r-Realcompact spaces

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Abstract. A new generalization of realcompactness based on ultrafilters of regular F_σ-subsets is introduced. Its relationship with realcompactness, almost realcompactness, almost* realcompactness, c-realcompactness is examined. Some of the properties of the newly introduced space is studied as well.

Keywords: regular F_σ-subsets, almost realcompactness, almost* realcompactness, r-weak cb, regular Oz, regular countably paracompact

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Introduction

Throughout this paper all the spaces are assumed to be Tychonoff (completely regular and Hausdorff). For basic definitions of zero-sets, cozero-sets, filter, ultrafilter, prime filter etc. we refer to [7]. An ultrafilter $\mathcal{F}$ is fixed *iff* $\bigcap \bar{\mathcal{F}} \neq \emptyset$. A non-empty family $\mathcal{F}$ of sets is said to have countable (resp. closed countable) intersection property provided that the intersection of any countable number of members (resp. of the closures of countable number of members) of $\mathcal{F}$ is non-empty. Likewise, finite intersection property (*fip*) may be defined.

We know that a space X is realcompact [9] if every ultrafilter of zero-sets with countable intersection property (*cip*) is fixed. A space X is said to be almost [5] (resp. almost* [15]) realcompact if every ultrafilter of open sets (resp. cozero-sets) with closed countable intersection property (*ccip*) is fixed.

The concept of regular G_δ-subsets was introduced by J. Mack [11]. A subset H of a topological space X is called a regular G_δ-subset if H is an intersection of a sequence of closed sets whose interiors contain H (or, equivalently, if $H = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \mathrm{Cl}_X\, G_n$, where each G_n is open in X). The complement of a regular G_δ-subset is called a regular F_σ-subset, i.e., a subset V of a space X is said to be a regular F_σ if $V = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \mathrm{Int}_X\, F_n$, where each F_n is closed in X. Properties of regular G_δ and regular F_σ-subsets have been studied in [1]. Every zero-set is a regular G_δ and every cozero-set is a regular F_σ. The intersection of two regular F_σ-subsets is regular F_σ and the countable union of regular F_σ-subsets is also regular F_σ. The sets of all zero-sets and regular F_σ-subsets of X are respectively denoted by $\mathcal{Z}(X)$ and $\mathcal{R}_f(X)$.

Frolík [5] and Schommer-Swardson [15] introduced and studied almost realcompactness and almost* realcompactness using the ultrafilter of open sets and cozero-sets, respectively. It would be interesting to study the structure defined in a similar fashion with the help of regular F_σ-subsets, when it is well known that every cozero-set is regular F_σ and every regular F_σ is open. The motivation of the present paper is to investigate whether strengthening (resp. weakening) the original definition of almost (resp. almost*) realcompactness with the introduction of regular F_σ-subsets produce a stronger (resp. weaker) result than that of almost (resp. almost*) realcompactness. However, we shall see that almost realcompactness, almost* realcompactness and r-realcompactness are all independent in Tychonoff spaces. Some of the properties of r-realcompact spaces have also been studied.

Definition 1. A space X is said to be *r-realcompact*, if whenever $\mathcal{F}$ is an ultrafilter of regular F_σ-subsets of X such that $\bar{\mathcal{F}}$ has the *cip*, then $\bigcap \bar{\mathcal{F}} \neq \emptyset$.

We first see that r-realcompactness is indeed a weak realcompactness condition. For this we need the following lemma.

Lemma 2. *Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X and $\mathcal{Z} = \{Z : Z$ is a zero-set of X and $Z \supseteq F$ where $F \in \mathcal{F}\}$. Then $\mathcal{Z}$ is a prime z-filter.*

Proof: Clearly $\mathcal{Z}$ is a z-filter. To prove that $\mathcal{Z}$ is prime, let us define $\tilde{\mathcal{Z}} = \{X - Z : Z \in \mathcal{Z}(X) - \mathcal{Z}\}$. If we can show that $\tilde{\mathcal{Z}}$ is a filter, then by Proposition 1(b) of [6], both $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ will be prime filters. For this we check the following:
(i) Clearly $\emptyset \notin \tilde{\mathcal{Z}}$.
(ii) Let $X - Z_1$, $X - Z_2 \in \tilde{\mathcal{Z}}$, then Z_1 and Z_2 do not contain any F, $F \in \mathcal{F}$. Under this condition, using the primeness of $\mathcal{F}$ it can be easily seen that $Z_1 \cup Z_2$ does not contain any $F \in \mathcal{F}$ and hence does not belong to $\mathcal{Z}$. Therefore $(X - Z_1) \cap (X - Z_2) \in \tilde{\mathcal{Z}}$.
(iii) Let $X - Z \in \tilde{\mathcal{Z}}$, then Z does not contain any F, for $F \in \mathcal{F}$ and also let $X - Z'$ contains $X - Z$. Then $Z' \subseteq Z$ and hence neither Z' contains any $F \in \mathcal{F}$. Therefore $X - Z' \in \tilde{\mathcal{Z}}$. □

Theorem 3. *Every realcompact space is r-realcompact.*

Proof: Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X with $\bigcap \bar{\mathcal{F}} = \emptyset$. Let $\mathcal{Z} = \{Z : Z$ is a zeroset of X and $F \subseteq Z$, where $F \in \mathcal{F}\}$. By Lemma 2, $\mathcal{Z}$ is a prime z-filter.

Since in a completely regular space the zero-sets form a base for the closed sets, we have $\bigcap \mathcal{Z} = \bigcap\{\mathrm{Cl}_X F : F \in \mathcal{F}\} = \bigcap \bar{\mathcal{F}} = \emptyset$. Therefore there exists a countable collection $\{Z_n : n \in \mathbb{N}\} \subseteq \mathcal{Z}$ such that $\bigcap_{n\in\mathbb{N}} Z_n = \emptyset$. By construction, for each $Z_n \in \mathcal{Z}$, there exists $F_n \in \mathcal{F}$ such that $F_n \subseteq Z_n$, i.e., $\mathrm{Cl}_X F_n \subseteq Z_n$. Thus $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X F_n \subseteq \bigcap_{n\in\mathbb{N}} Z_n = \emptyset$, i.e., $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X F_n = \emptyset$. Hence X is r-realcompact. □

To show that the converse of Theorem 3 is not always true, the following lemma will assist us in providing a counterexample.

Lemma 4. *Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X with ccip and let $U \in \mathcal{F}$. If $\mathrm{Cl}_X U$ is r-realcompact, then $\bigcap \bar{\mathcal{F}} \neq \emptyset$.*

Proof: Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X with ccip and let $U \in \mathcal{F}$. Further assume that $\operatorname{Cl}_X U$ is r-realcompact. We first show that if $H \in \mathcal{F}|\operatorname{Cl}_X U$, then $H \cap U \in \mathcal{F}$. Let $H \in \mathcal{F}|\operatorname{Cl}_X U$, then $H = V \cap \operatorname{Cl}_X U$, for $V \in \mathcal{F}$. So $H \cap U = V \cap \operatorname{Cl}_X U \cap U = V \cap U$, which further shows that $H \cap U = V \cap U \in \mathcal{F}$.

