A multidimensional distribution sampling theorem

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Abstract. Using Bochner-Riesz means we get a multidimensional sampling theorem for band-limited functions with polynomial growth, that is, for functions which are the Fourier transform of compactly supported distributions.

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1. Introduction

Let $S \in L^2(\mathbb{R})$ have support in $[-1/2, 1/2]$ and let $\mathcal{F}S(y) := \int_{\mathbb{R}} S(x)\, \mathrm{e}^{-2\pi ixy}\, dx$ be its Fourier transform. The classical sampling theorem states that

$$\mathcal{F}S(y) = \sum_{m=-\infty}^{+\infty} \mathcal{F}S(m) \frac{\sin \pi(y-m)}{\pi(y-m)}$$

uniformly on $\mathbb{R}$ (see [2] for the history of this result). When S is a distribution with support in $]-1/2, 1/2[$, its Fourier transform, which is still a function, is also determined by its values at the points $m \in \mathbb{Z}$; but the series above does not converge. However, it is possible to generalize the sampling formula in this case: Walter showed in 1988 that the series is summable in Cesàro and Abel means to $\mathcal{F}S(y)$ uniformly on bounded sets in $\mathbb{R}$ [5, Corollary 4.4, p. 1203], [6, Theorem, p. 353] ([5] was improved by Liu in 1996 [3, Theorem 5, p. 1155]).

Although extensions of the classical sampling theorem to several real variables are well known [2, pp. 76–82], the case of distributions in several variables does not seem to have been much studied, perhaps because of the mainly one-dimensional tools in the proofs of Walter and Liu.

Using Bochner-Riesz means we prove here the following multidimensional generalization.

Theorem. *Let V be a convex bounded open set in $\mathbb{R}^n$ such that $-V = V$ and $2V \cap \mathbb{Z}^n = \{0\}$. Let S be a distribution on $\mathbb{R}^n$ of order p with support in V. Then, for $k > p + (n-1)/2$,*

$$\mathcal{F}S(y) = \lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n,\, \|m\|\leq N} (1 - \|m\|^2/N^2)^k\, \mathcal{F}S(m)\, \mathcal{F}\chi_V(y-m),$$

uniformly on every compact set in $\mathbb{R}^n$ (with χ_V the indicator function of V).

If V is the cube $]-1/2, 1/2[^n$ this gives

$$\mathcal{F}S(y) = \lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n,\, \|m\|\le N} (1 - \|m\|^2/N^2)^k\, \mathcal{F}S(m) \prod_{j=1}^{n} \frac{\sin \pi(y_j - m_j)}{\pi(y_j - m_j)}\,;$$

and if V is the ball $B(0, 1/2)$ it gives

$$\mathcal{F}S(y) = \lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n,\, \|m\|\le N} (1 - \|m\|^2/N^2)^k\, \mathcal{F}S(m)\, \frac{J_{n/2}(\pi\|y - m\|)}{(2\|y - m\|)^{n/2}}\,,$$

where J_ν is the Bessel function of the first kind and order ν.

The proof of the theorem is given in Section 3. In Section 2 we introduce useful notations and study in some detail the Bochner-Riesz kernel.

2. Preliminaries

If f is a function on $\mathbb{R}^n$ and $a \in \mathbb{R}^n$, we write, for all $x \in \mathbb{R}^n$, $f^\vee(x) := f(-x)$, $\tau_a f(x) := f(x - a)$ and $\mathrm{e}_a(x) := \mathrm{e}^{2\pi i a\cdot x}$; moreover, if f is real valued we put $f_+(x) := \max(f(x), 0)$. We write $\omega_n := 2\pi^{n/2}/\Gamma(n/2)$, so that $\omega_n r^n/n$ is the Lebesgue measure (volume) of any ball $B(a, r)$ in $\mathbb{R}^n$ with radius $r > 0$.

