the set U, position of the interval J, an intersecting tile T, and tile parts W_l^+ and W_l^- with positive and negative u_x, and therefore with a jump.

$V^- \cap M \cap J = \emptyset$. This implies $\mathcal{H}^1(V^+ \cap M \cap J) = \mathcal{H}^1(J)$, moreover $\mathcal{H}^1(V^+ \cap M \cap W_l^-) = \mathcal{H}^1(W_l^-)$. Now we argue as in the previous section, but only for the fixed time t:

$$\int_{W_l^-} u_x = \int_{V^+ \cap M \cap W_l^-} u_x \geq |V^+ \cap M \cap W_l^-| z^- = |W_l^-| z^- > 0. \tag{4.13}$$

Like in the previous section we use the description of u as a sum, cf. (4.4), the zero-integral property of ϕ^2 in (4.5), and the bound for u_x in this particular tile part to estimate

$$\begin{aligned} \int_{W_l^-} u_x &= \int_{W_l^-} u_x^* + \phi_x^1 + g_x + \phi_x^2 \\ &= \int_{W_l^-} u_x^* + \phi_x^1 + g_x \leq \int_{W_l^-} (-1 + \frac{1}{2^l}) = |W_l^-|(-1 + \frac{1}{2^l}) < 0. \end{aligned}$$

This is a contradiction, so $(V^- \cap M \cap J) \neq \emptyset$ and $u_x \notin \mathrm{BV}$. $\square$

Remark: As in the previous subsection the argument is not restricted to $t > t_0$ but works for any open set U, in which the construction process is carried out.

4.3 Characterization of u_x. Theorem 9 implies that both $V^+ \cap M$ and $V^- \cap M$ are dense in $[t_0, T] \times [0, l]$. It seems reasonable that the limit derivative u_x should be continuous on both those dense sets and probably even differentiable in x-direction. The following characterization holds.

Theorem 10. *Let u be obtained as the limit of a sequence u_l constructed with the method described in Section 3. Then there exist two functions $d^+(t,x)$ and $d^-(t,x) \in C([t_0,T] \times [0,l])$ such that*

$$u_x = d^+ \cdot \mathbb{1}_{V^+ \cap M} + d^- \cdot \mathbb{1}_{V^- \cap M}.$$

Moreover, $d^\pm$ is weakly differentiable in x-direction and $d^\pm_x \in L^\infty$.

PROOF: It is sufficient to consider d^+ as the argument is analogous for d^-. Consider $u_x|_{V^+ \cap M}$. Since $u_x \geq z^- > 0$ and $\sigma \in C^\infty(\mathbb{R}^+)$,

$$u_x \in C(V^+ \cap M) \quad \Leftrightarrow \quad \sigma(u_x) = \psi_t \in C(V^+ \cap M). \tag{4.14}$$

We recall the definition of $\Psi_\eta = (\psi_\eta, u_\eta)$, compare (3.18), and the smoothness of the approximating sequence to see

$$(\psi_l)_t = \int_0^x (u_l)_t(t,s)\, ds = \int_0^x u_t^*(t,s) + \sum_{k=1}^{l} (u_g)_t^k \, ds.$$

Because of the derivative values of g given in (3.15)(iii) and the bounds for δ_i in (3.12) we have

$$|(\psi_l)_t - (\psi_{l-1})_t| = \left| \int_0^x (u_g)_t^l(t,s)\, ds \right| \leq \delta_i^l \cdot a_i^{l,k} \cdot \delta_i^l \leq 2z^+ \cdot (\eta^l)^2 \leq 2z^+ \frac{1}{2^{2l}}.$$

Therefore we obtain convergence to some yet unknown function ζ: $\psi_t^l \in C \xrightarrow{L^\infty} \zeta \in C$. Since $D\psi_l \to D\psi$ piecewise almost everywhere by Lemma 5, we have $\zeta = \psi_t$ almost everywhere, therefore ψ_t has the continuous representative ζ and we can say $\psi_t \in C(Q_T)$.