We next show that $\mathcal{F}|\operatorname{Cl}_X U$ is a base for an ultrafilter of regular F_σ-subsets on $\operatorname{Cl}_X U$. It is straightforward to check that $\mathcal{F}|\operatorname{Cl}_X U$ has fip and is also closed under fip. Let W be a regular F_σ-subset of $\operatorname{Cl}_X U$ and assume that $W \cap G \neq \emptyset$, for all $G \in \mathcal{F}|\operatorname{Cl}_X U$. In particular, $W \cap U \neq \emptyset$. Now since W is regular F_σ in $\operatorname{Cl}_X U$, by Theorem 28, $W \cap U$ is regular F_σ in U. Again U is regular F_σ in X and hence by Corollary 32, $W \cap U$ is a regular F_σ in X. Let $G' \in \mathcal{F}$. Then $G' \cap U \in \mathcal{F}$ and so $G' \cap U = (G' \cap U) \cap \operatorname{Cl}_X U \in \mathcal{F}|\operatorname{Cl}_X U$, which implies that $W \cap U \cap G' \neq \emptyset$. We conclude that $W \cap U \in \mathcal{F}$ and $W \cap U \in \mathcal{F}|\operatorname{Cl}_X U$. Since $W \cap U \subseteq W$, our claim is proved.

Let $\mathcal{G}$ be an ultrafilter of regular F_σ-subsets on $\operatorname{Cl}_X U$ with $\mathcal{F}|\operatorname{Cl}_X U \subseteq \mathcal{G}$. We show that $\mathcal{G}$ has ccip. Let $\{V_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$. Each $V_n \supseteq G_n$, where $G_n \in \mathcal{F}|\operatorname{Cl}_X U$. Now $\{G_n \cap U : n \in \mathbb{N}\} \subseteq \mathcal{F}$. Thus there is a point $p \in \bigcap_{n\in\mathbb{N}} \operatorname{Cl}_X(G_n \cap U)$. Clearly $p \in \bigcap_{n\in\mathbb{N}} \operatorname{Cl}_{\operatorname{Cl}_X U} V_n$. Since $\operatorname{Cl}_X U$ is r-realcompact, there is $p \in \bigcap \bar{\mathcal{G}}$. We claim that $p \in \bigcap \bar{\mathcal{F}}$. Let $P \in \mathcal{F}$. Then $P \cap U \in \mathcal{F}$ and so $P \cap U \in \mathcal{F}|\operatorname{Cl}_X U \subseteq \mathcal{G}$ and hence $p \in \operatorname{Cl}_X(P \cap U) \subseteq \operatorname{Cl}_X P$. This completes the proof. $\square$

Example 5. The Mysior plane is not realcompact, but is r-realcompact.

In [13], Mysior provides an example of an almost realcompact space that is not realcompact. He defines a topology on $\mathbb{R}^2$ by first isolating the points not on the x-axis. For every point $(x, 0)$ on the x-axis, a base of neighborhoods is defined to be the family $\{U_n(x,0) : n \in \mathbb{N}\}$, where each $U_n(x,0)$ is the union of three segments: $\{(x,y) : -\frac{1}{n} < y < \frac{1}{n}\}$, $\{(x+1+y, y) : 0 < y < \frac{1}{n}\}$ and $\{(x+\sqrt{2}+y, -y) : 0 < y < \frac{1}{n}\}$. Mysior demonstrates that the half-planes $X_+ = \{(x,y) : y \geq 0\}$ and $X_- = \{(x,y) : y \leq 0\}$ are both closed in X and realcompact, but their union $X = X_+ \cup X_-$ is not realcompact.

To show that X is r-realcompact, let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X with ccip. We see that the open half planes $U = \{(x,y) : y > 0\}$ and $L = \{(x,y) : y < 0\}$ are both cozero-sets and hence regular F_σ-subsets in X. Clearly $f : X \to \mathbb{R}$ defined by $f(x,y) = y$, if $(x,y) \in U$ and $f(x,y) = 0$, elsewhere, is continuous on X and also cozero $f = U$. Further $U \cup L$ is dense in X. Therefore, either U or L must belong to $\mathcal{F}$. Without loss of generality, assume $U \in \mathcal{F}$. But $X_+ = \operatorname{Cl}_X U$ is realcompact and hence r-realcompact and so $\mathcal{F}$ must be fixed by using the Lemma 4. Consequently X is r-realcompact.

However, we show that r-realcompactness with some additional condition implies realcompactness. For this we first define the following:

Definition 6. A space X is *r-weak cb* if for every decreasing sequence $\{P_n : n \in \mathbb{N}\}$ of regular F_σ-subsets with $\bigcap_{n\in\mathbb{N}} \operatorname{Cl}_X P_n = \emptyset$, there exists a decreasing sequence $\{Z_n : n \in \mathbb{N}\}$ of zero-sets such that $P_n \subseteq Z_n$ for every n, and $\bigcap_{n\in\mathbb{N}} Z_n = \emptyset$.

Remark 7. We recall here that a space X is a cb space [10] if for every decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of closed sets with $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, there exists a decreasing sequence $\{Z_n : n \in \mathbb{N}\}$ of zero-sets such that $F_n \subseteq Z_n$ for every n, and $\bigcap_{n\in\mathbb{N}} Z_n = \emptyset$. The concepts of weak cb and almost weak cb were introduced by Mack-Johnson [12] and Schommer-Swardson [15] respectively, as generalizations of cb spaces. A space X is weak cb space if for a given decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of regular closed subsets of X with empty intersection, there exists a decreasing sequence $\{Z_n : n \in \mathbb{N}\}$ of zero-sets with empty intersection such that $Z_n \supseteq F_n$ for each $n \in \mathbb{N}$. Similarly a space X is almost weak cb space if for a given decreasing sequence $\{P_n : n \in \mathbb{N}\}$ of cozero-sets of X with $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X P_n = \emptyset$, there exists a decreasing sequence $\{Z_n : n \in \mathbb{N}\}$ of zero-sets with empty intersection such that $Z_n \supseteq \mathrm{Cl}_X P_n$ for each $n \in \mathbb{N}$. It is straightforward to show that every weak cb space is r-weak cb and every r-weak cb space is almost weak cb.

Note 8. The hierarchy of the different spaces mentioned is as follows:

cb space $\Rightarrow$ weak cb $\Rightarrow$ r-weak cb $\Rightarrow$ almost weak cb.

The authors intend to study the properties of r-weak cb spaces elsewhere.

Theorem 9. *If X is r-realcompact and r-weak cb, then X is realcompact.*

Proof: Let $\mathcal{F}$ be a free z-ultrafilter on X. Let $\mathcal{B} = \{P : P$ is regular F_σ-subset and there exists $Z \in \mathcal{F}$ with $Z \subseteq P\}$. Clearly $\mathcal{B}$ is a filter of regular F_σ-subsets of X. Let $\mathcal{G}$ be an ultrafilter of regular F_σ-subsets of X containing $\mathcal{B}$. We show that $\bigcap \bar{\mathcal{G}} = \emptyset$. Let $p \in X$. Since $\mathcal{F}$ is free, $p \in X - Z$ for some $Z \in \mathcal{F}$. Again X is completely regular, so there exists a cozero-set Q and a zero-set Z' such that $p \in Q \subseteq Z' \subseteq X - Z$ [7]. Thus $Z \subseteq X - Z'$ and so $X - Z' \in \mathcal{G}$ as $X - Z'$ is a regular F_σ (being a cozero-set). But $p \notin \mathrm{Cl}_X(X - Z')$, as $p \in Q$ and $Q \cap (X - Z') = \emptyset$. Therefore $p \notin \bigcap \bar{\mathcal{G}}$. Since p is arbitrary, $\bigcap \bar{\mathcal{G}} = \emptyset$.