Let now $k \ge 0$ and $N > 0$. According to [4, Theorem IV.4.15],

$$\mathcal{F}[(1 - \|x\|^2/N^2)_+^k](y) = \frac{\Gamma(k+1)}{\pi^k}\, \frac{N^{-k+n/2}}{\|y\|^{k+n/2}}\, J_{k+n/2}(2\pi N\|y\|)$$

for any $y \in \mathbb{R}^n$. We now put

$${}_kK_N^n(y) := \frac{\Gamma(k+1)}{\pi^k}\, \frac{N^{-k+n/2}}{\|y\|^{k+n/2}}\, J_{k+n/2}(2\pi N\|y\|);$$

this defines ${}_kK_N^n$ not only on $\mathbb{R}^n$ but in fact on every $\mathbb{R}^q$, $q \in \mathbb{N}$. Clearly ${}_kK_N^n$ is analytic. If we differentiate it in $\mathbb{R}^n$, we find, because $(z^{-\nu}J_\nu(z))' = -z^{-\nu}J_{\nu+1}(z)$, that $(\partial/\partial_j)\,{}_kK_N^n(y) = -2\pi y_j \cdot {}_kK_N^{n+2}(y)$. Hence, for every multiindex $\alpha \in \mathbb{N}_0^n$ and all $y \in \mathbb{R}^n$,

$$D^\alpha\, {}_kK_N^n(y) = \sum_{r=0}^{|\alpha|} (-2\pi)^r P_r^\alpha(y) \cdot {}_kK_N^{n+2r}(y),$$

where the P_r^α are polynomials. We immediately have $P_0^0 = 1$. Put $P_r^\alpha := 0$ if $r < 0$ or $r > |\alpha|$; the P_r^α can be defined by the recurrence formula

$$P_l^{\alpha+e_j}(y) = y_j \cdot P_{l-1}^\alpha(y) + (\partial P_l^\alpha/\partial y_j)(y).$$

From this we get $P_{|\alpha|}^\alpha(y) = y^\alpha$ and, by induction, $2(|\alpha| - r)P_r^\alpha(y) = \Delta P_{r+1}^\alpha(y)$ if $r = 0, \ldots, |\alpha| - 1$. We then find $P_{|\alpha|-l}^\alpha(y) = \Delta^l y^\alpha / 2^l l!$. In particular, P_r^α is a polynomial of degree $\le r$ which only depends on α and r. Hence there exists $c_r^\alpha > 0$ such that $|P_r^\alpha(y)| \le c_r^\alpha(1 + \|y\|^r)$ for all $y \in \mathbb{R}^n$.

Given any $\nu \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, there exists $\ell_\nu > 0$ such that $|J_\nu(x)| < \ell_\nu/\sqrt{x}$ for all $x > 0$ [7, p. 199]. Put $L_k := \max\{\ell_\nu : \nu \in \frac{1}{2}\mathbb{Z}_{\geq 0}, \nu \leq \frac{n}{2} + k + p\}$. Then, if $0 \leq r \leq p$,

$$|{}_kK_N^{n+2r}(y)| \leq \frac{\Gamma(k+1)L_k}{\sqrt{2}\,\pi^{k+1/2}} \frac{N^{r-k+(n-1)/2}}{\|y\|^{r+k+(n+1)/2}}$$

for all $y \in \mathbb{R}^n \setminus \{0\}$. Hence, for any multiindex α with $|\alpha| \leq p$ and for all $y \in \mathbb{R}^n \setminus \{0\}$, we have:

$$|D^\alpha{}_kK_N^n(y)| \leq C_k^\alpha \frac{N^{|\alpha|-k+(n-1)/2}}{\|y\|^{k+(n+1)/2}},$$

where the constant $C_k^\alpha > 0$ also depends on p. It follows that the function ${}_kK_N^n$ is integrable on $\mathbb{R}^n$ if $k > \frac{n-1}{2}$, in which case all its derivatives are also integrable and moreover $(1 - \|x\|^2/N^2)_+^k = \mathcal{F}{}_kK_N^n(x)$ for any $x \in \mathbb{R}^n$.

3. Proof

We divide the proof of the theorem in seven steps.