Notice that the convergence argument for ψ_l is valid everywhere on Q_T, not just on the subset $V^+ \cap M$. This is important since it implies $\sigma(u_x) = \psi_t \in C(Q_T)$. The continuity of $\sigma(u_x)$ is necessary for u to be a weak solution.

The restriction to the subset $V^+ \cap M$ is necessary to apply the relation (4.14) and get $u_x \in C(V^+ \cap M)$. Since $V^+ \cap M$ is dense in $[t_0,T] \times [0,l]$, we can extend u_x continuously to find $d^+(t,x) \in C([t_0,T] \times [0,l])$. Analogously we find d^- and the first claim of the theorem is proved.

For differentiability in x-direction consider the sequence $(\psi_l)_t(t,x)$ for a fixed time t. With the definition of $\psi_l(t,x)$ as the integral in x-direction of the function u_l, compare (3.18), we can calculate $((\psi_l)_t)_x = (u_l)_t \in L^\infty$, at least for every fixed l. We obtain $(\psi_l)_t \in W^{1,\infty}(t \times [0,l])$. Notice that the argument is valid for all t and for all $x \in [0,l]$. Since

$$\begin{aligned} \|(u_l)_t\|_{L^\infty} &\leq \|(u_0)_t\|_{L^\infty} + \sum \|(u_g)_t\|_{L^\infty} \\ &= \|(u_0)_t\|_{L^\infty} + \sum a_i^{l,k} \cdot \delta_i^l \\ &\leq \|(u_0)_t\|_{L^\infty} + 2z^+ \sum \frac{1}{2^l} \leq c, \end{aligned}$$

the sequence $(\psi_l)_t$ is uniformly Lipschitz with respect to the x-variable, hence the limit is also Lipschitz and $\psi_t \in W^{1,\infty}(t \times [0,l])$. By the relation (4.14) we now have $u_x \in W^{1,\infty}$ with respect to x on $V^+ \cap M$. Since this set is dense, the extension has the same regularity, $d^+ \in W^{1,\infty}$. It follows that d_x exists weakly and $d_x \in L^\infty$. □

Let us add two general comments on the just proven results:

Remark I: The construction of Section 3 could also be carried out with $\sigma^*(s) = s$. The approximate solution u^* is then a solution to the heat equation. This approach is simpler since maximum principles, smoothing property and asymptotics are readily available. Further, the constructive process will take place in all of the domain since σ^* never touches the curve of σ, therefore the approximate solution is at no point a solution to the original problem. This implies that all structural results are true from $t = 0$.

Remark II: The method to construct sequences converging to Lipschitz functions has been developed by Zhang for the one-dimensional Perona-Malik equation [27] and generalized to similar diffusion equations of forward-backward type [28]. For these equations the construction appears when the derivative is close to the critical value separating the forward and backward parts of the diffusion. A typical example with critical value 1 is

$$u_t = [\sigma(u_x)]_x = \left(\frac{u_x}{1+u_x^2}\right)_x .$$

The proofs in this section require only the constructive process in some open set U. The constructions in [27], [28] to create solutions of the Perona-Malik equation and related problems are the same ones as used here. Because of this the structural results and their proofs transfer.

Further, these structural results are not exclusive to time dependent problems. They translate to Lipschitz solutions of elliptic problems achieved with similar constructions, for example the solutions constructed in [16], [17].

5. Conclusion

We have shown that the construction developed for the Perona-Malik equation and similar forward-backward diffusion equations [27], [29] works for singular logarithmic diffusion equations as well. All Lipschitz functions constructed in this way contain microstructure in the sense, that they are not locally smooth. We have further given an explicit description of this microstructure as the partitioning of two functions with higher regularity on two irregular dense sets. This gives insight into how such functions can solve diffusion problems even in cases where no regular solution exists, like the one-dimensional Perona-Malik equation and the one-dimensional singular fast diffusion equation.

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