Again X is r-realcompact and $\bigcap \bar{\mathcal{G}} = \emptyset$, thus there must exist a collection $\{P_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$ with $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X P_n = \emptyset$. Let $V_n = \bigcap\{P_i : i \leq n\}$. Then $\{V_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$ is a decreasing sequence of regular F_σ-subsets of X with $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X V_n \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X P_n = \emptyset$, and hence $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X V_n = \emptyset$. Since X is r-weak cb, there exists a collection $\{Z_n : n \in \mathbb{N}\}$ of zero-sets with $\mathrm{Cl}_X V_n \subseteq Z_n$ for each n, and $\bigcap_{n\in\mathbb{N}} Z_n = \emptyset$. Now we show that each Z_n meets every member of $\mathcal{F}$. If not, there exists a set $Z \in \mathcal{F}$ with $Z \cap Z_n = \emptyset$ for some n. Then $Z \subseteq X - Z_n$, and so $X - Z_n \in \mathcal{B} \subseteq \mathcal{G}$. Again $\mathrm{Cl}_X V_n \subseteq Z_n$ and so $\mathrm{Cl}_X V_n \cap (X - Z_n) = \emptyset$ and therefore $V_n \cap (X - Z_n) = \emptyset$. But it contradicts the fact that $\mathcal{G}$ is a filter. Thus $Z_n \in \mathcal{F}$ for each n, and $\bigcap_{n\in\mathbb{N}} Z_n = \emptyset$. This shows that X is realcompact. $\square$

Next we show that the notions of almost realcompactness and r-realcompactness are independent, neither of them implies the other. However, with some additional condition one could be obtained from the other.

Example 10. The Dieudonné plank D is almost realcompact, but not r-realcompact.

The Dieudonné plank D [14] is defined by $[0,\omega_1]\times[0,\omega]-\{(\omega_1,\omega)\}$, the points of the Tychonoff plank with the topology τ generated by declaring open each point of $[0,\omega_1)\times[0,\omega)$, together with the sets $U_\alpha(\beta)=\{(\beta,\gamma):\alpha<\gamma\leq\omega\}$ and $V_\alpha(\beta)=\{(\gamma,\beta):\alpha<\gamma\leq\omega_1\}$, where ω_1 (resp. ω) is the first uncountable (resp. infinite) ordinal. Thus points on the right edge have neighborhoods containing tails. The points on the top edge also have basic neighborhoods that contain tails (not "rectangles"). The Dieudonné topology τ on D is finer than the Tychonoff topology on $T=D$. Now in T, every closed G_δ-set is a zero-set [7]. Hence in T, every regular F_σ (open F_σ) is a cozero-set. Thus in Dieudonné plank D, every regular F_σ is a cozero-set and hence every ultrafilter of regular F_σ-subsets is an ultrafilter of cozero-sets and conversely. But Schommer [15] proved that in D, there exists an ultrafilter of cozero-sets with ccip having empty intersection. Hence there exists an ultrafilter of regular F_σ-subsets in D with ccip and empty intersection. Therefore D is not r-realcompact.

We now wish to search for the condition under which an almost realcompact space becomes r-realcompact.

Definition 11. X is said to be *regular countably paracompact* if for every decreasing sequence $\{F_n:n\in\mathbb{N}\}$ of closed sets with $\bigcap_{n\in\mathbb{N}}F_n=\emptyset$, there exists a decreasing sequence $\{H_n:n\in\mathbb{N}\}$ of regular F_σ-subsets with $F_n\subseteq H_n$ for each n, and $\bigcap_{n\in\mathbb{N}}\operatorname{Cl}_X H_n=\emptyset$.

Note 12. If X is regular countably paracompact, then X is countably paracompact. If X is normal and countably paracompact, then X is regular countably paracompact. Further if X is r-weak cb and regular countably paracompact, then X is a cb space.

Theorem 13. *If X is almost realcompact and regular countably paracompact, then X is r-realcompact.*

Proof: Following the usual technique let $\mathcal{H}$ be an ultrafilter of regular F_σ-subsets with $\bigcap\bar{\mathcal{H}}=\emptyset$. Let $\mathcal{R}=\{U: U$ is open and there exists $H\in\mathcal{H}$ with $H\subseteq U\}$. Clearly $\mathcal{H}$ is a subfamily of $\mathcal{R}$ and $\bigcap\bar{\mathcal{R}}\subseteq\bigcap\bar{\mathcal{H}}=\emptyset$, i.e., $\bigcap\bar{\mathcal{R}}=\emptyset$. Also $\mathcal{R}$ is a filter of open sets. Let $\mathcal{G}$ be an open ultrafilter containing $\mathcal{R}$. Next we show that $\bigcap\bar{\mathcal{G}}=\emptyset$ indeed. Let $p\in X$. Then $p\notin\operatorname{Cl}_X H$, i.e., $p\in X-\operatorname{Cl}_X H$ for some $H\in\mathcal{H}$. Since X is Tychonoff and hence regular, there exists an open set V such that $p\in V\subseteq\operatorname{Cl}_X V\subseteq X-\operatorname{Cl}_X H$. Thus $\operatorname{Cl}_X H\subseteq X-\operatorname{Cl}_X V$, i.e., $H\subseteq X-\operatorname{Cl}_X V$ and so $X-\operatorname{Cl}_X V\in\mathcal{R}\subseteq\mathcal{G}$. Also $p\notin X-\operatorname{Cl}_X V$, as $p\in\operatorname{Cl}_X V$. Further $p\in V$ has empty intersection with $X-\operatorname{Cl}_X V$, hence $p\notin\operatorname{Cl}_X(X-\operatorname{Cl}_X V)$. So $p\notin\bigcap\bar{\mathcal{G}}$. Since $p\in X$ is arbitrary, we conclude that $\bigcap\bar{\mathcal{G}}=\emptyset$.

Since X is almost realcompact and $\bigcap\bar{\mathcal{G}}=\emptyset$, $\bar{\mathcal{G}}$ does not have cip. So there exists a collection $\{V_n:n\in\mathbb{N}\}\subseteq\mathcal{G}$ with $\bigcap_{n\in\mathbb{N}}\operatorname{Cl}_X V_n=\emptyset$. Let $G_n=\bigcap\{V_i: i\leq n\}$. Then $\{G_n:n\in\mathbb{N}\}\subseteq\mathcal{G}$ is a decreasing sequence of open sets with $\bigcap_{n\in\mathbb{N}}\operatorname{Cl}_X G_n=\emptyset$. Since X is regular countably paracompact, there exists a decreasing sequence $\{F_n:n\in\mathbb{N}\}$ of regular F_σ-subsets with $\operatorname{Cl}_X G_n\subseteq F_n$ for each n, and $\bigcap_{n\in\mathbb{N}}\operatorname{Cl}_X F_n=\emptyset$. We now show that each F_n meets every member of $\mathcal{H}$. If not, there exists a set $H \in \mathcal{H}$ for which $H \cap F_n = \emptyset$ for some n. Then $H \subseteq X - \mathrm{Cl}_X F_n$ and so $X - \mathrm{Cl}_X F_n \in \mathcal{R} \subseteq \mathcal{G}$. Now $G_n \cap (X - \mathrm{Cl}_X F_n) = \emptyset$, contradicting the fact that $\mathcal{G}$ is a filter. Therefore $F_n \in \mathcal{H}$ for each n, and hence $\bar{\mathcal{H}}$ does not have cip. Hence X is r-realcompact. □

Here is an example to show that an r-realcompact space need not be almost realcompact.