Step 1. We have just seen that $(1 - \|m\|^2/N^2)_+^k = \mathcal{F}{}_kK_N^n(m)$. Moreover $\mathcal{F}\chi_V(m - y) = \mathcal{F}(\chi_V\,\mathrm{e}_y)(m)$. Since $\chi_V\,\mathrm{e}_y$ is integrable with compact support and ${}_kK_N^n$ is integrable and C^∞, their convolution, ${}_kK_N^n \star \chi_V\,\mathrm{e}_y$, is integrable and C^∞ with, for any multiindex α, $D^\alpha({}_kK_N^n \star \chi_V\,\mathrm{e}_y) = (D^\alpha{}_kK_N^n) \star \chi_V\,\mathrm{e}_y$. Hence $S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y) \in C^\infty(\mathbb{R}^n)$ and, for all $a \in \mathbb{R}^n$,

$$[S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)](a) = S(\tau_a[{}_kK_N^n \star \chi_V\,\mathrm{e}_y]^\vee).$$

From

$$\mathcal{F}[S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)] = \mathcal{F}S \cdot \mathcal{F}({}_kK_N^n \star \chi_V\,\mathrm{e}_y) = \mathcal{F}S \cdot \mathcal{F}{}_kK_N^n \cdot \mathcal{F}(\chi_V\,\mathrm{e}_y)$$

we deduce

$$\sum_{m\in\mathbb{Z}^n} (1 - \|m\|^2/N^2)_+^k\, \mathcal{F}S(m)\,\mathcal{F}\chi_V(y-m) = \sum_{m\in\mathbb{Z}^n} \mathcal{F}[S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)](m).$$

Step 2. There exists $0 \leq \lambda < 1$ such that $\operatorname{supp} S \subset \lambda V$. We define $U := \lambda V$; hence $\operatorname{supp} S \subset U \subset \overline{U} \subset V$. By assumption there exists $C > 0$ such that, for all $\varphi \in C^\infty(\mathbb{R}^n)$,

$$|S(\varphi)| \leq C \sup_{|\alpha|\leq p} \sup_{x\in\overline{U}} |D^\alpha\varphi(x)|. \tag{1}$$

We also define $\delta := d(\overline{U} + \overline{V}, \mathbb{Z}^n \setminus \{0\})$ and $\eta := d(\overline{U} + V^c, \{0\})$; remark that δ, $\eta > 0$. Finally, we choose $r > 0$ such that $\overline{U} + \overline{V} \subset \overline{B(0,r)}$.

Step 3. We have, for $a \in \mathbb{R}^n$,

$$\begin{aligned} |[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](a)| &= |S(\tau_a[{}_kK_N^n\star\chi_V\,\mathrm{e}_y]^\vee)| \\ &\le C\sup_{|\alpha|\le p}\sup_{x\in\overline{U}}|D^\alpha\,\tau_a[{}_kK_N^n\star\chi_V\,\mathrm{e}_y]^\vee(x)| \\ &= C\sup_{|\alpha|\le p}\sup_{x\in\overline{U}}|[(D^\alpha\,{}_kK_N^n)\star\chi_V\,\mathrm{e}_y](a-x)|. \end{aligned}$$

Take now $\|a\|\ge 2r$, so that in particular $a-\overline{U}-\overline{V}\subset B(0,\|a\|-r)^c$ and $\|a\|-r\ge\|a\|/2$. We get, for $x\in\overline{U}$,

$$\begin{aligned} |[(D^\alpha\,{}_kK_N^n)\star\chi_V\,\mathrm{e}_y](a-x)| &= \left|\int_{\mathbb{R}^n}(D^\alpha\,{}_kK_N^n)(t)(\chi_V\,\mathrm{e}_y)(a-x-t)\,dt\right| \\ &\le \int_{a-\overline{U}-\overline{V}}|(D^\alpha\,{}_kK_N^n)(t)|\,dt \\ &\le \sup_{\|t\|\ge\|a\|-r}|D^\alpha\,{}_kK_N^n(t)|\cdot\omega_n r^n/n \\ &\le C_k^\alpha\cdot 2^{k+(n+1)/2}\,\frac{N^{|\alpha|-k+(n-1)/2}}{\|a\|^{k+(n+1)/2}}\,\frac{\omega_n r^n}{n}\,. \end{aligned}$$