Example 14. The Fringed plank is r-realcompact, but not almost realcompact.

For this we consider the Fringed plank [15]. The Fringed plank X is defined by $X = T \cup \{x_{j,n} : j, n \in \mathbb{N}\}$, where $T = [0, \omega_1] \times [0, \omega] - \{(\omega_1, \omega)\}$ is the Tychonoff plank. Here we added a convergent sequence $\{x_{j,n} : n \in \mathbb{N}\}$ to each point (ω_1, j) on the right edge of T. In the topology of X all the adjoined points are isolated and the points (ω_1, j), $j \in \mathbb{N}$ on the right edge, have their usual neighborhoods plus enough tails of these sequences, i.e., $V \cup \{x_{j,n} : n > m, n \in \mathbb{N}\}$, for each $m \in \mathbb{N}$, is a neighborhood of (ω_1, j) in X, where V is the usual neighborhood of (ω_1, j) in T.

Schommer [15] proved that X is almost* realcompact, but not almost realcompact. Now in T, every regular F_σ (open F_σ) is a cozero-set [7]. Also the set of the added points P, say, is regular F_σ-subsets and also cozero in X, since countable union of regular F_σ-subsets (resp. cozero-sets) is regular F_σ (resp. cozero). Now let H be a regular F_σ-subset of X, then $H \cap T$ and $H \cap P$ are regular F_σ-subsets of T and P respectively (Theorem 28), and hence cozero-sets of T and P respectively. It can be easily seen that $H \cap T$ and $H \cap P$ are also cozero-sets of X. Hence $H = (H \cap T) \cup (H \cap P)$ is a cozero-set of X. Thus every regular F_σ-subset of X is a cozero-set. So in X, almost* realcompactness and r-realcompactness are identical. Therefore X is r-realcompact.

Is there any property that can be added to r-realcompact space to convert it into almost realcompact space? Yes, regular Oz is one such property and it is defined as follows:

Definition 15. X is said to be *regular Oz* if whenever $\mathcal{A}$ is an ultrafilter of open sets of X, then $\mathcal{F} = \{F : F \text{ is regular } F_\sigma \text{ and } F \in \mathcal{A}\}$ is an ultrafilter of regular F_σ-subsets of X.

Theorem 16. *If X is r-realcompact and regular Oz, then X is almost realcompact.*

Proof: Let $\mathcal{A}$ is an ultrafilter of open sets of X with $\bigcap \bar{\mathcal{A}} = \emptyset$. Now since X is regular Oz, the family $\mathcal{F} = \{F : F \text{ is regular } F_\sigma \text{ and } F \in \mathcal{A}\}$ is an ultrafilter of regular F_σ-subsets of X and $\mathcal{F} \subseteq \mathcal{A}$. As in Theorem 9, we can show that $\bigcap \bar{\mathcal{F}} = \emptyset$.

Again since X is r-realcompact and $\bigcap \bar{\mathcal{F}} = \emptyset$, there is a collection $\{F_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $\bigcap_{n \in \mathbb{N}} \mathrm{Cl}_X F_n = \emptyset$. But each $F_n \in \mathcal{A}$, so $\bar{\mathcal{A}}$ does not have the cip. Thus X is almost realcompact. □

Next we show that a weaker form of Oz space [3] possesses the regular Oz property.

Definition 17. A space X is *almost Oz* if the closure of every regular F_σ-subset is a zero-set.

Remark 18. Clearly every Oz space is almost Oz and every almost Oz space is weak Oz. (Recall that a space is Oz [3] if every regular closed set is a zero-set and the space is weak Oz [12] if the closure of every cozero-set is a zero-set.)

Theorem 19. *Every almost Oz space is regular Oz.*

Proof: Let X be an almost Oz space. To prove the theorem let us suppose the contrary. Then there is an ultrafilter $\mathcal{A}$ of open sets of X such that $\mathcal{F} = \{P : P$ is regular F_σ and $P \in \mathcal{A}\}$ is not an ultrafilter of regular F_σ-subsets of X. Clearly $\mathcal{F}$ is a filter of regular F_σ-subsets of X. Let $\mathcal{B}$ be an ultrafilter of regular F_σ-subsets of X containing $\mathcal{F}$. Thus there exists a regular F_σ-subset $U \in \mathcal{B}$ with $P \cap U \neq \emptyset$ for every $P \in \mathcal{F}$, but $U \notin \mathcal{F}$. Now $U \cap F = \emptyset$, for some $F \in \mathcal{A}$ and hence $\mathrm{Cl}_X U \cap F = \emptyset$. Thus $F \subseteq X - \mathrm{Cl}_X U$. Since X is almost Oz, so $\mathrm{Cl}_X U$ is a zero-set. Then $V = X - \mathrm{Cl}_X U$ is a cozero-set, i.e., a regular F_σ-subset containing F. This implies that $V \in \mathcal{A}$ and since V is a regular F_σ-subset, $V \in \mathcal{F}$ and hence $V \in \mathcal{B}$. But it contradicts the fact that U and V are two members of the ultrafilter $\mathcal{B}$ such that $U \cap V = \emptyset$. This shows that X is regular Oz. $\square$

To study the relationship between almost* realcompactness and r-realcompactness we define the following:

Definition 20. A space X is said to have the property RC if whenever $\mathcal{F}$ is an ultrafilter of regular F_σ-subsets of X, then $\mathcal{G} = \{P : P$ is cozero and $P \in \mathcal{F}\}$ is an ultrafilter of cozero-sets.

Theorem 21. *If X is almost* realcompact space with the property RC, then X is r-realcompact.*

Proof: Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X with ccip, i.e., $\bar{\mathcal{F}}$ has cip. Now by the property RC of X, $\mathcal{G} = \{P : P$ is cozero and $P \in \mathcal{F}\}$ is an ultrafilter of cozero-sets and $\mathcal{G} \subseteq \mathcal{F}$. Since X is almost* realcompact and $\bar{\mathcal{G}}$ has the cip, we must have $\bigcap \bar{\mathcal{G}} \neq \emptyset$.

Now we shall show that $\bigcap \bar{\mathcal{F}} \neq \emptyset$. Let us assume the contrary, i.e., $\bigcap \bar{\mathcal{F}} = \emptyset$. As in Theorem 9, we can easily show that $\bigcap \bar{\mathcal{G}} = \emptyset$, and we arrive at a contradiction. Hence $\bigcap \bar{\mathcal{F}} \neq \emptyset$ and X is r-realcompact. $\square$

One will be naturally interested to inquire the conditions under which r-realcompactness implies almost* realcompactness. In this direction we have a theorem (Theorem 23). Before this we recall the following definition.

Definition 22. X is said to be super countably paracompact [15] if for every decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of closed sets with $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, there exists a decreasing sequence $\{P_n : n \in \mathbb{N}\}$ of cozero-sets with $F_n \subseteq P_n$ for each n, and $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X P_n = \emptyset$.

Theorem 23. *If X is r-realcompact, super countably paracompact and weak Oz, then X is almost* realcompact.*

Proof: Let $\mathcal{F}$ be an ultrafilter of cozero-sets with $\bigcap \bar{\mathcal{F}} = \emptyset$. Let $\mathcal{A} = \{U : U$ is regular F_σ-subset and there exists $F \in \mathcal{F}$ with $F \subseteq U\}$. Clearly $\mathcal{A}$ is a filter of regular F_σ-subsets of X. Let $\mathcal{G}$ be an ultrafilter of regular F_σ-subsets of X containing $\mathcal{A}$. As before (Theorem 9) we can verify that $\bigcap \bar{\mathcal{G}} = \emptyset$.

Again X is r-realcompact and $\bigcap \bar{\mathcal{G}} = \emptyset$, so there must exist a collection $\{G_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$ with $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X G_n = \emptyset$. Now let $V_n = \bigcap\{G_i : i \leq n\}$. Then $\{V_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$ is a decreasing sequence of regular F_σ-subsets of X with $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X V_n \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X G_n = \emptyset$ and hence $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X V_n = \emptyset$. By super countable paracompactness of the space X, there exists a collection $\{P_n : n \in \mathbb{N}\}$ of cozero-sets with $\mathrm{Cl}_X V_n \subseteq P_n$ for each n, and $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X P_n = \emptyset$.

Next our aim is to show that $P_n \in \mathcal{F}$ for each $n \in \mathbb{N}$. To show this we need to prove that $P_n \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. Let us suppose the contrary, i.e., there exists a set $F \in \mathcal{F}$ with $P_n \cap F = \emptyset$ for some n. Then $F \subseteq X - \mathrm{Cl}_X P_n$. By weak Oz property of X, $\mathrm{Cl}_X P_n$ is a zero-set. Hence $X - \mathrm{Cl}_X P_n$ is a cozero-set and so a regular F_σ-subset of X which contains $F \in \mathcal{F}$. Thus $X - \mathrm{Cl}_X P_n \in \mathcal{A} \subseteq \mathcal{G}$. Again $\mathrm{Cl}_X V_n \subseteq P_n \subseteq \mathrm{Cl}_X P_n$ and so $\mathrm{Cl}_X V_n \cap (X - \mathrm{Cl}_X P_n) = \emptyset$. Therefore $V_n \cap (X - \mathrm{Cl}_X P_n) = \emptyset$. But it contradicts the fact that V_n and $X - \mathrm{Cl}_X P_n$ are two members of $\mathcal{G}$. Therefore $P_n \in \mathcal{F}$ for each $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X P_n = \emptyset$. This shows that X is almost* realcompact. □

From Theorem 21 and Theorem 23, it appears that the two concepts of almost* realcompactness and r-realcompactness are independent. However, for confirmation we are in search for an example. So far in examples considered every regular F_σ is a cozero. Our aim is to find a space (of course completely regular) wherein not all regular F_σ are cozero-sets.

Next we study some properties of r-realcompact spaces.

That r-realcompactness is not closed hereditary is shown in the following example.

Example 24. We consider the Fringed plank X which is r-realcompact (Example 14). But Tychonoff plank T is a closed subspace of X which is not r-realcompact (Example 38).

However, the closure of regular F_σ-subset in an r-realcompact space is r-realcompact. For this we require a few results that we prove first.

Lemma 25. *If $Y \subseteq X$ and F is closed in X, then* $\mathrm{Int}_X F \cap Y \subseteq \mathrm{Int}_Y(F \cap Y)$.

Proof: The proof is straightforward. □

For the reverse inclusion we have the following theorem:

Theorem 26. *If F is closed in X and Y is dense in X, then* $\mathrm{Int}_Y(F\cap Y) \subseteq \mathrm{Int}_X F$.

Proof: Let $p \in \mathrm{Int}_Y(F \cap Y)$, then there exists an open set U_p in Y such that $p \in U_p \subseteq F \cap Y$. Let U'_P be open in X such that $U'_P \cap Y = U_P$. Therefore $p \in U'_P \cap Y \subseteq F \cap Y$. We now show that $U'_P \subseteq F$. If not, then $U'_P - F$ is an open set of X lying in $Y - X$, which contradicts the fact that Y is dense in X. Hence $U'_P \subseteq F$, i.e., $U'_P \subseteq \mathrm{Int}_X F$. Thus $p \in U'_P \subseteq \mathrm{Int}_X F$. Hence the result follows. □

The above result may not hold if Y is not dense in X.

Example 27. Let $X = [0,1]$ and $Y = (0,1) - \{\frac{1}{2}\}$. X and Y have their usual topologies and refined so that $\{\frac{1}{2}\}$ is an isolated point in X. Now $F = [0, \frac{1}{2}]$ is closed in X. Then $F \cap Y = (0, \frac{1}{2})$ and so $\mathrm{Int}_Y(F \cap Y) = (0, \frac{1}{2})$. Again $\mathrm{Int}_X F = (0, \frac{1}{2}]$. Thus $\mathrm{Int}_Y(F \cap Y) \neq \mathrm{Int}_X F$.

Theorem 28. *If G is a regular F_σ-subset of a space X and $Y \subseteq X$, then $G \cap Y$ is a regular F_σ-subset of Y.*

PROOF: Let $G = \bigcup_n F_n = \bigcup_n \mathrm{Int}_X F_n$, where each F_n is closed in X. Let $F_n \cap Y = K_n$; then each K_n is closed in Y. Now we define $V = \bigcup_n K_n$. Next we show that V is indeed a regular F_σ-subset, i.e., we prove that $V = \bigcup_n K_n = \bigcup_n \mathrm{Int}_Y K_n$. Since $\mathrm{Int}_Y K_n \subseteq K_n$, we always have $\bigcup_n \mathrm{Int}_Y K_n \subseteq \bigcup_n K_n$.

To prove the reverse inclusion, let $p \in \bigcup_n K_n$. Then $p \in K_n$ for some n, and hence $p \in F_n$ for some n. This implies that $p \in \bigcup_n F_n = \bigcup_n \mathrm{Int}_X F_n$ and hence $p \in \mathrm{Int}_X F_n$ for some n. Thus $p \in \mathrm{Int}_X F_n \cap Y \subseteq \mathrm{Int}_Y(F_n \cap Y) = \mathrm{Int}_Y K_n$ (Lemma 25), i.e., $p \in \mathrm{Int}_Y K_n$ and hence $p \in \bigcup_n \mathrm{Int}_Y K_n$. Thus $\bigcup_n K_n \subseteq \bigcup_n \mathrm{Int}_Y K_n$. Therefore $V = \bigcup_n K_n = \bigcup_n \mathrm{Int}_Y K_n$ and it is a regular F_σ-subset of Y such that $V = G \cap Y$. Hence, the theorem follows. □

The converse of the above theorem is not always true as we have the following example:

Example 29. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, e\}, \{a, b, c, e\}\}$. Then τ-closed sets of X are $\{c, d, e\}$, $\{b, c, d, e\}$, $\{d\}$ and $\{b, d\}$. Now we consider a subspace $Y = \{b, c, e\}$ of X. Then open sets of Y are $\emptyset$, Y, $\{b\}$, $\{c, e\}$, $\{b, c, e\}$ and so closed sets of Y are $\emptyset$, Y, $\{c, e\}$ and $\{b\}$. Clearly $B = \{c, e\}$ is an open, as well as, closed subset of Y and hence a regular F_σ-subset of Y. The subsets of X that have B as the intersection with Y, are given by $\{a, c, e\}$, $\{c, e\}$, $\{c, d, e\}$ and $\{a, c, d, e\}$. But none of these is a regular F_σ-subset of X.