Hence, for all $a\in\mathbb{R}^n$ with $\|a\|\ge 2r$,

$$|[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](a)|\le\widetilde{C}_k^p\,\frac{N^{p-k+(n-1)/2}}{\|a\|^{k+(n+1)/2}}\,,$$

where the constant $\widetilde{C}_k^p>0$ also depends on C, r and n. Since $k>p+\frac{n-1}{2}$, $k+\frac{n+1}{2}>n$ and we may apply the Poisson summation formula [4, Corollary VII.2.6]:

$$\sum_{m\in\mathbb{Z}^n}\mathcal{F}[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](m)=\sum_{m\in\mathbb{Z}^n}[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](m).$$

Step 4. Because $k>p+\frac{n-1}{2}$, we get

$$\lim_{N\to+\infty}\sum_{\substack{m\in\mathbb{Z}^n\\ \|m\|\ge 2r}}|[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](m)|\le\lim_{N\to+\infty}\sum_{\substack{m\in\mathbb{Z}^n\\ \|m\|\ge 2r}}\widetilde{C}_k^p\,\frac{N^{p-k+(n-1)/2}}{\|m\|^{k+(n+1)/2}}=0.$$

Take now $m\in\mathbb{Z}^n$ with $0<\|m\|<2r$. From Step 3 we know that

$$|[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](m)|\le C\sup_{|\alpha|\le p}\sup_{t\in m-\overline{U}-\overline{V}}|(D^\alpha\,{}_kK_N^n)(t)|\cdot\omega_n r^n/n.$$

From Section 2 we deduce that

$$\sup_{t\in m-\overline{U}-\overline{V}}|(D^\alpha\,{}_kK_N^n)(t)|\le C_k^\alpha\,\frac{N^{|\alpha|-k+(n-1)/2}}{\delta^{k+(n+1)/2}}\,.$$

Therefore

$$\lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n\setminus\{0\}} [S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](m)=0,$$

uniformly (in y) on the whole $\mathbb{R}^n$.

Step 5. We must now study the limit

$$\lim_{N\to+\infty}[S\star({}_kK_N^n\star\chi_V\,\mathrm{e}_y)](0)=\lim_{N\to+\infty} S([{}_kK_N^n\star\chi_V\,\mathrm{e}_y]^\vee).$$

We use an auxiliary function $\psi\in C^\infty(\mathbb{R}^n)$ with compact support such that $\psi=1$ on V and $0\le\psi\le 1$. Let $W=B(0,\rho)\supset\operatorname{supp}\psi$. We have $0\le\psi-\chi_V\le 1$ and $(\psi-\chi_V)(u)=0$ if $u\in V\cup W^c$. Then, for all $x\in\overline{U}$,

$$\begin{aligned}
|D^\alpha[{}_kK_N^n\star(\psi-\chi_V)\,\mathrm{e}_y]^\vee(x)| &= \left|\int_{\mathbb{R}^n} D^\alpha{}_kK_N^n(t)\cdot\{(\psi-\chi_V)\,\mathrm{e}_y\}(-x-t)\,dt\right|\\
&\le \int_{t\in-\overline{U}-(\overline{W}\setminus V)} |D^\alpha{}_kK_N^n(t)|\,dt;
\end{aligned}$$

and we get

$$\begin{aligned}
S([{}_kK_N^n\star(\psi-\chi_V)\,\mathrm{e}_y]^\vee)| &\le C\sup_{|\alpha|\le p}\sup_{x\in\overline{U}}|D^\alpha[{}_kK_N^n\star(\psi-\chi_V)\,\mathrm{e}_y]^\vee(x)|\\
&\le C\cdot\operatorname{vol}(\overline{U}+(\overline{W}\setminus V))\cdot\sup_{|\alpha|\le p} C_k^\alpha\,\frac{N^{|\alpha|-k+(n-1)/2}}{\eta^{k+(n+1)/2}}\,.
\end{aligned}$$

Hence

$$\lim_{N\to+\infty} S([{}_kK_N^n\star(\psi-\chi_V)\,\mathrm{e}_y]^\vee)=0$$

uniformly (in y) on all $\mathbb{R}^n$.