Thus we have seen that if G is a regular F_σ-subset of a space X and $Y \subseteq X$, then $G \cap Y$ is a regular F_σ-subset of Y. But the converse of this result is not true. This prompts us to define the following:

Definition 30. A subspace X of a space T is said to be *regular F_σ-embedded* in T if for each regular F_σ-subset B of X there exists a regular F_σ-subset A of T such that $B = A \cap X$.

The regular F_σ-embedded property of a subspace will be studied elsewhere.

Theorem 31. *Every regular F_σ-subset of a topological space is regular F_σ-embedded.*

PROOF: Let Y be a regular F_σ-subspace of X and H be a regular F_σ-subset of Y. Then $Y - H$ is a regular G_δ-subset of Y. Thus there exists a regular G_δ-subset A of X such that $Y - H = A \cap Y$, [2]. Hence $H = Y - (A \cap Y) = (X - A) \cap Y = B \cap Y$, where $B = X - A$ is a regular F_σ-subset of X. Therefore for each regular F_σ-subset H of Y, there exists a regular F_σ-subset B in X such that $H = B \cap Y$. Thus Y is regular F_σ-embedded in X. □

Corollary 32. *If Y is a regular F_σ-subset of a space X, then every regular F_σ-subset of Y is also a regular F_σ-subset of X.*

PROOF: Let H be a regular F_σ-subset of Y. Then by Theorem 28, there exists a regular F_σ-subset B of X such that $H = B \cap Y$. Y being regular F_σ in X, $B \cap Y$ is also a regular F_σ-subset of X and hence H is also a regular F_σ-subset of X. □

Corollary 33. *If Y is a regular F_σ-subset of a space X, then for each ultrafilter $\mathcal{F}$ of regular F_σ-subsets of $P = \mathrm{Cl}_X Y$, $\mathcal{F}|Y$ is a filter of regular F_σ-subsets of Y.*

PROOF: To prove that $\mathcal{F}|Y = \{F \cap Y : F \in \mathcal{F}\}$ is a filter of regular F_σ-subsets of Y, we observe the following:

(i) Each member $F \in \mathcal{F}$ being open in $P = \mathrm{Cl}_X Y \subseteq X$, there exists an open subset G of X such that $G \cap P = F$. Now $G \cap Y \subseteq G \cap P = F$ and hence $G \cap Y \subseteq F \cap Y$. Now $G \cap Y \neq \emptyset$, since G is open in X and $G \cap P = G \cap \mathrm{Cl}_X Y = F$. So $F \cap Y \neq \emptyset$ and $\emptyset \notin \mathcal{F}|Y$.

(ii) Let $A_1, A_2 \in \mathcal{F}|Y$. Then $A_1 = F_1 \cap Y$ and $A_2 = F_2 \cap Y$, where $F_1, F_2 \in \mathcal{F}$. Now $A_1 \cap A_2 = (F_1 \cap Y) \cap (F_2 \cap Y) = (F_1 \cap F_2) \cap Y \in \mathcal{F}|Y$.

(iii) Let $A \in \mathcal{F}|Y$ and A_1 be a regular F_σ-subset in Y such that $A \subseteq A_1$. Since Y is regular F_σ in X, by Corollary 32, A and A_1 are regular F_σ-subsets of X and hence of P, as $Y \subseteq P \subseteq X$. Since $A \in \mathcal{F}|Y$, there exists $B \in \mathcal{F}$ such that $A = B \cap Y$. Lastly, Y is a regular F_σ-subset of P and we claim that $Y \in \mathcal{F}$ (an ultrafilter). Otherwise, there will be some $C \in \mathcal{F}$ such that $C \cap Y = \emptyset$, which is impossible. Hence $A = B \cap Y \in \mathcal{F}$, which in turn implies that $A_1 (\supseteq A)$ must belong to $\mathcal{F}$ and hence $A_1 \in \mathcal{F}|Y$. □

Theorem 34. *Let X be r-realcompact and Y be a regular F_σ-subspace of X. Then $\mathrm{Cl}_X Y$ is r-realcompact.*

PROOF: Let $F = \mathrm{Cl}_X Y$, where Y is a regular F_σ-subspace in X, and let $\mathcal{F}$ be an ultrafilter of all regular F_σ-subsets of F with ccip. By Corollary 33, $\mathcal{F}|Y$ is a filter of regular F_σ-subsets of Y. Since every regular F_σ-subset of Y is also a regular F_σ-subset of X, let us consider an ultrafilter $\mathcal{G}$ of regular F_σ-subsets of X such that $\mathcal{F}|Y \subseteq \mathcal{G}$. Then $\mathcal{G}$ has ccip. To prove this, let us consider a regular F_σ-subset $P \in \mathcal{G}$. Now since $H \cap Y$ is a member of $\mathcal{F}|Y$ for every $H \in \mathcal{F}$, we have $P \cap H \cap Y \neq \emptyset$. Thus $P \cap H \cap Y \in \mathcal{F}|Y$ and since $P \cap H \cap Y \subseteq P$, $\mathcal{F}|Y$ must be a base for $\mathcal{G}$. Now let $\{V_n : n \in \mathbb{N}\}$ be a collection of regular F_σ-subsets of $\mathcal{G}$. Since $\mathcal{F}|Y$ is a base, there exists $U_n \subseteq V_n$, for each $n \in \mathbb{N}$ with $U_n \in \mathcal{F}|Y$. For each $n \in \mathbb{N}$, there exists a $H_n \in \mathcal{F}$ such that $H_n \cap Y = U_n$. Since $\mathcal{F}$ has ccip, there exists a $p \in \bigcap_{n \in \mathbb{N}} \mathrm{Cl}_F (H_n \cap Y) = \bigcap_{n \in \mathbb{N}} \mathrm{Cl}_F U_n \subseteq \bigcap_{n \in \mathbb{N}} \mathrm{Cl}_F (V_n \cap Y) \subseteq \bigcap_{n \in \mathbb{N}} \mathrm{Cl}_X V_n$ and so $\mathcal{G}$ has ccip as well. Since X is r-realcompact and $\mathcal{G}$ has ccip, $\bigcap \bar{\mathcal{G}} \neq \emptyset$. Now $\bigcap_{P \in \mathcal{G}} \mathrm{Cl}_X P \subseteq \bigcap_{H \in \mathcal{F}} \mathrm{Cl}_X (H \cap Y) \subseteq \bigcap_{H \in \mathcal{F}} \mathrm{Cl}_X H$ and it follows that $\bigcap \bar{\mathcal{F}} \neq \emptyset$. Hence $F = \mathrm{Cl}_X Y$ must be r-realcompact. □

Theorem 35. *If Y is regular F_σ-embedded in X, then for each ultrafilter $\mathcal{F}$ of regular F_σ-subsets of X which meets Y, $\mathcal{F}|Y$ is an ultrafilter on Y.*