Step 6. We will now show that

$$\lim_{N\to+\infty} S([{}_kK_N^n\star\psi\,\mathrm{e}_y]^\vee)=S([\psi\,\mathrm{e}_y]^\vee)$$

uniformly (in y) on every compact set L in $\mathbb{R}^n$. In view of (1) it will suffice to prove that, for every multiindex α with $|\alpha|\le p$,

$$\lim_{N\to+\infty}\sup_{x\in\mathbb{R}^n}|[D^\alpha({}_kK_N^n\star\psi\,\mathrm{e}_y)-D^\alpha(\psi\,\mathrm{e}_y)](x)|=0,$$

uniformly in $y\in L$. But since $D^\alpha({}_kK_N^n\star\psi\,\mathrm{e}_y)={}_kK_N^n\star D^\alpha(\psi\,\mathrm{e}_y)$, we only have to show that, given any $\varphi\in C^\infty(\mathbb{R}^n)$ with compact support,

$$\lim_{N\to+\infty}\sup_{x\in\mathbb{R}^n}|[({}_kK_N^n\star\varphi\,\mathrm{e}_y)-\varphi\,\mathrm{e}_y](x)|=0,$$

uniformly in $y \in L$. Now

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} |[({}_kK_N^n \star \varphi\,\mathrm{e}_y) - \varphi\,\mathrm{e}_y](x)| \\
&= \sup_{x\in\mathbb{R}^n} |\mathcal{F}\{(1-\|t\|^2/N^2)_+^k \cdot \overline{\mathcal{F}}(\varphi\,\mathrm{e}_y) - \overline{\mathcal{F}}(\varphi\,\mathrm{e}_y)\}(x)| \\
&\le \int_{\mathbb{R}^n} |(1-\|t\|^2/N^2)_+^k - 1| \cdot |\overline{\mathcal{F}}\varphi(t+y)|\,dt,
\end{aligned}$$

which tends to 0 uniformly in $y \in L$ when $N \to +\infty$ by the dominated convergence theorem, since $\overline{\mathcal{F}}(\varphi)$ vanishes at infinity.

Step 7. We deduce from the last two steps that

$$\lim_{N\to+\infty} [S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)](0) = S([\psi\,\mathrm{e}_y]^\vee)$$

uniformly (in y) on every compact set in $\mathbb{R}^n$. Now

$$S([\psi\,\mathrm{e}_y]^\vee) = S(x \mapsto \psi(-x)\,\mathrm{e}^{2\pi i(-x|y)}) = S(x \mapsto \mathrm{e}^{-2\pi i(x|y)}) = \mathcal{F}S(y),$$

since $\psi = 1$ on $V = -V \supset U \supset \operatorname{supp} S$. Finally we calculate:

$$\begin{aligned}
&\lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n} (1-\|m\|^2/N^2)_+^k\, \mathcal{F}S(m)\, \mathcal{F}\chi_V(y-m) \\
&= \lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n} \mathcal{F}[S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)](m) \\
&= \lim_{N\to+\infty} \sum_{m\in\mathbb{Z}^n} [S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)](m) \\
&= \lim_{N\to+\infty} [S \star ({}_kK_N^n \star \chi_V\,\mathrm{e}_y)](0) \\
&= \mathcal{F}S(y),
\end{aligned}$$

uniformly on every compact set in $\mathbb{R}^n$, and the proof is complete.

Remarks. 1. The theorem is also true if we use $(1-\|m\|/N)_+^k$ instead of $(1-\|m\|^2/N^2)_+^k$; however, the asymptotic estimate of $D^\alpha\mathcal{F}[(1-\|x\|/N)_+^k]$ is more difficult to obtain (see [1]).

2. The theorem is false if we only assume $\operatorname{supp} S \subset \overline{V}$. For example, when $n = 1$ and $V =]-1/2, 1/2[$, $S = \delta_{-1/2} - \delta_{1/2}$ (where δ_q is the Dirac measure at q) gives $\mathcal{F}S(y) = 2i\sin\pi y$, which is null on every $m \in \mathbb{Z}$.

3. The theorem is false if we only assume $k = p + (n-1)/2$: consider the counter-example on $\mathbb{R}$ of $S = \delta_0^{(l)}$ $(l \in \mathbb{Z}_{\ge 0})$.

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