PROOF: Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X, which meets $Y \subseteq X$. We show that $\mathcal{F}|Y = \{F \cap Y : F \in \mathcal{F}\}$ is an ultrafilter of regular F_σ-subsets of Y. We know that if F is a regular F_σ-subset of X and $Y \subseteq X$, then $F \cap Y$ is a regular F_σ-subset of Y. Thus $\mathcal{F}|Y = \{F \cap Y : F \in \mathcal{F}\}$ is a family of regular F_σ-subsets of Y. Clearly $\mathcal{F}|Y$ is a filter. Because $\emptyset \notin \mathcal{F}|Y$, since every regular F_σ-subset of X which belongs to $\mathcal{F}$ meets Y. Again let $(F_1 \cap Y)$ and $(F_2 \cap Y) \in \mathcal{F}|Y$ for $F_1, F_2 \in \mathcal{F}$. Then $(F_1 \cap Y) \cap (F_2 \cap Y) = (F_1 \cap F_2) \cap Y \in \mathcal{F}|Y$, since $F_1 \cap F_2 \in \mathcal{F}$. Lastly, let $F \cap Y \in \mathcal{F}|Y$ and V be a regular F_σ-subset of Y containing $F \cap Y$. Let G be a regular F_σ-subset of X such that $G \cap Y = V$, since Y is regular F_σ-embedded in X. Now $F \cup G$ is a regular F_σ-subset of X and hence $(F \cup G) \in \mathcal{F}$, as $F \in \mathcal{F}$. Also $(F \cup G) \cap Y = (F \cap Y) \cup (G \cap Y) = (F \cap Y) \cup V = V \in \mathcal{F}|Y$. Hence $\mathcal{F}|Y$ is a filter. Now we will show that $\mathcal{F}|Y$ is indeed an ultrafilter. Let K be a regular F_σ-subset of Y that meets each member of the filter $\mathcal{F}|Y$. We want to show that $K \in \mathcal{F}|Y$. By our assumption, there exists a regular F_σ-subset K' of X such that $K' \cap Y = K$. As K meets each member of $\mathcal{F}|Y$ and K' contains K, K' meets each member of $\mathcal{F}|Y$ and hence each member of $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter it follows that $K' \in \mathcal{F}$. Hence $K = K' \cap Y \in \mathcal{F}|Y$. Thus $\mathcal{F}|Y$ is an ultrafilter. □

In the following, the relationship between r-realcompactness and c-realcompactness will be studied.

Definition 36. A space X is *c-realcompact* [4] *iff* for every $p \in \beta X - X$ there exists a normal lower semi continuous (*nlsc*) function f on βX such that $f(p) = 0$ and f is positive on X. Equivalently, a space X is c-realcompact [8] *iff* for every point $p \in \beta X - X$, there exists a decreasing sequence $\{A_n\}$ of regular closed subsets of βX with $p \in \bigcap_{n\in\mathbb{N}} A_n$ while $\bigcap_{n\in\mathbb{N}}(A_n \cap X) = \emptyset$.

Theorem 37. *Every r-realcompact space is c-realcompact.*

PROOF: Let us consider a point $p \in \beta X - X$. Let $\mathcal{F}$ be an ultrafilter of all regular F_σ-subsets of βX containing p. Now we claim that $\bigcap \bar{\mathcal{F}} = \{p\}$. If possible let $q \in \bigcap \bar{\mathcal{F}}$, $q \neq p$. Now since βX is Hausdorff, there exists disjoint open sets U_p and U_q containing p and q respectively. Again in a completely regular space every neighborhood of a point contains a zero-set neighborhood of the point, so we can find two zero-sets Z_1 and Z_2 such that $p \in X - Z_1 \subseteq Z_2 \subseteq U_p$. Then $X - Z_1$ is a cozero-set and hence regular F_σ-subset containing p and is disjoint from U_q. Thus $X - Z_1 \in \mathcal{F}$. Hence $q \notin \bigcap \bar{\mathcal{F}}$, a contradiction. Then $\mathcal{G} = \mathcal{F}|X$ is an ultrafilter of regular F_σ-subsets of X with $\bigcap \mathrm{Cl}_X \mathcal{G} = \emptyset$. By hypothesis, there exists a sequence (which may be supposed to be decreasing) $\{F_i : i \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $\bigcap_{i\in\mathbb{N}} \mathrm{Cl}_X(F_i \cap X) = \emptyset$. Now we define $f_i(x) = 0$ if $x \in \mathrm{Cl}_X F_i$ and $f_i(x) = 1$ otherwise, with $0 \le f_i \le 1$ for all $i \in \mathbb{N}$. Now let $f = \sum_{i\in\mathbb{N}} \frac{f_i}{2^i}$. Then f is nlsc function [4], such that $f(p) = 0$ and f is positive on X. Hence X is c-realcompact. □

In the following, the preservation of r-realcompact under mappings will be studied. First we show that r-realcompactness is not preserved under perfect map (continuous, closed and compact).

Example 38. We recall that the 'Fringed plank' X is r-realcompact. Let $f : X \to T$ (Tychonoff plank) be the identity mapping on $T \subset X$, while all the added points in each sequence go to the point to which the sequence converges. Schommer [15] observed that this map is perfect, X is almost* realcompact while T is not. Now in X, as well as in T, every regular F_σ is a cozero-set. Hence, the two concepts of r-realcompactness and almost* realcompactness are identical in these spaces. Thus under the above perfect mapping f, r-realcompactness is not preserved. It may be mentioned here that the set P of all added points in X is a regular F_σ-subset of X. But under f, P is mapped to the right edge of T, which is not a regular F_σ-subset of T. This example also shows that under a perfect map, the image of a regular F_σ-subset may not be regular F_σ.

Definition 39. A mapping from a space X to a space Y is said to be *regular F_σ-preserving* if the image of every regular F_σ-subset of X is regular F_σ in Y.

Theorem 40. *The image of an r-realcompact space under a countably compact, continuous, onto and regular F_σ-preserving mapping is r-realcompact.*

PROOF: Let X be a r-realcompact space and $f : X \to Y$ be a countably compact, continuous, onto and regular F_σ-preserving mapping. Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of Y with $\bigcap \bar{\mathcal{F}} = \emptyset$. Since inverse image of a regular F_σ-subset under a continuous mapping is regular F_σ [11], $f^{-1}(\mathcal{F}) = \{f^{-1}(F) : F \in \mathcal{F}\}$ is a family of regular F_σ-subsets of X closed under finite intersection and does not contain $\emptyset$. Thus $f^{-1}(\mathcal{F})$ is a filter base. Now there exists an ultrafilter $\mathcal{A}$ of regular F_σ-subsets of X containing $f^{-1}(\mathcal{F})$. It can be seen that $\bigcap \bar{\mathcal{A}} \subseteq \bigcap \mathrm{Cl}_X(f^{-1}(\mathcal{F})) = \emptyset$. Now let $V \in \mathcal{A}$. Since f is regular F_σ-preserving, $f(V)$ is regular F_σ in Y. If $f(V) \notin \mathcal{F}$, there exists a regular F_σ-subset F' of Y such that $F' \subseteq Y - f(V)$ and $F' \in \mathcal{F}$. It follows that $f^{-1}(F') \subseteq f^{-1}(Y - f(V)) = X - V$, i.e., the two members $f^{-1}(F')$ and V of $\mathcal{A}$ are disjoint, which is impossible. Hence for $V \in \mathcal{A}$, $f(V) \in \mathcal{F}$. Since X is r-realcompact, there exists a countable sequence $\{V_i\}$ of $\mathcal{A}$ such that $\bigcap_i \mathrm{Cl}_X V_i = \emptyset$. The sequence $\{V_i\}$ can be supposed to be decreasing. Again f being countably compact, for each $y \in Y$, the family $\{f^{-1}(y) \cap \mathrm{Cl}_X V_i\}$, $i \in \mathbb{N}$, does not have the fip. So there exists a $k \in \mathbb{N}$ such that $f^{-1}(y) \cap \mathrm{Cl}_X V_k = \emptyset$, which implies that $y \notin f(V_k) \in \mathcal{F}$. Thus $\{f(V_i)\}$ is a countable subfamily of $\mathcal{F}$ such that $\bigcap_i \mathrm{Cl}_Y f(V_i) = \emptyset$. Hence Y is r-realcompact. □

Remark 41. Next we see that r-realcompactness is neither inversely preserved by perfect maps $f : X \to Y$. For this, as in [15], we construct the range space Y to be the Fringed plank while the domain X consists of the disjoint union of the Tychonoff plank together with ω many copies of the convergent sequence. Let us consider the mapping $f : X \to Y$ which is the identity map for the points on Tychonoff plank while under f each point of the first copy of the convergent sequence is mapped to the corresponding point of the sequence $\{x_{j,0} : j \in \mathbb{N}\}$

of Y, points of the second copy are mapped to the corresponding points of the sequence $\{x_{j,1} : j \in \mathbb{N}\}$ of Y and so on. This map between X and Y is perfect [15]. But the Tychonoff plank T is the closure of a cozero-set, say P of X, i.e., $T = \mathrm{Cl}_X P$ (recall that every cozero-set is regular F_σ). By Theorem 34, X is not r-realcompact, since T is not r-realcompact.

Before we conclude, let us study the productivity of r-realcompactness. It is not known whether r-realcompact is productive or not. However, under certain condition on the factor spaces, an arbitrary product becomes r-realcompact. The property called RFP is such a condition defined as follows:

Definition 42. A topological space X is said to satisfy the property RFP if, whenever an ultrafilter $\mathcal{F}$ of regular F_σ-subsets of X contains a prime filter of regular F_σ-subsets of X with ccip, then $\mathcal{F}$ has ccip.

Before proceeding to the main theorem we first prove the following lemmas.

Lemma 43. *If $f : X \to Y$ is continuous and $\mathcal{F} \subseteq \mathcal{R}_f(X)$ is a prime filter, then the family $\mathcal{A} = \{A \in \mathcal{R}_f(Y) : f^{-1}(A) \in \mathcal{F}\}$ is also a prime filter.*

Proof: Since the inverse image of a regular F_σ-subset under a continuous map is regular F_σ, $\mathcal{A}$ is a family of regular F_σ-subsets of Y closed under finite intersection and does not contain $\emptyset$. Thus $\mathcal{A}$ is a filter base. Let $A \in \mathcal{A}$ and $B \in \mathcal{R}_f(Y)$ such that $A \subseteq B$. Then $f^{-1}(A) \subseteq f^{-1}(B)$, $f^{-1}(A) \in \mathcal{F}$ and hence $f^{-1}(B) \in \mathcal{F}$, which in turn implies that $B \in \mathcal{A}$. Thus $\mathcal{A}$ is a filter. To prove that the filter $\mathcal{A}$ is indeed prime, let $A_1 \cup A_2 \in \mathcal{A}$ and $A_1 \notin \mathcal{A}$. Since $A_1 \cup A_2 \in \mathcal{A}$, $f^{-1}(A_1 \cup A_2) \in \mathcal{F}$, i.e., $f^{-1}(A_1) \cup f^{-1}(A_2) \in \mathcal{F}$. Also $f^{-1}(A_1) \notin \mathcal{F}$, so $f^{-1}(A_2) \in \mathcal{F}$ and hence $A_2 \in \mathcal{A}$. Therefore $\mathcal{A}$ is prime. $\square$

Lemma 44. *Let $\mathcal{F}$ be a prime filter of regular F_σ-subsets of the product space $X = \prod_\alpha X_\alpha$. Then the family $\pi^\sharp_\alpha \mathcal{F} = \{F_\alpha \in \mathcal{R}_f(X_\alpha) : \pi_\alpha^{-1}(F_\alpha) \in \mathcal{F}\}$ is a prime filter of regular F_σ-subsets of X_α, where π_α is the α-th projection map from the product space X to X_α.*

Proof: The proof follows from Lemma 43, since the projection mappings are continuous. $\square$

Lemma 45. *Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets on $X = \prod_\alpha X_\alpha$. If every prime filter $\pi^\sharp_\alpha \mathcal{F}$ is fixed, then $\mathcal{F}$ is also fixed.*

Proof: For each α, we choose $x_\alpha \in \bigcap \pi^\sharp_\alpha \mathcal{F}$ and let $x = \{x_\alpha\}$, then $x \in X$. To prove the assertion it suffices to show that $x \in \bigcap \mathcal{F}$. From the construction of x, x belongs to every member of $\mathcal{F}$ of the form $\pi_\alpha^{-1}(F_\alpha)$, where $F_\alpha \in \pi^\sharp_\alpha \mathcal{F}$, since $\pi_\alpha^{-1}(x_\alpha) = \{t \in X : \pi_\alpha(t) = x_\alpha\} \subseteq \pi_\alpha^{-1}(F_\alpha)$. Again every cozero-set being regular F_σ, x belongs to every member of $\mathcal{F}$ having the form $\pi_{\alpha_k}^{-1}(X_{\alpha_k} - Z_k)$, where Z_k is a zero-set in X_{α_k}. Further it is known that the collection of all finite intersections like $\bigcap_{k=1}^n \pi_{\alpha_k}^{-1}(X_{\alpha_k} - Z_k)$ of cozero-sets is a base for the open sets in X [7] and contains x. Now an arbitrary member F of $\mathcal{F}$ is a union of members of this base and hence it also contains x. Since F is arbitrary, $x \in \bigcap \mathcal{F}$. $\square$

Theorem 46. *An arbitrary product of r-realcompact spaces, where each factor space has RFP, is r-realcompact.*

PROOF: Let $X = \prod_\alpha X_\alpha$, where each X_α is r-realcompact. Now X is completely regular. Let $\mathcal{F}$ be an ultrafilter of regular F_σ-subsets of X with ccip. By Lemma 44, the family $\pi^\sharp_\alpha \mathcal{F} = \{G_\alpha \in \mathcal{R}_f(X_\alpha) : \pi_\alpha^{-1}(G_\alpha) \in \mathcal{F}\}$ is a prime filter of regular F_σ-subsets of X_α, for each α. Since $\mathcal{F}$ has ccip, so has $\pi^\sharp_\alpha \mathcal{F}$. Now by the RFP property of X_α, the ultrafilter containing the prime filter $\pi^\sharp_\alpha \mathcal{F}$ also has ccip. Now X_α being r-realcompact, $\bigcap \pi^\sharp_\alpha \mathcal{F}$ is fixed. Hence by Lemma 45, $\mathcal{F}$ is fixed and the theorem follows. □

To conclude, the authors would like to examine the relationship of r-realcompact spaces with another class of generalized realcompact spaces, namely, $\aleph_1$-ultracompact spaces introduced by J. van der Slot [16]. We recall that a space X is said to be m-ultracompact for an infinite cardinal m and relative to a closed subbase $\mathcal{C}$ of X, *iff* each ultrafilter $\mathcal{F}$ in X, for which $\mathcal{F} \cap \mathcal{C}$ satisfies the m-intersection property, is convergent. In particular, for $m = \aleph_1$ we have $\aleph_1$-ultracompact spaces. Now in [6], Frolík has shown that for regular spaces $\aleph_1$-ultracompactness is equivalent to almost realcompactness. But the r-realcompactness and almost realcompactness are independent in Tychonoff spaces (Examples 10 and 14). From these it follows immediately that $\aleph_1$-ultracompactness is a property independent of r-realcompactness in Tychonoff spaces.

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